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Abstract

We review the de Boer-Verlinde-Verlinde formalism for the renormalization group in the context of Dp brane vacua. We comment on various aspects of the dictionary between bulk and boundary and relate the discussion to the Randall-Sundrum scenario. We find that the gravitational coupling for the Randall-Sundrum gravity on the Dp brane worldvolume is proportional to the central charge of the Yang-Mills theory. We compute the beta function and find the expected uneventful flow prescribed by the classical dimension of the Yang-Mills operator. Finally, we argue for a dynamical mechanism for determining the cosmology on the brane.

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1 Introduction

At the foundation of the string theoretical incarnation of the holographic principle [1, 2, 3, 4] is an intriguing relation between renormalization group (RG) flow and the equations of motion of gravity. This connection was made recently more transparent in the work of [5] by casting the Einstein equations into the form of Hamiltonian evolution across timelike foliations. In this note, we present a series of comments regarding this approach in the context of studying RG flow in Dp brane geometries.

The proposal [5] raises many intriguing questions regarding the dictionary between bulk and boundary. We list a sample of questions that drove us to investigate this subject:

- Given that the prescription involves an explicit interpretation of the gravitational Hamiltonian equations as describing RG evolution, one may wonder whether this approach amounts to reconstructing the boundary theory within the gravitational initial value problem. There is then an apparent mismatch between the amount of data required to define a boundary theory and that required to uniquely specify the bulk.

- One would like to clarify the proper interpretation of the Randall-Sundrum gravity [6] on the boundary in the context of the flow equations. This has to do with studying the bulk/boundary correspondence with a *finite* cutoff and understanding the meaning of the slicing of a space from the perspective of renormalization of the boundary theory.

- The principle of renormalization group [7] prescribes the general behavior of a quantum field theory as a function of energy scale. As a statement in its most fundamental form, it relates the theory at different energy scales through diffeomorphisms in the space of its couplings. It is known however that this flow may also be endowed with additional structures [8, 9, 10, 11]. Of particular physical interest is a possible mechanism for driving it by a scalar function, the so-called c-function [8, 12]. Beta functions of a theory for example may be measures of the gradients of this function in coupling space. In this context, we expect that such additional features of RG flow play an important role in its relation to gravitational physics. One would like to determine how much structure the bulk equations impose on the boundary theory.

- There is room to better understand the dynamical aspects of cosmology in the Randall-Sundrum gravity [13, 14, 16] from the bulk point of view; the tuning of the brane tension so as to cancel the cosmological constant must be determined by the physics. Unravelling the content of the canonical momentum variable of the induced metric may provide part of the answer.

- The Hamiltonian evolution equations do not care about a preferred direction or "arrow" for the flow; whereas renormalization group does. Introducing this arrow into the formalism is a highly non-trivial statement related to a principle introduced by Bousso [17, 18].

We will attempt to comment on these issues in the process of examining extremal and near extremal Dp brane geometries. Let us mention as well that recently an interesting tangential exploration of some of these issues were carried out in an attempt to find a relation directly between closed and open string theories [19]. Related interesting work also appeared in [20] and [21]. The preprint [22] contains overlapping and complimentary material to this work.

We present our discussion as a review of [5] generalized to arbitrary dimensions, interdispersing the text with our examples and comments.

2 The de Boer-Verlinde-Verlinde formalism

We consider a bulk gravitational theory in D space-time dimensions, with matter added in the form of an arbitrary number of scalar fields [5]

$$S_D = \alpha_G \int_D \left(R^{(D)} + 2\Lambda \right) - \alpha_M \int_D \frac{1}{2} G_{IJ}(\phi) \nabla_\mu \phi^I \nabla^\mu \phi^J - V(\phi) \ . \tag{1}$$

We assume we are given some vacuum of this action corresponding to a trajectory (or a section of a trajectory) of renormalization group flow of a $D-1$ dimensional "boundary theory". We embed an arbitrary *timelike* surface in this space and choose Gaussian normal coordinates with respect to it (see the appendix for definitions and conventions used). We then cut the space along this surface while focusing on excitations in the region bounded by the cut. There will generally be two such regions, and the criterion is to pick the one through which null geodesics projected from the boundary tend to converge. This statement was first proposed by Bousso in the context of entropy bounds on gravitational vacua [17, 18]. The convergence of the geodesics is a statement related to the monotonicity of the c-function of the boundary theory [23]; regions of spacetime away from the boundary and deep into the bulk are then to be associated with physics at lower energy scales. Slicing space in this manner can be done at the expense of introducing a boundary term to the action

$$S_{tot} = S_D + 2\alpha_G \int_{D-1} K \ . \tag{2}$$

$K_c^c \equiv K$ is the trace of the extrinsic curvature of the boundary [2]. We then have

$$\frac{\delta S_{tot}}{\delta h^{ab}} = \alpha_G \sqrt{h} \left(K_{ab} - h_{ab} K \right) \equiv \alpha_G \sqrt{h}\ \tau_{ab} \ , \tag{3}$$

[2]Our definition for the normals to the boundary is such that they point *inward* toward the bulk.

where h_{ab} is the induced metric on the boundary. The additional piece assures that vacua of (1) are at minima of (2) when perturbations of the metric vanish at the boundary. The extrinsic curvature K_{ab} can be viewed from the perspective of the D dimensional bulk as an energy-momentum tensor with delta function support on the boundary. The space outside this region is thrown away and the interface at the cut may be thought of as sourcing the bulk. One is also left with boundary conditions on the scalar fields. In parallel to (3), we also define

$$\Omega_I \equiv \frac{1}{\alpha_M \sqrt{h}} \frac{\delta S}{\delta \phi^I} \ . \tag{4}$$

This setup is a low energy approximation of string theory with an IR cutoff; it is proposed that this string theory has a dual description through a $D-1$ dimensional effective boundary quantum field theory with a UV cutoff. The partition functions of the bulk and boundary theories are to be equated; in the low energy supergravity approximation, we have then the statement [2, 3, 5]

$$i\ S = \ln\ Z_{bndry}^{(\mu)} \ , \tag{5}$$

where μ is the UV cutoff scale, related to the location of the cut, and the dual theory is naturally coupled to the metric h_{ab}. In this approximation, we are treating the latter as a classical background to the boundary quantum field theory.

The forms of equations (3) and (4) suggest writing the bulk equations of motion in the Hamiltonian formalism. Hamiltonian flow from the boundary is a constrained system, as is typical of theories with gauge symmetries. The choice of the cut is arbitrary, and, fixing one, the system is still endowed with redundancies; these are handled with two sets of constraint equations on the initial value data at the boundary. The first is a statement regarding $D-1$ dimensional Poincaré invariance

$$D^a \tau_{ab} = \frac{\alpha_M}{2\alpha_G} \Omega_J D_b \phi^J \ ; \tag{6}$$

the second and the more interesting one is less transparent [5]

$$\begin{aligned} &\alpha_G \left(R + 2\Lambda\right) + \alpha_G \left(\tau_{ab}\tau^{ab} - \frac{\left(\tau_c^c\right)^2}{D-2}\right) \\ - \ &\alpha_M \left(\frac{1}{2} G_{IJ} D_a \phi^I D^a \phi^J - \frac{1}{2} G^{IJ} \Omega_I \Omega_J\right) + \alpha_M V = 0 \ , \end{aligned} \tag{7}$$

The bulk/boundary correspondence proposes to replace the bulk action appearing in these equations through Ω_I and τ_{ab} with that of an effective boundary theory in $D-1$ dimensions. Equation (7) is one of the most important statements of [5]. Recently, its relevance in perturbative regimes was also investigated [19].

Typically, renormalization group flow of a quantum field theory coupled to a classical background metric induces an Einstein gravity term in the effective action. A computation of the expectation value for the quantum field theory energy momentum tensor renormalizes both the Einstein tensor and the cosmological constant [24]. Therefore, a general form for the effective action at scale μ is given by [5]

$$S_{bndry} = \int_{D-1} \sqrt{h} \left(\Phi(\phi) R + U(\phi) + \frac{1}{2} M_{IJ}(\phi) D_a \phi^I D^a \phi^J \right) + \Gamma^{\mu}_{eff} + \cdots \ . \tag{8}$$

R denotes the scalar curvature constructed from the metric h_{ab}; we will write D^a for the covariant derivatives associated with this metric with the latin indices running over the $D-1$ boundary coordinates. Γ^{μ}_{eff} is the effective action of the boundary theory at scale μ with

$$\frac{1}{\sqrt{h}} \frac{\partial \Gamma^{\mu}_{eff}}{\partial \phi^I} \equiv \langle \mathcal{O}_I \rangle \ ; \qquad \frac{1}{\sqrt{h}} \frac{\partial \Gamma^{\mu}_{eff}}{\partial h^{ab}} \equiv \langle t_{ab} \rangle \propto \langle T_{ab} \rangle \ , \tag{9}$$

the $\mathcal{O}_I$'s being the operators appearing in the boundary theory, and $\langle T_{ab} \rangle$ being the part of the theory's energy momentum tensor that, by definition, acquires only the beta function anomaly

$$\langle T^c_c \rangle = \beta^I \langle \mathcal{O}_I \rangle \ . \tag{10}$$

The normalization $2 t_{ab} = T_{ab}$ will be determined from the Hamiltonian equations later. The boundary values for the scalars ϕ^I are to be equated with the *dimensionless* coupling constants of the boundary theory as measured at the scale μ. We have absorbed any explicit appearances of μ into the definition of the $\langle O_I \rangle$'s. The effective action is an infinite sum where all operators and higher derivative terms of the background metric are allowed subject to the symmetries inherited from the bulk. The $\Phi\ R$ term is referred to in the literature as the Randall-Sundrum gravity [6]. In this context, it acquires dual interpretations: from the point of view of the bulk, it is the zero mode of the D dimensional graviton dynamically confined to the boundary. From the boundary point of view, it is generated at low energies by the flow of the boundary quantum field theory. Note that the latter is part of a holographic image of the D dimensional gravitational bulk. Both viewpoints then eventually amount to the same physical interpretation for the origin of the Randall-Sundrum gravity.

It is easy to see that the form of equation (7) determines the coefficients of all the local terms in the boundary action. One treats the metric h_{ab} and the scalars ϕ^I as arbitrary classical fields thus generating a set of relations for the unknown functions U, Φ and M_{IJ} of (8)

$$2\alpha_G^2 \Lambda + \alpha_M \alpha_G V = \frac{D-1}{4(D-2)} U^2 - \frac{1}{2} \frac{\alpha_G}{\alpha_M} G^{IJ} \partial_I U \partial_J U \ ; \tag{11}$$

$$\alpha_G^2 = \Phi U \frac{D-3}{2(D-2)} - \frac{\alpha_G}{\alpha_M} G^{IJ} \partial_I \Phi \partial_J U \ ; \tag{12}$$

$$-\beta^K \left(\partial_K M_{IJ} - \partial_I M_{KJ} - \partial_J M_{KI}\right) = 2(D-2)\frac{\alpha_G \alpha_M}{U} G_{IJ} + (D-3) M_{IJ} \ . \tag{13}$$

One is also left, to this order in the expansion, with expressions originating from Γ^μ_{eff}

$$\left\langle t^c_c \right\rangle = -\left(\frac{\alpha_G}{\alpha_M}\right) \frac{D-2}{U} G^{IJ} \partial_I U \left\langle \mathcal{O}_J \right\rangle \ , \tag{14}$$

and

$$\left\langle t_{ab} \right\rangle \left\langle t^{ab} \right\rangle = \frac{1}{D-2} \left\langle t^c_c \right\rangle^2 - \frac{\alpha_G}{2\alpha_M} G^{IJ} \left\langle \mathcal{O}_I \right\rangle \left\langle \mathcal{O}_J \right\rangle \ . \tag{15}$$

Relations involving non-local data about the boundary theory arise from the first order differential equations describing the evolution of the system; we will later on explore this additional information in some detail. For now, let us apply what we have up to now to AdS and Dp brane geometries.

3 Gravity on the worldvolume of Dp branes

We take the AdS metric in D dimensions as

$$ds^2 = \frac{u^2}{l^2}\left(dx^2_{D-1}\right) + \frac{l^2}{u^2} du^2 \ . \tag{16}$$

We choose the foliation $u =$ constant, and cut the space along $u = u_c$. The space $u < u_c$ is candidate for holography; the $D-1$ dimensional boundary theory is conformal with a UV cutoff u_c breaking this symmetry in a trivial manner. Setting $V = 0$ and $D^a \phi^I = 0$ in (1), we obtain from (11) and (12)[3]

$$U = -\alpha_G \sqrt{8\Lambda \frac{D-2}{D-1}} \ , \quad \Phi = -\alpha_G \sqrt{\frac{(D-1)(D-2)}{2\Lambda (D-3)^2}} \ . \tag{17}$$

These relations were also found in [25] by requiring the finiteness of the energy content of the bulk vacuum as the boundary is taken to infinity. In the renormalization group language,

[3] The sign of U cannot be determined from (11); but once its sign is fixed, that of Φ follows. We have determined these signs by required positivity of the boundary theory's energy momentum tensor; this will become clear in equation (35).

the method corresponds to adding counterterms so as to cancel divergences resulting from taking the UV cutoff to infinity. The boundary action then takes the form

$$-S_{bndry} = \frac{\alpha_G^{(D-2)} l}{D-3} \int_{D-1} \sqrt{h} \left(R + 2\Lambda^{(D-1)} \right) + \cdots \ , \tag{18}$$

with the $D-1$ dimensional parameters

$$\alpha_G^{(D-1)} = \frac{\alpha_G l}{D-3} (\mu l)^{D-3} \ , \quad 2\Lambda^{(D-1)} = 2\frac{(D-2)(D-3)}{l^2} (\mu l)^2 \ ; \tag{19}$$

The scale l is defined by $2\Lambda \equiv \frac{(D-1)(D-2)}{l^2}$, which, in our convention, is positive for AdS spaces. From the UV-IR relation [26], we have used $\mu \sim u_c/l^2$. The UV cutoff scale μ appears in these equations since we have rescaled the metric so as to make h_{ab} Minkowski. This is because the boundary theory corresponding to our choice of foliation lives in flat space. More on this issue later. In the literature, the space is often arbitrarily cut at the energy scale $\mu = 1/l$, and we recognize the Randall-Sundrum relation [6] $G_4 = 2G_5/l$ for AdS_5 ($\alpha_G \equiv 1/16\pi G_D$). More generally, collecting the factors of l together in (19), we are fixing the value of the dimensionless gravitational coupling and the cosmological constant as measured at a cutoff scale μ. Note also that all terms appearing in (18) remain finite in the decoupling limit (as we shall also see explicitly later on).

Along the same line of thought, one may look at R^2 corrections to this boundary gravity. These would also be interpreted as being generated by the flow of the boundary quantum field theory to lower energy scales. Adding to the action the terms

$$-S_{bndry} \to \int_{D-1} \sqrt{h} \left(A(\phi) R^2 + B(\phi) R_{ab} R^{ab} + C(\phi) R_{abcd} R^{abcd} \right) \ , \tag{20}$$

and substituting this in the constraint equation (7), one obtains, in the case where the boundary theory is conformal,

$$A = -\frac{\alpha_G}{4} \frac{(D-1)}{(D-3)^2 (D-5)(D-2)} l^3 \ , \quad B = \alpha_G \frac{l^3}{(D-3)^2 (D-5)} \ , \quad C = 0 \ . \tag{21}$$

These are precisely the expressions proposed in [27, 28] as the counterterms needed to keep the action finite when the boundary is sent to infinity[4]. We note that the pole for $D = 5$ in these expression is harmless; as shown in [27], when evaluated for the AdS vacuum configuration, these factors cancel. The duality between bulk and boundary would be one expanded about this vacuum configuration, so that we expect these terms to remain as finite

[4]In comparing the coefficient, note our normalization of the Einstein term in (8).

corrections to the $D-1$ dimensional Einstein gravity. Rescaling the boundary metric so as to make it Minkowski for the AdS vacuum, we see that the R^2 terms scale as $R^2 l^2 (\mu l)^{-2}$; and with μ_0 being the energy scale of a process under consideration in the boundary theory, an infinite expansion in higher derivative of the metric will correspond to an expansion in powers of μ_0/μ (μ being the cutoff energy). Therefore, these corrections become important as we study processes in the boundary theory with energy progressively closer to the cutoff scale. This effect may be related to observations made in [29, 30] regarding non-local effects in the holographic image of the bulk; we will talk more on this issue in the next section.

While in the subject of higher order corrections to the Randall-Sundrum gravity, let us qualitatively analyze the effects of R^2 corrections to the *bulk* action; the physical origin of such corrections being expansions in the string tension and the string coupling. The D dimensional scalar curvature $R^{(D)}$ splits, as outlined in the appendix, into the $D-1$ dimensional scalar curvature R and the extrinsic curvature of the boundary K, schematically, as $R^{(D)} \sim R + K^2$. For corrections to Einstein gravity in the bulk of the form R^α, where α is a positive integer, we induce in the Lagrangian, among a series of other corrections, a term of the form $K^{2\alpha}$. This means that the canonical momentum τ will appear in the expression for K as $K \sim \tau^{2\alpha-1}$, amongst an infinite sequence of powers of τ and R. The Hamiltonian will then involve a term of the form $\tau^{2\alpha}$, with essentially the same coefficient as in the original R^2 term of the bulk action. Tracing carefully the lapse function in this process, one finds that the constraint equation (7) will get corrections involving powers of $\delta S/\delta h^{ab}$ and R; in particular, the term we traced above leads to a contribution of the form

$$\rightarrow \left(\frac{\delta S}{\delta h^{ab}} \right)^{2\alpha} . \tag{22}$$

In string theories with 32 supersymmetries, the first corrections will have $\alpha = 4$, and the coefficient will be proportional to α'^3. This implies that the gravitational coupling of the Randall-Sundrum gravity will in general get stringy corrections. We will argue later on that this coupling is also expected to be proportional to the c-function of the boundary theory. For Dp brane geometries with $p \neq 3$, we will see that the first non-trivial contribution to the Yang-Mills beta function comes from such contributions [5]. Finally, let us note that the R^2 terms in the boundary action that we just discussed will generally mix with terms of similar form arising from our discussion in the previous paragraph. There may however be a hierarchy between these two sets of contributions if we assume that operators in the

[5] In the case of D3 branes, it then better be the case that the coefficients of the scalar curvature arising from terms such as (22) sum to zero; for otherwise the central charge of the conformal Yang-Mills theory would get corrections of order $({g_Y}^2 N)^{-3/2}$ for $\alpha = 4$. Perhaps this is related to the observation that the $AdS_5 \times S^5$ solution does not get modified in the presence of higher order string corrections as shown in [31].

boundary theory coupling to the "tails" of stringy states in the bulk will have much higher mass dimension than those responsible for the R^2 terms of the previous discussion.

We next look for the Randall-Sundrum gravity for the case of Dp brane geometries; *i.e.* the classical gravitational sector coupled to $p+1$ dimensional Super Yang-Mills (SYM) theory; the extremal Dp brane solution is used in the bulk and corresponds to a canonical choice for the SYM vacuum in Minkowski space. To apply the previous formalism to this case, we would need to do a little of juggling to put the supergravity action in the desired form (1). This can easily be achieved by using the results of [32, 33]. One starts from IIA or IIB theory in the string frame, and goes to the so called "holographic dual frame"

$$g_{ab}^{(dual)} = \left(Ne^{\phi}\right)^{\frac{2}{p-7}} g_{ab}^{(str)} \ , \tag{23}$$

where ϕ is the dilaton and N is the number of Dp branes. For extremal Dp brane vacua, we also have

$$Ne^{\phi} = \left({g_Y}^2 N \left(\frac{5-p}{2}u\right)^{p-3}\right)^{\frac{7-p}{2(5-p)}} \equiv \left(g_{\rm eff}^2\right)^{\frac{7-p}{2(5-p)}} \ , \tag{24}$$

where ${g_Y}^2 = g_{str} l_{str}^{p-3}$, and u appears in the dual holographic frame as

$$ds_{dual}^2 = \alpha' \left(\frac{4u^2}{(5-p)^2}\left(-dt^2 + dx_{(p)}^2\right) + \frac{(5-p)^2}{4u^2}du^2 + d\Omega_{8-p}^2\right) \ . \tag{25}$$

We identify u as the energy scale of the SYM theory in flat $p+1$ dimensional space; and $g_{\rm eff}^2$ is the effective dimensionless Yang-Mills coupling measured at energy scale u. One then dimensionally reduces on the transverse $8-p$ sphere of constant radius to $D = p+2$ dimensions. Rescaling to the Einstein frame

$$g_{ab}^{(dual)} = \left(Ne^{\phi}\right)^{\frac{4}{p}\frac{p-3}{p-7}} g_{ab} \ , \tag{26}$$

one gets the action

$$S_{Dp} = \frac{N^2\ \Omega_{8-p}}{(2\pi)^7 l_{str}^p} \int d^{p+2}x\ \sqrt{|g|}\left(R - \frac{1}{2}\left(\partial\Pi\right)^2 + V\right) \ . \tag{27}$$

Here Π is related to the dilaton field by

$$\Pi = \frac{2\sqrt{2(9-p)}}{\sqrt{p}(7-p)}\phi \ , \tag{28}$$

and

$$V \equiv \frac{1}{2\alpha'}(9-p)(7-p)\left(Ne^{\phi}\right)^{\frac{4}{p}\frac{p-3}{p-7}} \ . \tag{29}$$

We foliate the $p+2$ dimensional space with $u=$ constant surfaces and cut it at a certain $u=u_c$. The region $u<u_c$ has a dual description through the effective $p+1$ dimensional SYM theory with UV cutoff u_c. We then apply the previous formalism, and find

$$U=-\frac{\alpha_G(9-p)}{l_{str}}\left(Ne^{\phi}\right)^{\frac{2}{p}\frac{p-3}{p-7}}\ ,\quad \Phi=-\frac{\alpha_G}{2}l_{str}\left(Ne^{\phi}\right)^{-\frac{2}{p}\frac{p-3}{p-7}}\ , \tag{30}$$

$$M_{g^2_{\rm eff}}=l_{str}\frac{\Omega_{8-p}}{(2\pi)^7}\frac{9-p}{p(5-p)^2}\frac{c(\phi)}{g^4_{\rm eff}}\ . \tag{31}$$

Here, $M_{g^2_{\rm eff}}$ is the metric over the one dimensional Yang-Mills coupling space; to obtain it, we have rescaled the boundary metric to the holographic dual frame, and used the relation between Π and $g^2_{\rm eff}$; *i.e.* this term appears in the boundary theory as $\int M_{g^2_{\rm eff}}\partial_a g^2_{\rm eff}\partial^a g^2_{\rm eff}$. We have introduced the function $c(\phi)$

$$c(\phi)\equiv N^2\left(Ne^{\phi}\right)^{2\frac{p-3}{7-p}}\ . \tag{32}$$

The gravitational part of the boundary action then becomes

$$S_{Dp}=\frac{\Omega_{8-p}}{2(2\pi)^7 l_{str}^{p-1}}\left(\frac{2l_{str}}{5-p}u_c\right)^{p-1}c(\phi)\int_{p+1}\sqrt{h}\ \left(R+2\frac{(9-p)}{l^2_{str}}\left(\frac{2l_{str}}{5-p}u_c\right)^2\right)+\cdots\ . \tag{33}$$

We have also rescaled the metric to a flat Minkowski form. Now, if one uses (24), the function c dressing the gravitational coupling gets identified with the c-function

$$c=N^2\left(g^2_{\rm eff}\right)^{\frac{p-3}{5-p}}\ . \tag{34}$$

This expression was found in [23] using a proposal that relates the c-function to the rate of acceleration of null geodesics in the geometry. It was also shown to interpolate correctly between the known asymptotics from string theory. For the AdS_5 case, this dressing of G_4 is well known. We see here that it generalizes for all Dp branes with $p<5$. It is a statement which is consistent with our intuitive picture that gravity would be induced by all of the degrees of freedom of the holographic boundary theory. The appearance of $u_c\sim\mu$ is similar in physical content to the AdS cases above: The effective gravitational coupling of the Randall-Sundrum gravity on the Dp branes is given by the c-function of the Super-Yang Mills theory measured at the cutoff scale μ. Note also that all factors of the string tension cancel in (33) and all remains finite in the decoupling limit. Finally, we note that the $p=5$ scenario is problematic as the near horizon geometry is Minkowski space and the relation between the bulk extent and energy scale on the boundary is less understood. Technically, the conformal transformations we applied break down for $p=5$ as can be seen in the equations above. We hope we will return to this case in the future.

4 The initial value formulation

In the Hamiltonian formulation of the dynamics, the canonical variables and their momenta are are h_{ab}, ϕ^I, τ_{ab} and Ω_I. The initial value problem for gravity assures the existence of a bulk vacuum, unique up to diffeomorphisms, once the boundary values of these variables are specified. We will argue below that the need for this initial value data adapts well into a renormalization group interpretation. However, this will be only a partial understanding of the consistency of the second order nature of the bulk equations of motion with renormalization group flow and we will have to come back to this issue in the last section.

The roles of the canonical variables h_{ab} and ϕ^I in the boundary theory are straightforward to understand. h_{ab} is the classical background metric to which one couples the boundary quantum field theory, and the ϕ^I's map onto the couplings of this quantum field theory. In cases where one encounters relevant or irrelevant operators in the $D-1$ dimensional effective theory, the boundary values for the scalars are to be taken as the dimensionless couplings of these operators measured at the cutoff scale, as we will show explicitly in the Dp brane scenarios below. The interpretation of the canonical momenta is also straightforward when one writes them, using equations (3), (4) and (8), in terms of the boundary action parameters

$$\begin{aligned}
\alpha_G\left(K_{ab}-h_{ab}K\right) &= \Phi G_{ab}-\frac{1}{2}h_{ab}U+\langle t_{ab}\rangle \\
&+ \frac{1}{2}M_{IJ}D_a\phi^I D_b\phi^J-\frac{1}{4}h_{ab}M_{IJ}D_c\phi^I D^c\phi^J \\
&+ h_{ab}\partial_I\partial_J\Phi D_c\phi^I D^c\phi^J+h_{ab}\partial_I\Phi D_c D^c\phi^I \\
&- \partial_I\partial_J\Phi D_a\phi^I D_b\phi^J-\partial_I\Phi D_a D_b\phi^I+\cdots\ ,
\end{aligned} \tag{35}$$

$$\alpha_M\Omega_I = R\partial_I\phi+\partial_I U\langle\mathcal{O}_I\rangle+\frac{1}{2}\partial_I M_{JK}D_a\phi^J D^a\phi^K-D^a\left(M_{IJ}D_a\phi^J\right)+\cdots\ . \tag{36}$$

A simpler form of equation (35), with $D_a\phi^I=0$ was written in [25, 14]. It is an energy balance equation between boundary and bulk/boundary interface. Comments relating to equation (36) were eluded to in [30, 34, 35]. In the Hamiltonian formulation, these two statements are simply the definitions of the canonical momenta at the boundary. Specifying the initial values for τ_{ab} and Ω_I is necessary to determine a physically unique bulk solution, in addition to the specification of the induced metric and the boundary values of the scalar fields. It may be expected that, to define the effective dual boundary theory, all one needs is the classical background metric and the dimensionless values of the couplings at the cutoff scale. One then singles out a renormalization group trajectory in the space of couplings, and hence a bulk solution describing physics at lower energy scales. It is obvious however that such an interpretation is inconsistent with the bulk initial value formulation. A proper resolution to this problem may be reached if we slightly revise the statement of the bulk/boundary correspondence: The latter needs to be taken as an identification between two theories as each is expanded about a "common vacuum" configuration; *i.e.* one needs also to specify a map between the corresponding states. The boundary theory in general may be expanded about a unique vacuum, or a finite temperature one, or one which is a choice out of many if the theory is endowed with a moduli space (as in the case of bulk vacua representing multicenter BPS configurations). In general, it is then necessary to specify a map between bulk and boundary vacua. Focusing on (36), specifying the value of Ω_I in addition to h_{ab} and ϕ^I amounts to fixing the one point correlation functions of the boundary operators, *i.e.* one specifies the boundary vacuum of interest. Fixing τ_{ab} as well would amount to specifying the vacuum expectation value of the energy momentum tensor.

However, this is not the whole story. Let us focus on (35). Given a bulk solution, we should expect that cutting the space along any arbitrary timelike surface does not affect the energy momentum tensor's vev, except for a redshift factor arising from the different choice of time. From the point of view of the boundary theory, this phenomenon is highly non-trivial; we will illustrate this statement through the example of near extremal D3 branes. We take the metric as

$$ds^2 = \frac{u^2}{l^2}\left(-h(u)dt^2 + dx_{(3)}^2\right) + \frac{l^2}{u^2}h(u)^{-1}du^2 \ . \tag{37}$$

$l = l_{str}(g_{str}N)^{1/4}$ is the AdS scale, $h(u) = 1-(u_0/u)^4$ with $u_0 = c_1\sqrt{g_{str}N}T$ where T is the Hawking temperature and c_1 is a numerical constant. For all cuts of this space we consider, we define the coordinate τ by putting the boundary metric in the synchronous coordinate system

$$ds_{bound}^2 = -d\tau^2 + \frac{u(\tau)^2}{l^2}dx_{(3)}^2 \ . \tag{38}$$

This simplifies comparisons between boundary vacua by factoring in the redshift factor between the coordinate t and the boundary time variable τ uniformly in all expressions for the energy momentum tensor we will write. We first cut the space at a constant value $u = u_c$; then using (35), we obtain an expression for the vev of the $\tau\tau$ component of the energy momentum tensor (we also have $D_a\phi^I = 0$)

$$\langle T_{\tau\tau}\rangle = \frac{6}{l}\alpha_G\left(1 - h(u_c)^{1/2}\right) \simeq 3\frac{\alpha_G}{l}\frac{u_0^4}{u_c^4} + O\left[\left(\frac{u_0}{u_c}\right)^8\right] \ . \tag{39}$$

The first term in the expansion was computed in [25] and [33]; it is the contribution of a gas at temperature T in the strongly interacting conformal field theory. The higher order corrections at finite u_c become important as the temperature approaches the UV cutoff.

This phenomenon was also pointed out in the scaling of boundary correlation functions in [29] and [30]. It has been suggested that it indicates that the conformal field theory is not enough to account for the holographic image of the bulk, and new unknown physics, probably non-local, is needed to remedy the mismatch. Perhaps the same degrees of freedom are responsible for both this phenomena and for the presence of higher order corrections in powers of R in (20). In the conventional AdS/CFT correspondence, the boundary is taken to infinity and these effects are dropped [6]. Looking back at equation (35), we note that the boundary metric is flat, and the conformal field theory energy is being heated by the bulk/boundary interface. More generally, the boundary gravity will sourced by the K and U terms in addition to the conformal field theory contribution. Fixing the bulk solution and tempering with the embedding of the boundary, one will get different four dimensional space time and different contributions from the K and U terms, but we must still see the same conformal field theory vacuum. Following [14], let us cut the space at $u = u_c(t)$ such that the K terms in (35) cancel the U term; then one has, by construction, a pseudo-realistic cosmology sourced by the conformal field theory with

$$\langle T_{\tau\tau} \rangle = 3 \frac{\alpha_G}{l} \frac{u_0^4}{u_c^4} \ . \tag{40}$$

This is the whole animal; no expansion in u_0/u_c arises. And it is equal to the leading term of (39). And hence the boundary theory's energy momentum tensor vev does not change. For an arbitrary cut, the four dimensional Einstein tensor $G_{\tau\tau} = 3\dot{u}^2/u^2 \equiv 3\alpha_G H$ (or the Hubble parameter H for the metric (38)) will involve an expansion in powers of u with always a term of the form (40) identifying the vacuum state of the conformal field theory. The additional terms should be given another physical interpretation that we will study to some extent below. Yet another example for a boundary is given in equation (43) where the reader can now identify the now familiar conformal field theory contribution. Equation (35) then determines the vev of the boundary theory's energy momentum tensor modulo this effect.

Let us recap our observations. In the traditional AdS/CFT correspondence, the boundary is fixed as in (39) and taken to infinity. The holographic image to the bulk is just a conformal field theory. When the space is sliced at finite cutoff, as in the renomalization group picture (or equivalently the Randall-Sundrum scenario), the holographic projection of the bulk cannot be accounted for by the conformal field theory alone: finite cutoff effects arise as we approach the boundary as in (39) or (20); and the extrinsic curvature of the embedding of the boundary in the bulk introduces new contributions in the boundary theory.

[6]Equation (39) for example will get multiplied by four powers of u_c to measure mass in the flat space at infinity; hence the first term remains finite when $u_c \to \infty$.

We will term collectively this additional dynamics beyond the conformal field theory as interface effects. Fixing a bulk solution and tempering with the embedding of the boundary does not change the dual conformal field theory, but rather alters the boundary theory through these interface effects. This may be interpreted as a manifestation of coordinate invariance in the bulk in the dynamics of the boundary theory. In some sense, it is reminiscent of the Unruh effect. In foliating space with timelike boundaries, we are dealing however with a somewhat perverse manifestation of this physics, in that we are "boosting" the boundary in a direction which is also identified with renormalization group scale. We conclude that the initial value formulation in the bulk is consistent with the formulation required to define an effective boundary theory dual to the bulk. In this context, it is important to emphasize that the bulk/boundary correspondence is to be established at the level of vacua on both sides of the duality. We will try next to find a physical interpretation for some class of interface effects in the boundary theory.

5 A little cosmology

One can play games with equation (35) by adjusting the cut so as to obtain one's favorite cosmology on the boundary, and hope to obtain a physical understanding for the additional powers of u that will typically contribute to the Hubble parameter $G_{\tau\tau}$. We will argue next in favor of a scenario where these terms may be associated to the dynamics of a probe in the background geometry of a large number of D3 branes.

Consider the following picture[7]: we are in the near horizon region of a large number N of D3 branes at finite temperature, represented by a bulk geometry described in the previous paragraph. The cosmological constant in the bulk is negative and large in string units so as to keep the curvature scale small. Another D3 brane (or a few of them) described by a U(1) (or SU(2), or SU(3), etc.) Yang-Mills theory is freely moving in this space in parallel orientation under the gravitational spell of the N D3 branes. We also set initial conditions such that this probe is moving away from the horizon with energy scale much greater than the Hawking temperature. In a classical approximation scheme with the back reaction of the light probe on the geometry being ignored, the dynamics is described by the Dirac-Born-Infeld action. We assume the probe brane has only dynamics given by the vevs of the scalar field corresponding to the bulk u direction; the fluctuations of the gauge fields and other

[7] This probe brane scenario was instigated by comments made by P. Argyres. *Note added:* After this work was complete, we also learned of [15] where a more detailed analysis along this line of thought was presented. The authors there studied dynamics of a Dp brane probe in the background geometry of a large number of Dp' branes; they also investigated the effect of exciting the matter sector on the brane, and argued for the resolution of the singularity in the cosmology on the probe.

scalars are frozen to zero by initial conditions. Obviously, we are not shooting for a realistic scenario by making this assumption, as cosmology of interest is to be driven by the SYM theory of small gauge group rank. The dynamics is purely gravitational; the RR flux induces a constant shift in the energy, and the dilaton is constant. The trajectory of this probe in the $u-t$ plane is obtained by minimizing the word-volume; the action is[8]

$$S_{\mathrm{DBI}} \sim \int dt \ \sqrt{h(u)\frac{u^8}{l^8} + h^{-1}(u)\frac{u^4}{l^4}\dot{u}^2 + 1} \ ; \tag{41}$$

which leads to

$$\frac{du}{dt} = \frac{h(u)u^2}{l^2}\left(1 - \left(\frac{u}{u_m}\right)^8 \frac{h(u)}{h(u_m)}\right)^{1/2} \ . \tag{42}$$

This probe samples the bulk space at progressively larger values of u before turning around and falling back into the horizon. Its maximum reach is denoted by u_m and is a constant parameter related to a measure of the energy of the projectile. We may now replace the bulk space by one with a cut just outside the trajectory of this probe. This does not affect the dynamics in any away. It however allows us to replace the bulk space by the boundary theory given by equation (8) subject to (35) using the bulk/boundary duality. The left hand side of equation (35) then encodes the dynamical information about the trajectory of the probe; essentially, this is the matter sector of the probe SYM theory with the contribution coming from the evolving vev of the u field. We may then propose that a miserable group of people confined to the surface of the probe brane would witness a cosmology given by the induced metric h_{ab} with the Yang-Mills sector gluons frozen in their vacuum state; the four dimensional Hubble constant is

$$H = \frac{u_0^4}{l^2u^4} + \frac{u_m^8}{l^2u^8}h(u_m) - \frac{1}{l^2} \ . \tag{43}$$

Note that we have $u_o \ll u \leq u_m$. H is defined by

$$H \equiv \left(\frac{\dot{u}(\tau)}{u}\right)^2 \ . \tag{44}$$

The first term in (43) is the contribution from the conformal field theory we saw on two other occasions earlier; and we conclude that the vev of the conformal field theory energy momentum tensor is unchanged as before. The additional pieces coming from the K terms

[8] There is a subtlety in this description due to the infinite extent of the probe branes. One reguralizes the dynamics by wrapping all D3 branes on a three torus of finite size, finds the trajectory, then takes the limit of infinite torus size at the end.

in (35) are to be interpreted as contributions from the probe dynamics. We propose that this interpretation for the meaning of the cut is a general one. The timelike bulk/boundary interface defines a time dependent UV cutoff (with respect to the canonical choice of time with u = constant) which should be regarded as a D dimensional dynamical effect. The choice of a cut conveys to the boundary theory through (35) both the vev of the energy momentum tensor of the conformal field theory as well as the dynamics of the trajectory, all packaged together in the extrinsic curvature term. This idea was hinted at in [14] by adding a matter sector to (35) to cancel the extrinsic curvature contribution. The cosmology on the brane for H given by (43) looks like [9]

$$ds^2_{bound} = -d\tau^2 + \sqrt{u_0^4 + \frac{1}{2}\sqrt{4u_m^8 h(u_m) + u_0^8}\sin\frac{4}{l}(\tau - \tau_0)}dx^2_{(3)} \ , \tag{46}$$

This universe initially expands quickly as the projectile recedes away from the horizon, then slows down reaching a maximum size, and turns around to head back for the singularity inside the horizon to end in a spectacular crunch. The bottom line of this discussion is that understanding four dimensional cosmology in such a picture involves understanding the proper dynamics of probes in the near horizon geometry of a large number of D3 branes. Fluctuations on the probe must be crucial to any realistic scenario and these have been ignored[10]. The suggestion is that there may exist a dynamical problem in the near horizon geometry of D3 branes whose solution leads to a realistic scenario for cosmology on the brane.

6 Hamiltonian equations and speculations

In the previous section, we argued that the initial value data required to specify a bulk vacuum is consistent with the prescription needed to define an effective boundary theory at finite cutoff. However, given that the Hamiltonian formulation specifies two sets of evolution equations, a set for the canonical variables, and a set for the momenta, one may worry

[9] It is amusing that, had the trajectory been that of a point particle, the metric would have been

$$ds^2_{bound} = -d\tau^2 + \left(\frac{\varepsilon^2}{2} + \sqrt{\frac{\varepsilon^4}{4} + u_0^4}\sin(\tau - \tau_0)\right)dx^2_{(3)} \ , \tag{45}$$

where ε is the energy of the probe. The difference arises due to the fact that the space parallel to the D3 branes warps as a function of u as well; and that timelike geodesics for spacetimes related to each other by conformal rescaling are different.

[10] We note that in the solution of [14] the cosmology is also sourced by the conformal field theory, not the matter sector.

about the consistency of the generated flow with our expectations regarding the behavior of a generic quantum field theory as a function of energy scale. For example, on the field theory side, the flow of the expectation values of operators in the boundary theory is related to that of the couplings since the product in the action must have a mass dimension exactly equal to $D-1$ along the flow. This implies a relation between the evolution equations for the scalar fields and their canonical momenta, which we already know exists in the Hamiltonian formulation. That Hamiltonian flow, with its property to preserve phase space volume, may realize the structural form of renormalization group flow equations in this sense was first observed in [36]. More generally, perhaps the interpretation may be that the full form of the first order Hamiltonian differential equations do not present any puzzles with regards to what we know about quantum field theories, but point to new characteristics of theories with dual gravitational descriptions[11]. In this section, we make a mostly unsuccessful attempt to address these issues. We present a set of representative formulas for readers to stare at and possibly get inspired by.

Let us systematically look at the first order differential equations arising in the Hamiltonian formalism. The two canonical variables h_{ab} and ϕ^I satisfy the equations

$$\dot{h}_{ab} = 2\left(\tau_{ab} - h_{ab}\frac{\tau_c^c}{D-2}\right)\ ,\quad \dot{\phi}^I = -G^{IJ}\Omega_J\ . \tag{47}$$

The dot stands for differentiation with respect to the coordinate u normal to the boundary; the reader is referred to the appendix for the details. We note that, the gauge in which we have written these equations corresponds to the choice of Gaussian normal coordinates where u is the parameter along spacelike geodesics projected from the boundary. Bousso's criterion for holography is a statement about the rate of convergence of null geodesics projected from a boundary. It was shown in [23] that this statement can be connected to the monotonicity of the c-function of the boundary theory. In general, it is natural to expect that one needs to pick an "arrow" in Hamiltonian flow to map the flow equations onto renormalization group given that the latter naturally runs from higher energies to lower. In the context of a timelike boundary, noting that the normals to the foliations are tangents to spacelike geodesics, we may propose a statement of the same form as [17, 18]; we require the trace of the extrinsic curvature to be non-positive $K \leq 0$. This may follow from the criterion with respect to null geodesics, but we will not worry about the connection, and instead propose a stronger version; we will require that the eigenvalues of K_{ab} are non-positive if the bulk is to have a holographic image on the boundary. Since K_{ab} is the Lie derivative of the induced metric h_{ab} along the flow lines (see appendix), the physical content of this statement is that one needs the induced boundary metric to warp in a monotonically decreasing manner as one moves

[11] This possibility was pointed out by P. Argyres.

deep into the bulk space. The scenario in [5] assumed that the warping is isotropic. More importantly, this new statement regarding monotonicity of the flow of the metric allows us to implicitly invert the u dependence of all the variables to one with respect to the boundary metric

$$\frac{d}{du} = \int \dot{h}_{ab} \frac{\delta}{\delta h_{ab}} \ . \tag{48}$$

Let us use this statement on $\dot{\phi}^I$ with equations (47) and (8); we then get

$$\int h_{ab} \frac{\delta \phi^I}{\delta h_{ab}} = -\frac{\alpha_G}{\alpha_M} \frac{D-2}{U} G^{IJ} \partial_J U \ . \tag{49}$$

A natural definition for the beta functions of the boundary theory in this context is

$$\beta^I \equiv 2 \int h_{ab} \frac{\delta \phi^I}{\delta h_{ab}} \ . \tag{50}$$

The factor of two comes from the fact that the metric scales with two powers of length. Equations (49), (50), (10) and (14) allow us to determine the correct normalization for the energy momentum tensor of the boundary theory

$$2 \langle t_{ab} \rangle = \langle T_{ab} \rangle \ , \tag{51}$$

which we already made use of in the text. We emphasize that our definition for T_{ab} is part of the whole animal, the part that acquires only an anomaly of the $\beta^I \langle O_I \rangle$ form. In particular, gravitational anomaly terms have been factored away.

The other two sets of equations are those specifying the evolution of the canonical momenta

$$\begin{aligned} \dot{\Omega}_I &= \partial_I V - \frac{1}{2} \partial_I G_{KL} D_c \phi^K D^c \phi^L + \partial_K G_{IJ} D_c \phi^K D^c \phi^J + G_{IJ} D_c D^c \phi^J \\ &+ \frac{1}{2} \partial_I G^{KL} \Omega_K \Omega_L + \frac{1}{D-2} \Omega_I \tau_c^c \ ; \end{aligned} \tag{52}$$

$$\begin{aligned} \dot{\tau}_{ab} &= G_{ab} - h_{ab} \Lambda - \frac{1}{2} h_{ab} \left(\tau_{cd} \tau^{cd} - \frac{(\tau_c^c)^2}{D-2} \right) + 2 \left(\tau_{ac} \tau_b^c - \frac{1}{2} \frac{\tau_c^c \tau_{ab}}{D-2} \right) \\ &+ \frac{1}{2} h_{ab} \left(\frac{1}{2} G_{IJ} D_c \phi^I D^c \phi^J - \frac{1}{2} G^{IJ} \Omega_I \Omega_J - V(\phi) \right) \\ &- \frac{1}{2} G_{IJ} D_a \phi^I D_b \phi^J \ . \end{aligned} \tag{53}$$

Trading derivatives with respect to u for derivatives with respect to h_{ab} in (52), and after somewhat lengthy manipulations, one gets

$$\int_x \beta^J \langle \mathcal{O}_I(x)\mathcal{O}_J(0)\rangle + (\partial_J\beta_I)\, G^{KJ} \langle \mathcal{O}_K\rangle + \left(\partial_I G^{KL}\right) \beta_K \langle \mathcal{O}_L\rangle + (D-1)\langle \mathcal{O}_I\rangle = 0 \ . \tag{54}$$

In handling these expressions, the process is somewhat tedious due to the fact that, at various stages of the expansion, one gets relations or derivatives of relations obtained independently from other Hamiltonian equations or the constraint equation. So, one gets a sea of useless information, buried within it a few interesting statements such as (54). The structure of the latter is suggestive; if one could replace $\beta^I\mathcal{O}_I$ with T_c^c (which one cannot except when the vevs are taken), and if the boundary theory is conformal, this is just the Ward identity for the dilation current and one identifies the anomalous dimension of the operator $\mathcal{O}_I$ as

$$\Delta_I = (\partial_J\beta_I)\, G^{KJ} + \left(\partial_I G^{KL}\right)\beta_K + D - 1 \ , \tag{55}$$

which is exactly as expected from an effective $D-1$ dimensional field theory on the boundary. Note that the scalars are to be identified with the dimensionless couplings. The middle piece is non-zero when the kinetic term for the scalars in the bulk is dressed by additional powers of the scalar fields (as in non-linear sigma models). Equation (54) is however some sort of integral of these statements and it is not apparent to us how to extract the suggestive information about the operator's dimension from this expression. We may however speculate that it appears structurally that the evolution of the ϕ^I's and the Ω_I's "know" about each other in the manner required by a $D-1$ dimensional quantum field theory. We also have not been able to pinpoint to a general principle assuring consistency of the content of the Hamiltonian equations with renormalization group flow in general.

Let us comment briefly on some of the other physics in equations (52) and (53). If we assume monotonicity of the couplings as well (thus restricting to a class of operators along a particular flow line), we can invert the u derivative in $\dot{\Omega}_I$ for one for ϕ^J and obtain an equation involving the correlators of two operators

$$\begin{aligned}\int_x G^{JK} M_{KA}\Big|_x \langle \mathcal{O}_J(x)\mathcal{O}_I(0)\rangle \ &= \ \partial_I M_{AJ} G^{JK}\langle \mathcal{O}_K\rangle - \partial_I G^{KL} M_{AL}\langle \mathcal{O}_K\rangle \\ &- \ \frac{\alpha_M}{2(D-2)\alpha_G}\beta^J M_{AI}\langle \mathcal{O}_J\rangle + \frac{\alpha_M}{\alpha_G}\partial_A\Phi\langle \mathcal{O}_I\rangle \ .\end{aligned} \tag{56}$$

We remind the reader that M_{IJ} is the metric over coupling space. Again this can be argued to have a suggestive form, but we are not able to extract the OPE coefficients from this expression. A myriad of other relations may also be written between the correlation functions, and we write this one as a representative for the form these equations generically take.

Playing the same game with the $\dot{\tau}_{ab}$ equation, one gets relations for the correlators for the energy momentum tensor with itself and all the operators in the theory.

One may hope that the Hamiltonian evolution equations can be used to systematically *construct* an effective boundary theory by extracting the expectation values of all the correlators of the theory from such equations; for higher point correlators, one looks at higher derivatives of the canonical variables and trades the u derivatives with variations with respect to the other variables. The effective action piece in (8) then spits out higher point correlation functions. This is a laborious but straightforward procedure; at every stage, one gets pieces of new physics about the boundary. It is in this sense that one may hope to systematically construct a $D-1$ dimensional boundary theory dual to a D dimensional gravitational bulk. One would even hope to say then that this constitutes a proof that the equations of motion of any gravitational theory are equivalent to defining a $D-1$ dimensional quantum field theory. The equations satisfied by the correlators, such as (56), may also be viewed as restrictions on the class of theories which admit gravitational duals.

We end the discussion in this section by applying some of the previous equations to Dp branes. We are looking for the beta function of the Yang-Mills coupling as encoded in the extremal geometries. The trace of the energy momentum tensor then becomes

$$\langle T_c^c\rangle_{h_{ab},f} = \frac{p}{2}\frac{(p-3)(7-p)}{(9-p)}\partial_\phi f \ \ \langle \mathcal{L}_{SYM}\rangle_{h_{ab},f}\ \ , \tag{57}$$

where

$$f(\phi) \equiv -N\left(Ne^\phi\right)^{2\frac{p-9}{p(7-p)}}\ \ . \tag{58}$$

Rescalings this expression to the frame where the Yang-Mills is coupled to a Minkowski metric, most factors cancel and one obtains the simple result[12]

$$\beta_{g^2_{\rm eff}} = (p-3)g^2_{\rm eff} + \cdots\ . \tag{60}$$

This is precisely the evolution of the dimensionless Yang-Mills coupling according to its classical dimension. Higher order quantum corrections are then encoded in string theoretical corrections to the bulk action. We saw earlier that R^2 terms in the bulk will readily lead to corrections to the gravitational coupling, and hence one would expect to the c-function. The beta functions being gradients of the c-function [5], we expect an expansion in negative powers of the Yang-Mills effective coupling corresponding to α' corrections in the bulk.

[12] In particular, the transformations are

$$\langle \mathcal{L}_{SYM}\rangle_{h_{ab},f} \to \Omega^{-4}\langle \mathcal{L}_{SYM}\rangle_{h^{dual}_{ab},\Omega^{p-3}f}\ \ ,\quad \langle T_{ab}\rangle_{h_{ab},f} \to \Omega^{-p-1}\langle T_{ab}\rangle_{h^{dual}_{ab},\Omega^{p-3}f}\ \ . \tag{59}$$

Furthermore, it can be checked that the beta function we obtained at this order is the gradient of the c-function (34) with the metric over the one dimensional coupling space taken as $M_{g^2_{\text{eff}}}$ in (31)

$$\beta_{g^2_{\text{eff}}} = \frac{\Omega_{8-p}}{(2\pi)^7}\frac{9-p}{p(5-p)}M^{-1}_{g^2_{\text{eff}}}\partial_{g^2_{\text{eff}}}c(g^2_{\text{eff}}) \ . \tag{61}$$

7 Appendix A: Conventions and notation

We review in this appendix the Hamiltonian formalism used in the text. We note in particular important flips of signs between our formulas and the ones found in standard textbooks due to the fact that the boundary is timelike. Otherwise, we follow closely [37].

The bulk action is given by

$$S = \alpha_G \int_D (R + 2\Lambda) + 2\alpha_G \int_{D-1} K + \alpha_M \int_D V(\phi) - \frac{1}{2}G_{IJ}\nabla_\mu\phi^I\nabla^\mu\phi^J \ . \tag{62}$$

An arbitrary foliation is chosen and the space is cut along a timelike surface of this foliation generating a $D-1$ dimensional boundary; the foliations are defined by the scalar function $u =$ constant; and the vector u^a is chosen such that $u^a\nabla_a u = +1$. The spacelike *inward pointing* normals to the foliations, n_a, are normalized $g_{ab}\ n^a n^b = +1$ and the boundary metric h_{ab} is given by $g_{ab} = h_{ab} + n_a n_b$. The extrinsic curvature is $K_{ab} \equiv h^c_a\nabla_c n_b$; this tensor is symmetric by virtue of the surface orthogonality of the normals and it is transverse on both of its indices by construction. We define the lapse function N and the lapse vector N^a by $u^a \equiv N^a - Nn^a$. The metric then takes the form

$$ds^2 = \left(N^2 + N_a N^a\right) du^2 + 2N_a du dx^a + h_{ab}dx^a dx^b \ . \tag{63}$$

Splitting the action along the foliations yields

$$\mathcal{L}_{tot} = -\alpha_G\sqrt{h}N\left(R^{\{D-1\}} + K^2 - K_{ab}K^{ab} + 2\Lambda\right) - \alpha_M\sqrt{h}N\mathcal{L}_M \ . \tag{64}$$

Here $R^{\{D-1\}}$ is the $D-1$ dimensional Ricci scalar; in the text and hereafter, we drop the $D-1$ superscript to avoid cluttering formulas. We have

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab} \ , \Rightarrow \ 2N\ K^{ab} = \dot{h}^{ab} + D^a N^b + D^b N^a \ . \tag{65}$$

The canonical momenta are defined by

$$\tau_{ab} \equiv \frac{1}{\alpha_G\sqrt{h}}\frac{\delta\mathcal{L}_{tot}}{\delta\dot{h}^{ab}} = (K_{ab} - Kh_{ab}) \ , \quad \Omega_I \equiv \frac{1}{\alpha_M\sqrt{h}}\frac{\delta\mathcal{L}_{tot}}{\delta\dot{\phi}^I} = \frac{1}{N}G_{IJ}\left(\dot{\phi}^J - N^a D_a\phi^J\right) \ . \tag{66}$$

The Hamiltonian then becomes

$$
\begin{aligned}
\mathcal{H} &= \alpha_G \sqrt{h} N \left(R + 2\Lambda\right) + \alpha_G \sqrt{h} N \left(\tau_{ab}\tau^{ab} - \frac{\left(\tau_c^c\right)^2}{D-2}\right) \\
&- \alpha_G \sqrt{h} \tau_{ab} \left(D^a N^b + D^b N^a\right) + \alpha_M \sqrt{h}\, \Omega_J N^a D_a \phi^J \\
&- \alpha_M \sqrt{h} N \left(\frac{1}{2} G_{IJ} D_c \phi^I D^c \phi^J - \frac{1}{2} G^{IJ} \Omega_I \Omega_J - V(\phi)\right) \ . \qquad (67)
\end{aligned}
$$

The constraint equations result from varying N and N^a; they are given in (7) and (6) respectively.

In the text, we choose throughout the gauge $N^a = 0$ and $N = -1$ corresponding to Gaussian normal coordinates. The normals are then tangents to spacelike geodesics and u is the parameter along these geodesics. The equations of motion are given by (47), (52) and (53) with $N^a = 0$, $N = -1$.

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