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Cover Page Footnote

We would like to thank Zoe Ashton, Jessica Carter, Matthew Inglis, David Waszek, and two anonymous reviewers for comments on earlier versions of this draft. Note by Juan: In addition to Gilbert Labelle and my teachers named in this article, I would like to thank the architecture professors who have doodled homothetic paths with me for their inspiration that goes beyond architecture and geometry: Theodora Vardouli, Ricardo L Castro and Radoslav Zuk, as well as David M. Covo, who lent me his copy of The Psychology of Invention in the Mathematical Field (Hadamard 1945), motivating me to write this paper. I would also like to thank Martin Bressani and David Krawitz for caring about the story behind my article The Polygon of Euler's Circle, and for sharing it with the McGill community. Lastly, I would like to thank Dirk Schlimm for having welcomed a stranger who randomly walked into his office (me), for having talked with me, and for having proposed to write an article about a doodle that became a theorem.

From a Doodle to a Theorem: A Case Study in Mathematical Discovery

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Abstract

We present some aspects of the genesis of a geometric construction, which can be carried out with compass and straightedge, from the original idea to the published version (Fernández González 2016). The Midpoint Path Construction makes it possible to multiply the length of a line segment by a rational number between 0 and 1 by constructing only midpoints and a straight line. In the form of an interview, we explore the context and narrative behind the discovery, with first-hand insights by its author. Finally, we discuss some general aspects of this case study in the context of philosophy of mathematical practice.

Keywords: compass and straightedge constructions, diagrams, geometry, mathematical discovery, philosophy of mathematical practice.

1. Introduction

Many philosophers of mathematics have shown an interest in mathematical discovery and other aspects of mathematical practice. The most famous discussion of the process of discovery in mathematics is Lakatos' reconstruction of the historical development of Euler's formula $V - E + F = 2$, which relates the number of vertices, edges, and faces in convex polyhedra [16].

More recently, VanHattum [24] and Livingston [19] have described their own attempts at solving mathematical problems and proving established theorems. However, only few in-depth, *first-hand* accounts of the creative processes behind published mathematical results exist in the literature. These include Poincaré's famous report of his invention in the theory of Fuchsian functions ([21]; discussed in [10]), Henkin's "The discovery of my completeness proofs" [12], Hersh's account of his discovery of a new proof for Heron's area formula [14], and Villani's extended account *Birth of a Theorem* [26]. In addition, philosophers have also followed mathematicians in the wild, so to speak, to observe and discuss their methods. Two examples of such investigations are Leng's studies of the development of the definition of C^*-algebras [17] and Carter's discussion of the genesis of a theorem in free probability theory [3]. In all of the cases mentioned, the agents were professional mathematicians and the mathematics involved mostly far from trivial. Thus, the case study we present in this paper stands out by being an account of a discovery, from the initial conception of the mathematical idea to the publication of a mathematical article, for a case where the agent is not a professional mathematician and where the novel geometric construction is relatively simple (in fact, it can be done with straightedge and compass alone).[1] By supplying personal context to the discovery process, this paper gives insight into a single episode in the development of mathematics, showing real mathematical practice beyond the purely technical results. We believe that adding this particular case study to the literature contributes to displaying the diversity of the processes underlying the development of new mathematics and encourages further investigations into mathematical creativity.

Straightedge and compass constructions date back to ancient Greek mathematics. The first three postulates of Euclid's *Elements* [11] can be interpreted as allowing only the following constructions: drawing a line segment between two points, extending a line segment by any length in a straight line, and drawing a circle with a given point as center, passing through another given point. Constructions based on these have been taught and studied extensively for over two millennia, but it is rare that entirely new constructions

[1] We leave aside the studies on students uncovering mathematical content, as this would go beyond the scope of our presentation. Nevertheless, we think that a comparison with this literature would provide a welcome addition to our discussion. Two overview articles that could serve as starting points for such an endeavor are [18] and [27].

are proposed. Most famously, eighteen-year-old Carl Friedrich Gauss discovered in 1796 (in the morning of March 29, before getting up from bed, to be exact) a way of constructing a regular heptadecagon, i. e., a 17-sided regular polygon, which he published at the age of 19 [29, page 180]. Two hundred and twenty years later, nineteen-year-old Juan Fernández González published a new construction, named the Midpoint Path Construction, which makes it possible to multiply the length of a line segment by a rational number between 0 and 1 by constructing only midpoints and a straight line [5].[2] Since antiquity, it has been known that the problem of multiplying the length of a line segment by a rational number can be solved on the basis of the Intercept Theorem (also known as Thales's Theorem) by constructing an additional line segment, a series of circles or circle arcs, and parallel lines (see Figure 1). This construction, however, requires the use of parallel lines, whereas the new construction does not. In "Chemins homothétiques" [5], Fernández González justified the Midpoint Path Construction by introducing a more general notion, namely that of *homothetic paths*, which consist of a finite series of points defined by a finite series of homotheties in any spatial dimension.[3] An algorithm for the Midpoint Path Construction is presented in Appendix A, together with an example; the interested reader is invited to take a look before reading on.

In the present paper, we discuss the discovery process behind this geometric construction, from the early stages until its publication, on the basis of notebooks and a personal account of the discoverer. Next, in Section 2, we present Juan's youthful excitement about his discovery and his mission to turn it into a published paper in his own words. The section is divided according to five themes: the genesis of the original idea of the geometric construction, the use of notebooks during its development, the importance of focus and other people, the process of turning the ideas into a mathematical publication, and a glimpse into future endeavors. Finally, we briefly discuss some aspects of mathematical discovery that are raised in the presentation of this case study (Section 3).

[2] The midpoint between two given points can be constructed with a compass and a straightedge (see Book I, Proposition 10 of Euclid's *Elements* [11]). Moreover, the midpoint between two given points can be constructed with a compass alone with the Mascheroni Construction; see [8, pages 219–220].

[3] A homothety is an affine transformation having a homothetic center A and a ratio $\lambda \in \mathbb{R}$, which associates any point M_0 to a point M_1 such that $\overrightarrow{AM_1} = \lambda\overrightarrow{AM_0}$.

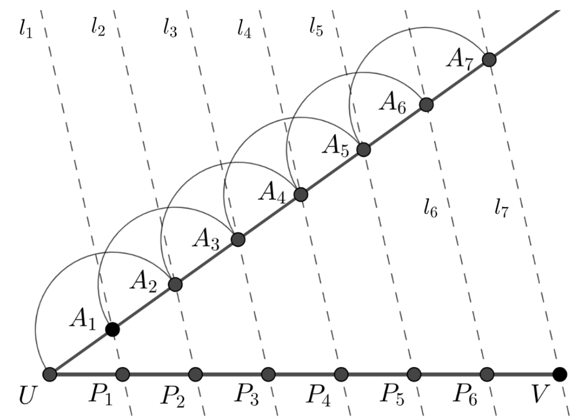

Figure 1: Dividing a line segment into seven equal parts on the basis of the Intercept Theorem (Thales's Theorem): Given the line segment $[UV]$, pick a point A_1, not on (UV), and draw the ray $[UA_1)$. Using a compass, determine the points $A_2, A_3, \ldots, A_7$ on $[UA_1)$, such that each segment $[A_iA_{(i+1)}]$ is equal in length to $[UA_1]$. Construct the straight line l_7 through A_7 and V, and then construct the lines l_1, l_2, …, l_6, all parallel to l_7 and passing respectively trough A_1, A_2, …, A_6. These parallel lines intersect $[UV]$ at $P_1, P_2, \ldots, P_6$, respectively, dividing it into seven equal parts.

2. Interview

2.1. Ideas

Dirk: Good morning, Juan. Let's begin with some background: Can you tell us something about your mathematical upbringing and your desire to come up with new things?

Juan: Good morning, Dirk. Since I was a child, I've loved doodling and sketching, and I've dreamt of inventing something like the paper clip, the revolving door, or the geodesic dome. I've also liked math for as long as I can remember. I used to draw geometrical shapes on the classroom's chalkboard with my high school math teacher Christophe Brun[4], after the end of class. I tried really hard to come up with something new and share it with him, but it always led to something that was already known.

Dirk: Do you remember when you had the first idea about your new construction?

[4] Christophe Brun teaches mathematics at *Collège international Marie de France*, a private international high school in Montréal.

Juan: In Fall 2014, when I was eighteen years old, I was doodling points and lines on my agenda in class.[5] Suddenly, something caught my attention. Given a line segment, it seemed possible to draw a random point and a series of midpoints in such a way that these midpoints eventually ended up dividing the line segment into three equal parts. This was a very simple construction, but I had never seen it before! My gut feeling told me that I was up to something new, that I was just starting to uncover it with the doodle.

Dirk: Can you explain this construction a bit more?

Juan: Of course! (See Figure 2.) Imagine a line segment with endpoints U and V, and an arbitrary point M_0 not on the straight line (UV). Then, draw the segment $[M_0U]$ and determine its midpoint M_1. From here, draw the segment $[M_1V]$ and determine its midpoint M_2, and so on, always alternating between U and V for each construction of a new midpoint. These midpoints, which form the *midpoint path* $M_0, M_1, M_2, \ldots$, get closer and closer to the line segment $[UV]$. If this construction would be continued forever, they would end on $[UV]$, dividing it into three equal parts.

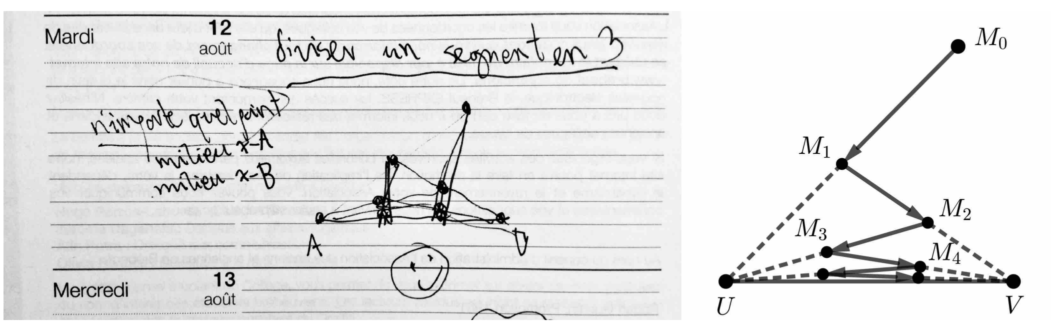

Figure 2: Left: A doodle on Juan's school agenda. This was one of the first doodles related to the Midpoint Path Construction. In French, Juan wrote "dividing a segment in 3. Any point. Middle x-A. Middle x-B." Right: Reconstruction of the doodle with additional (re)naming of the points to match the later publication.

Dirk: This sounds quite straightforward, indeed. When you used the word "doodle" earlier, I had to think of something drawn absentmindedly: is that what you mean by it?

[5] At the time, Juan was studying Science, Letters and Arts at *Collège Jean-de-Brébeuf* in Montréal, two years before attending university.

Juan: Some of them can be drawn absentmindedly, but the important thing is that they are spontaneous. A doodle, a geometrical one in this case, is an imprecise representation of a thought by drawing points and lines. The lines don't have to be straight and the points don't have to be round. For example, if I drew a midpoint that happened to not be exactly at an equal distance from two given points, I knew it was, by definition. Inaccuracy in the doodle does not mean inaccuracy in the idea. It is the logic behind the construction that made sense. If "doodle" sounds too trivial, the drawings could also be called "diagrams," especially when they are more elaborate.

Dirk: So, now I'm curious: what came first, the thought or the doodle?

Juan: That's a hard question ... a bit like the chicken and the egg. In this particular case that sparked the discovery, the doodle came first. The observation that the midpoint path would divide the line segment into three equal parts came only while doodling, since I started drawing without expecting a particular result.

Dirk: How did you convince yourself that your construction would always work?

Juan: The first doodle only sparked the idea, but I had to draw it many more times to be convinced that it really worked. I drew the construction in notebooks, on napkins, and on a bus's frozen window with my finger. I noticed that, regardless of the initial position of the starting point of the midpoint paths, the construction always tended to the same result after a sufficiently large number of steps. As much as I trusted my geometrical doodles, I wanted a machine to verify it, and this led me to GeoGebra[6], a software for geometric constructions. When I constructed a point at 33.333333% of the length of the initial line segment, it was pretty certain that it corresponded to 1/3, and that it was not just a lucky coincidence. At that time, I didn't understand that a computer result was far from a formal proof. However, these results still made me more confident about my construction. A couple of weeks into the discovery, I showed it to my mathematics teacher, Philippe Dompierre. He was very enthusiastic about it and told me that I had to find a real proof. Eventually, I developed an algebraic proof which used coordinates of points and line equations to prove this particular scenario. But the final proof only came several months later.

[6] https://www.geogebra.org, last accessed on January 29, 2023.

Dirk: That's great. In addition, you were also able to generalize the construction method for dividing a line segment into a different number of equal parts. How did that come about?

Juan: Initially, I would always draw the same construction. The midpoints would go toward $U, V, U, V, \ldots$ endlessly and in that order. Then, at some point, I tried to change the order in which the midpoints would be drawn, for example, by drawing the series of midpoints to go toward $V, U, U, V, U, U, \ldots,$ endlessly and in that order. In this way I realized that different variations of the construction divided the line segment in different ways. For example, with this particular construction I could find the point at 1/7 of the line segment $[UV]$. But, the mystery was that I didn't know *why* it resulted in 1/7, it just did! By trying different combinations of midpoints I was able to construct their corresponding points on the given line segment, so that I could divide a given line segment into 3, 5, and 7 equal parts. In some cases, I needed to use more than one midpoint path, each one defined by a different midpoint series, to obtain all the desired points. (See Figure 3). Later on, I figured out how to divide the line segment into any desired number of equal parts.

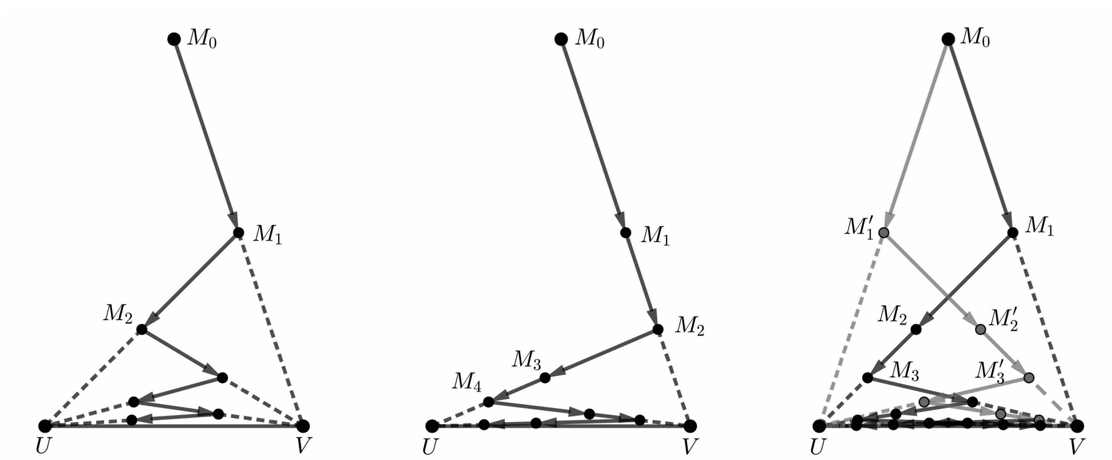

Figure 3: Dividing a line segment into 3, 5, and 7 equal parts with midpoint paths whose lengths tend to infinity. The midpoint paths start at M_0 and are composed of successive constructions of midpoints M_i, midpoint of either $[M_{i-1}U]$ or $[M_{i-1}V]$, for $i = 1, 2, \ldots$. The rightmost diagram shows two midpoint paths simultaneously: one with points $M_0, M_1, M_2, \ldots$ and one with points $M_0, M_1', M_2', \ldots$. Both are needed to construct all the points on $[UV]$ that divide it into 7 equal parts.

Dirk: You mentioned that your construction would require an infinite series of midpoints. But, that would be impossible to complete in a finite amount of time, no?

Juan: Yes, and straightedge and compass constructions requiring an infinite number of steps are not really admissible. That's why I initially didn't think that my discovery was of much value. The fact that it required an infinite number of steps did intrigue me, because the ancient Greeks would have not accepted it.

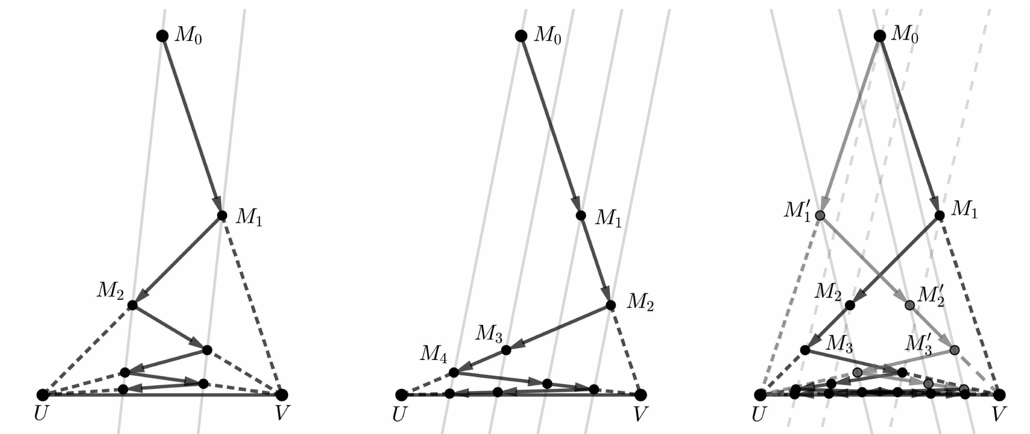

Figure 4: The points M_i of these midpoint paths, which are the same as those in Figure 3, lie on parallel lines (shown in grey).

Dirk: But, ultimately you were able to find a construction that requires only a finite number of steps. How did you arrive at this result?

Juan: Yes, the finite construction simplified everything. Although the midpoint paths that I was constructing were endless, I realized that the midpoints that composed them laid on parallel lines. (See Figure 4). Moreover, these straight lines intersected the line segment $[UV]$ at the points of interest, dividing it into equal parts. So, it seemed only necessary to construct two midpoints which lie on one of these lines to define the line and to find its intersection with $[UV]$. Then, instead of constructing an infinite number of midpoints towards $V, U, V, U, V, U, \ldots$, I studied more carefully the case where only two midpoints M_1 and M_2 were constructed toward V and U, respectively. Almost magically, the *convergence line* (M_0M_2) intersects $[UV]$ at one third of its length. Thus, only a finite number of steps is required to obtain the same result that before had required an infinite process.

The same reasoning worked for other fractions, such as 1/5 and 1/7. (See Figure 5.)

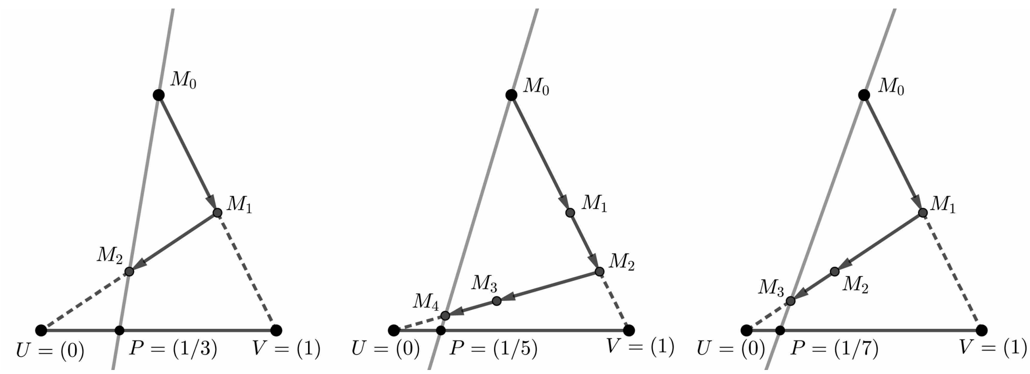

Figure 5: Three line segments $[UV]$ and midpoint paths allowing the construction of points $P = (1/3)$, $P = (1/5)$ and $P = (1/7)$, respectively from left to right, with $U = (0)$ and $V = (1)$. Only a finite series of midpoints and the convergence line (M_0M_n) are needed for the construction, where $P = (M_0M_n) \cap [UV]$.

Dirk: The ancient Greeks would have liked this construction better. But then, how did you determine the series of midpoints behind each construction?

Juan: That took me a long time to figure out. One day, I was listening to my psychology teacher, Sébastien Bureau, during his class, while he taught us the parts of the brain and their functions. I had a Eureka moment and jotted my idea down right away, to verify it. What I came up with was a special type of midpoint paths, which I would eventually name *midpoint loops*. These are midpoint paths that start and end at the same point. For example, take the points $U = (0)$, $V = (1)$ and $P_0 = (1/3)$ on a straight line. If P_0 moves halfway toward V, it reaches $P_1 = (2/3)$. Now, from P_1 halfway to U lies P_0 again, so that P_0 and P_2 coincide. Since the starting point and endpoint are the same, we have a midpoint loop, beginning and ending in P. (This example is shown in the first diagram in Figure 6, where $P_0 = P_2 = P$.) The most interesting fact about midpoint loops is that they are unique. Any midpoint path that begins in $M_0 \neq P$, goes halfway toward V, and then halfway toward U, will end at a point M_2, distinct from M_0. However, there exists only one single fixed point P which returns to itself after going halfway toward V and then halfway toward U. In short, while many midpoint paths can be constructed for P, there is only one such midpoint loop.

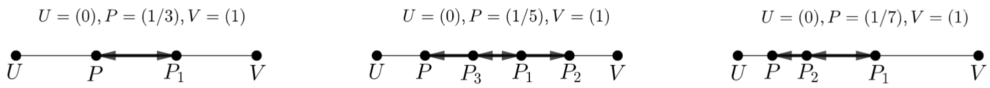

Figure 6: A line segment $[UV]$, with $U = (0)$ and $V = (1)$ and three midpoint loops that start and end at point P. Left: A midpoint loop with $P = (1/3)$ and $P_1 = (2/3)$. Middle: A midpoint loop with $P = (1/5)$, $P_1 = (3/5)$, $P_2 = (4/5)$, and $P_3 = (2/5)$. Right: A midpoint loop with $P = (1/7)$, $P_1 = (4/7)$, and $P_2 = (2/7)$. The arrows illustrate the consecutive midpoint constructions. They overlap because all the points P_i are collinear.

Understanding this relationship allowed me to look for and find the midpoint loops that are associated to a given fixed point P on the line segment $[UV]$. Such loops, in turn, give rise to a *midpoint series*, i. e., a finite sequence of 'U's and 'V's, which characterize the construction of a midpoint path. (An algorithm for determining such a midpoint series is presented in Appendix A.)

Dirk: Ok, so now we have midpoint paths, midpoint loops, and midpoint series. What are *homothetic paths*, then, which are at the core of your article?

Juan: They are a further generalization of the idea of midpoint paths. At first, I was only thinking in terms of midpoint constructions. But, at one point I realized that ratios other than one half were possible as well. For example, a path could go 1/5 toward U and 4/7 toward V. Moreover, the points M_i of a path didn't have to be limited to a plane containing a line segment $[UV]$. Observations like these led me to define and explore what I later named "homothetic paths." These consist of finite sequences of points defined by a finite series of homotheties. The ratio of the homothety can take any real value, and the points can be in any spatial dimension. Because homothetic paths are much more general than midpoint paths, I decided to present them first in the article, thereby reversing the order in which they were discovered.

Dirk: Since you've mentioned your article, can you briefly explain its two main theorems, the Homothetic Path Theorem and the Midpoint Path Theorem? In particular, how are they related to the construction?

Juan: Yes, those are the two main theorems used for the construction.[7]

[7] There are two other theorems in the article. The Homothetic Path Convergence Theorem describes how certain homothetic paths converge to a fixed point P when their

In a way, they represent the two main results of the discovery which, together, justify the Midpoint Path Construction. The Homothetic Path Theorem establishes a relationship between an infinite number of homothetic paths, a single homothetic loop, and a fixed point P, all defined by a given finite series of homotheties. The Midpoint Path Theorem indicates which series of midpoints is used to define a fixed point P on a line segment $[UV]$, so that it multiplies its length by a rational number between 0 and 1. I like to think of them metaphorically in terms of maps: Whereas the Homothetic Path Theorem states that you can reach any point on a map, the Midpoint Path Theorem gives you the directions to follow to reach that point.

2.2. Notebooks

Dirk: Let's return to the process of developing your ideas. In particular, how was your writing process?

Juan: At first, I wrote down my thoughts wherever I could. The diagrams were quite simple and were often repetitions of the same construction with small variations. My school agenda had the most notes. Some doodles, diagrams, and sentences appeared on random pages. The back of the agenda had my first algebraic proof of the construction that divides a line segment into three equal parts, using the coordinates of points and straight lines. However, as I started to run out of space on my agenda, I decided to buy small Moleskine notebooks dedicated only for work on my discovery. These were much more organized, since the order of the pages established a chronological progression of the work. Though they were written only for myself, I still cared for the way in which the ideas were presented visually. (As examples of pages from these notebooks, see Figures 7 and 8).

Dirk: It looks like you have a good amount of text in your notebooks, as opposed to just diagrams. What do you think are the main differences between diagrams and text?

Juan: The diagrams are ideal for showing an example and for testing different possibilities, but a diagram cannot always express everything.

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length n tends to infinity. The Centroid Homothetic Loop Theorem shows a way of defining a series of homotheties associated to a fixed point which is the center of gravity of a group of points of arbitrary weights in any spatial dimension.

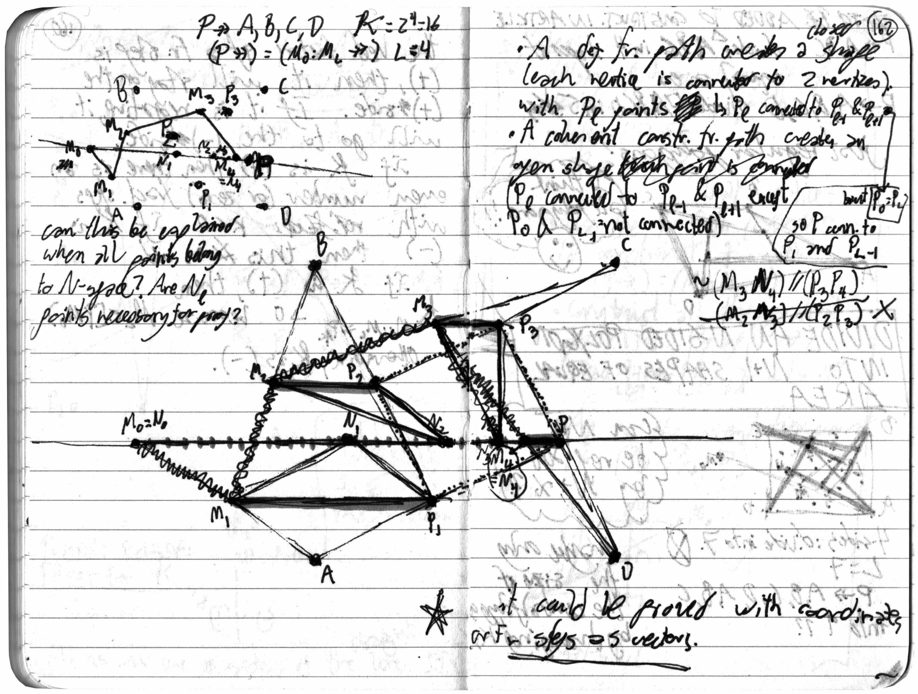

Figure 7: Two pages of one of Juan's notebooks showing a crucial step in the development of the Homothetic Path Theorem's first proof. We can see a diagram of a homothetic path, with points M_i and a homothetic loop with points P_i, which would lead Juan to find the similar triangles $A_iP_iM_i$ and $A_iP_{i-1}M_{i-1}$, which are essential to that proof.

Words and mathematical expressions can express a generality that encompasses a number of possible doodles. So, the diagrams or doodles are for experimenting with different instances and the words are more for trying to find generalized truths and for writing the theorems. Even though a picture can be worth a thousand words, we could say that a theorem is worth a thousand doodles, or an infinite amount of doodles.

Dirk: What explains the choice of English or French for the text in your notebooks?

Juan: Even though my first language is Spanish, my high school education was in English and French. I simply knew more mathematical terms in these two. I chose English at first, since it seemed better for publication, to reach a wider public once the article would be written. When I realized that publishing an article in the *Bulletin AMQ* of the Mathematical Association

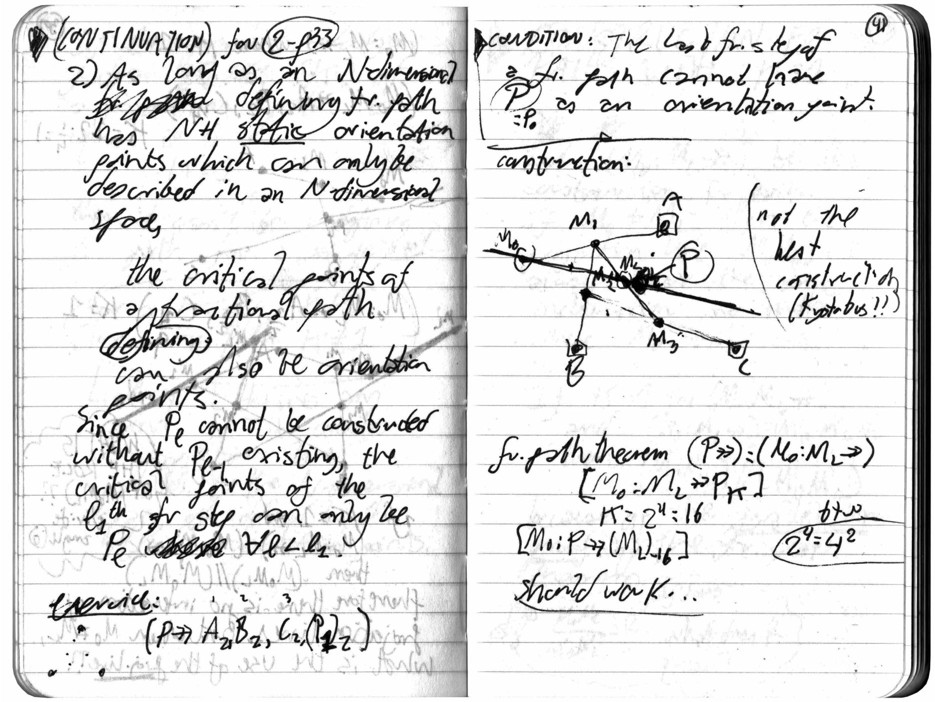

Figure 8: These two pages show some of Juan's thought process, which would lead to the discovery of a higher type of homothetic paths: interdependent homothetic paths. (They are discussed at the end of the *Bulletin AMQ* article.) We can read "not the best construction (Kyoto bus!!)" on the page on the right, written in 2015 while riding on a bus in Japan.

of Québec[8] became a possibility, I changed my writing to French. I wrote sentences and paragraphs to capture my ideas. Sometimes I would even record the place where I happened to be. (For example, on the right page shown in Figure 8 we can read "Kyoto bus!!", which was written while riding a bus in Japan.)

Dirk: What are your favourite places to work and write?

Juan: At first, I wanted my notebooks to look organized and clean, so I needed a table and a chair. Later on, I became used to carrying the small

[8] Published by Association mathématique du Québec (AMQ), *Bulletin AMQ* is available at https://www.amq.math.ca/bulletin/, last accessed on January 29, 2023..

notebook everywhere. With time, my favourite places to doodle and write about my discoveries became any form of transport: the metro, the buses, or even an airplane, on a special occasion. Because of that, the notes are often very shaky, but I believe that this captures best the moment, since the setting and context becomes part of them. In fact, I do think that the messy notes and doodles look beautiful nonetheless. I use a pen for my math notebooks because I don't want any of the process to be erased. It is not trial and error, because nothing is wrong. Since it's all a creative process, I believe that "trial and better" is a more fair description.

Dirk: But later you also used a computer to write up your ideas, didn't you?

Juan: Yes, at some point I worked almost entirely on my computer. Even though the main ideas were mostly developed in the notebooks, the well-articulated ideas were sometimes only written in typed form. For producing the drafts of the article I learned to use LaTeX, and I recall that seeing a professionally-made PDF of my ideas was very exciting.

Dirk: Speaking of computers, you also mentioned using GeoGebra at some point for your diagrams. Do you think you could have used it, or some other computer program, from the start, instead of using pen and paper?

Juan: Yes, I could have done everything using GeoGebra, except maybe the doodle that started it all. However, not doodling because I have a computer program would be like not walking because I have a car. I really like drawing on paper, because it feels more natural and can be done anywhere.

2.3. Focus and people

Dirk: What motivated you to try to publish your discovery?

Juan: Pretty early during the process, I felt the need to find out if it already existed. The internet seemed like the right place to look for my answers. After looking for "dividing a line segment" and many similar things, I was extremely surprised to not find my construction. I was shocked! I then knew that I had the responsibility, and the honour, to understand my discovery and to share it. Publishing an article about it became my mission.

Dirk: But, you were a full-time student at the time, so how were you able to stay focused on your research?

Juan: My two years as a student at *Collège Jean-de-Brébeuf* were amazing, but they also were not easy. It was hard to decide whether I should work on my idea or study for an exam. Surprisingly, it was easier to focus on my own research than on school work. It became a habit to do my homework late at night, since I would sometimes spend all the evening doing math. Even though I have always really cared about school, at that moment I preferred to doodle, discover, and give shape to the mathematical article. I was in some sort of a trance... drawing, writing and looping songs like "The Swimmer" by Phil France for many consecutive hours.

Dirk: So, it sounds like you were always thinking about your discovery?

Juan: Yes, I gradually became obsessed with it. One day I was walking in Montréal's downtown and stopped right before crossing a street. A car passed right next to me, not really that fast, but it still scared me. The first thing that came to my mind was: "I can't die! I haven't published my geometry discovery!" When I played basketball, I would imagine the movement of the ball around the court as a homothetic path. I once even dreamt of a bridge whose structure was defined by a homothetic path. It touched the ground at only one end, drawing a polygonal spiral toward the sky, and I could walk on it. I'd also like to add that the moments between being awake and falling asleep have been very important to my discoveries. It somehow mixes the real with the imaginary in a very natural way and makes unexpectedly interesting combinations. During these moments, with my eyes closed, I felt like I could imagine spatial configurations and move them around much more easily than while being fully awake or with my eyes open. I kept one of my notebooks next to my bed and sometimes wrote down ideas in the middle of the night.

Dirk: Did you also share your ideas with others?

Juan: When I first realized that I had discovered something new, I was pretty cautious about sharing it, since I didn't want anyone to publish it before me. However, it only took a few days for me to want to talk about it, particularly with my mathematics teachers. Philippe Dompierre, Anne-Marie Lorrain, and Marie-Claude Périgny used to stay after class to talk with me and write and draw on the chalkboard. They always encouraged me to pursue my discovery, and our wonderful discussions helped me develop it further. I also talked about it with Louis-Philippe Giroux, a mathematics teacher who loves geometry, my philosophy teacher Dave Anctil, and my physics teacher François Meunier.[9] I also exchanged a few emails with my high school teacher Christophe Brun. I am very thankful to all of them for appreciating their students' curiosity and for helping me. My parents, my brother, and my friend Antony Diaz[10] also talked with me and were very encouraging.

By the time I was Anne-Marie Lorrain's student, the discovery was much more advanced. She found it pretty interesting and, after giving me some comments, recommended me to show it to her retired university professor, Gilbert Labelle[11]. I was particularly excited to meet him because he used to invent the problems for the competition of the Mathematical Association of Quebec, which I had enjoyed as a participant and which had given me the opportunity to assist their summer camp as a prize at the end of high school. He is also a very accomplished mathematician and professor, who has published countless discoveries and had even ranked among the top ten participants in the famous Putnam Competition in 1964.[12] So I eagerly sent him a really long email with some bold and some green highlighted text, which is actually the first message I ever sent with my current personal email address.

2.4. Publishing

Dirk: So you wrote an email to Professor Labelle and then what happened?

Juan: I received a reply and we scheduled a meeting! I found it very touching that he took the time to answer a kid who claimed to have made a discovery in geometry and wanted to talk to me. During our meeting at the *Université du Québec à Montréal*, I was afraid of him saying "Oh yes, I know this construction," but that didn't happen. Instead, he said that he had never seen my construction before and that the whole discovery was very interesting.

[9] The teachers mentioned in this paragraph currently teach at *Collège Jean-de-Brébeuf* in Montréal along with Sébastien Bureau, except for Anne-Marie Lorrain, whose last year before retirement was with Juan's class.

[10] Antony Diaz was awarded a Gold Medal at the Canadian Senior Math Contest while studying at *Collège Jean-de-Brébeuf* in 2016.

[11] Gilbert Labelle is an emeritus professor at the Department of Mathematics at the *Université du Québec à Montréal*.

[12] For a brief account of the life and accomplishments of Labelle, see [7].

We talked for a while and drew on his chalkboard. I kept saying "one more thing..." and he generously stayed longer than planned. This is a memory that I treasure a lot: I felt like a real mathematician! Then, Professor Labelle told me that he would read my article draft more carefully at his cottage and would send me some comments. A couple of months later, after having shared new advances with him, Professor Labelle sent me a completely revised version of the article while congratulating me for the improvement of the presentation of my ideas. This was basically a draft of my article, rewritten by him: it explained the essence of the discovery in a clearer way using existing mathematical terminology; it was also accompanied by rigorous proofs that were missing or that had to be improved, as well as a couple of notions that were not part of the original discovery. This is one of the greatest gifts I have ever received! Then, inspired by the mathematical style of the draft, I worked on it for several weeks, restructuring different parts, while adding more pages and ideas.

Dirk: What were the main differences between the article's drafts before and after Professor Labelle's help?

Juan: The article drafts that I had shared with him were, in a way, more naive. The first draft that I showed Professor Labelle, when I visited his office, was over fifty pages long! It somehow resembled the structure of the math textbooks that I had studied at school, since I thought that this was the only professional or valid way to explain math. The draft began with some definitions. It then presented numerous examples with diagrams that got increasingly more complex, with the intention that the reader would understand the new ideas gradually. Finally, a couple of theorems were presented, but not all of them had complete proofs. On the other hand, the improved draft that Professor Labelle sent to me was clear, straightforward, and concise. In thirteen pages, he found a way to encompass all and more that I had written about in over fifty pages. Definitions were followed by theorems, rigorous proofs, and only one diagram accompanied each theorem as an example. One could say that my first draft was "from a doodle to a theorem," while Professor Labelle's presentation was "from a theorem to a doodle."

Dirk: Can you point your finger on what was different between the way you approached the problems and how Professor Labelle did it?

Juan: One of the main differences is the mathematics used to prove the theorems. Whereas my proofs were entirely geometrical, such as a proof of the Homothetic Path Theorem using series of similar triangles, Professor Labelle's involved many areas of mathematics, such as linear algebra, binary representation, and modular arithmetic. Even though I had learned some similar things on my own time, I hadn't even taken a linear algebra course back then. Reading his proofs seemed magical. It felt like I had been trying to build a building using only a hammer, whereas he seemed to have all the tools and machines you could imagine. In some cases, his proofs appeared in the article, whereas mine became two-sentence notes to illustrate that it could also be proved geometrically. It was also very interesting to see how he attacked the problem. He described it to me in this way:

> My research approach consists essentially of examining the studied concrete or abstract object from all its possible *angles*, while selecting analogies with my accumulated past knowledge. Just as in the theory of evolution, the possible angles play the role of *gene mutations* and the selection of analogies from my accumulated knowledge play the role of the *natural selection*.[13]

It is with this approach that he discovered one of the most beautiful properties of midpoint paths: there is a relationship between the construction of multiplying the length of a line segment by a rational number between 0 and 1 and the binary development of r/s. To find this, he saw that the successive homotheties could be represented as one simple affine transformation, which coupled with the condition of constructing only midpoints, suggested the binary representation. His wise words made me realize that if I wanted to examine mathematical objects from different *angles*, I had to learn more math.

Since the construction is relatively simple, I believe it is quite possible that someone else came up with this construction before me. However, the generalization as a theorem and the proof are not obvious. I have a theory, which has made me ponder a lot: Maybe somebody did discover my construction in the past, let us say a geometer in Ancient Greece, fascinated by compass

[13] The source of this quote is an email that Labelle sent to Juan on July 31, 2019. The translation from French to English is by Juan.

and straightedge constructions. However, since the proof in my article uses notions of more recent mathematics such as linear algebra and binary representation, introduced by Professor Labelle, this particular proof could not have been developed with the existing mathematical tools of Ancient Greece and, more particularly, with those of Euclid's *Elements*. This might have led to the construction being forgotten. But who knows? Maybe nobody had ever done this construction, or maybe they did find a proof that was tragically lost with the burning of the Library of Alexandria or some other historical event... It's just a theory.

Dirk: Do you think you could have published your idea without Professor Labelle's help?

Juan: I don't think so. I recognized the fact that his help was essential to the advancement of the discovery and would eventually lead to it being published. I asked him how I could thank him and give credit to him for all his help. He humbly said that a thank-you note at the end of the article would be enough, and that is what I did. I have expressed my thanks to him on different occasions and we have stayed in touch, talking about mathematics and trips around the world. I also drew a caricature of him as a gift.

Dirk: How did the writing process change the way you thought about your work?

Juan: I realized how the same thing can be said in many different ways, even for something as specific as a mathematical theorem. Among other things, I had to create a new language, so that my ideas could fit the standards of a mathematical publication. The definitions established the vocabulary, giving meaning to different concepts and structures. The theorems and proofs established the grammar, creating rules for what could happen and what could not. I believe that, whereas the existence and the behaviour of homothetic paths (truths stated by theorems) I found were a discovery, the new language used to define them (definitions, graphic representations, and notations), and the way of formulating the ideas (narrative of the article, presentation of the examples, and structure of the theorems) in "Chemins homothétiques" were an invention.

Dirk: What do you mean by "a new language"?

Juan: Whereas a doodle can be interpreted differently by anyone, giving it a name limits this interpretation. By naming my doodles "homothetic paths" or "midpoint paths," I was specifying their nature with a new language, and they felt somehow less free but more real.

Dirk: How was it to give a title to your paper and to name the new concepts you introduced?

Juan: The title of the article changed many times. Even though I love poetic and abstract names, I wanted the title of the article to be very descriptive and clear. The word "paths" (*chemins* in French) had to be in the title, since that was the first image that came to my mind when I drew these series of points. "Paths" was, of course, too general, so I needed an adjective. One of the first names was "fractional paths," since points moved according to fractions. When I discovered that they didn't have to be fractions, but that they could also be irrational numbers, I named it "*chemins pondérés*," a name suggested by Professor Labelle, meaning "weighted paths." Finally, when I acknowledged that the homothety as a transformation was the building block of all the paths, I ended up naming them "*chemins homothétiques*," "homothetic paths."

Finding names for all the elements of the paths was also very interesting. I tried to give names that meant something visually, such as the "convergence line," but always aiming for something simple and descriptive. I also had to invent a new mathematical notation, since homothetic paths were a new type of mathematical structure. I chose $[M_0 : M_n \twoheadrightarrow \mathcal{A}]$ to represent a homothetic path starting at point M_0, ending at M_n, and defined by the series of homotheties $\mathcal{A}$. The square brackets were inspired by the notation of a line segment and meant that it was a finite structure. The double arrow came naturally, representing a succession of steps, or homotheties.

Dirk: After submitting the article to the *Bulletin AMQ*, did you receive some comments from anonymous referees that you found useful and perhaps unexpected?

Juan: Yes, and I learned a lot from their corrections and suggestions. They made the article more concise and logical, both "locally" at the scale of definitions, theorems and proofs, and "globally" at the scale of the entire article. One of the referees wrote: "Your definitions and theorems are sometimes too long and contain [side elements/results]. I would recommend you to rewrite them so that each definition and each theorem contains a single *punch*." This single punch comment had a big impact on me in many ways. It made me more aware of having a single punch while writing new math, writing essays, developing and presenting architecture projects, among other things. However, I sometimes have a hard time applying the single punch method while talking to people, since I usually want to say many things at once!

There were also some very interesting comments about the figures, which corresponded to a selection of doodles translated to computer drawings. The ones I had chosen for the first submission showed diverse examples, because I thought that showing a wide range of unrelated results would emphasize how flexible and complex homothetic paths could be. However, the referees suggested that the figures should build upon each other, to make it clearer to the reader as a story. This made me think of the article as a movie, which I had to reduce to six frames. Whereas I had chosen the six most different frames to show the scope of the movie, the referees wanted to see the six most meaningful frames that would narrate the essence of the story.

Dirk: How would you characterize the main differences between your notebooks, your first drafts, and the final published paper?

Juan: The main difference would be the inversion of the order of the ideas. The doodle that started the entire discovery, which I redrew countless times, was reduced to a single image as an example in the published article, towards the end of it. Since all of my thinking was aimed at understanding and demonstrating how this original doodle worked, the process of discovery went backwards from the result, illustrated by the doodle, to the theorems and the definitions. In the article, the order of presentation is opposite to the order in which I developed my ideas.

Dirk: What do you think about this inversion of the order of presentation?

Juan: After studying the structure of some mathematical articles, I was surprised to see that they all followed similar patterns. For example, definitions were followed by theorems, which were followed by proofs, discussions, and examples. I wanted to know how the authors had made their discoveries, but the process seemed to be hidden. Since there have been so many mathematical papers in history, there have to be good reasons for this "theorem to doodle" method; it is probably more didactic. However, I don't think it should be the only way of publishing new mathematics.

I believe that explaining mathematical ideas in the order of the process of discovery could benefit the readers of mathematical articles who are used to understanding results (from a theorem to a doodle) instead of understanding processes of discovery (from a doodle to a theorem).

Dirk: When you tell people about your discovery, which order do you think works best?

Juan: My favourite way to explain my discovery is from the doodle to the theorem, as a story. Usually, I doodle it on paper, but I have also explained my discovery with random objects that I find around me, like glasses on a table. I've also explained the construction by walking halfway toward two different objects that represent the endpoints of a line segment. I find it very lucky that the construction can be shared with people in a pretty simple way. I have shown the simplest cases to some curious kids at the art school where I used to work, and they understood the doodles.[14] I could show the published article to someone, but I prefer showing a doodle on a paper and talking about how the discovery happened. It feels more engaging and memorable.

2.5. More ideas

Dirk: In the meantime, you also published a second mathematical article, "The Polygon of Euler's Circle" [6]. Could you briefly tell me how this discovery came about?

Juan: Of course. Right after my first paper was published, I felt the need to discover and invent more things! So, I kept doodling and thinking, this time with the intention of making a geometry discovery, but without knowing what it could possibly be. At that time, I was working on an architecture project at McGill University. Professor David Covo's[15] cardboard chair project involved drawing a lot of shapes, in order to design and build a functioning cardboard chair. As I doodled lots of triangles beside my drafting tools, which included different types of orange triangular rulers, I told myself: "If I have a chance of making a discovery about anything, it's probably about triangles, or some simple geometric shape." Then, I drew circles on the triangles and ended up defining the polygon of Euler's circle, which was something new. It's defined like this: "For any triangle, the polygon of Euler's circle that is associated to it is the convex polygon whose summits are all the points defined by the intersection of the perimeter of the triangle and its Euler circle."

[14] Juan was a Teaching Assistant at the Visual Arts Centre, Montréal in summer 2016.

[15] David M. Covo is an Associate Professor and past Director (1996–2007) of the School of Architecture at McGill University, where he has taught since 1977. He has geometry notebooks and architecture sketchbooks similar to Juan's.

This simple definition led to an entire article, and I found it beautiful. It also involved a lot of doodling in the bus and metro, but I wrote the article entirely on my own this time.

Dirk: How would you describe the difference between your first mathematical discovery and its path to its publication, and the corresponding process that lead to your second article?

Juan: Even though both were extremely enjoyable to write, the second article was much faster and easier, for a couple reasons. I had to learn many new things to write my first article, whereas the second one was almost entirely about simple geometry and combinatorics, which I knew better. This made the discovery process much shorter. Whereas the first article was about a new mathematical structure, which could behave in unexpected ways, the second article was about something more limited, a polygon. This made it simpler to study all the possible scenarios and to organize the presentation of my ideas. Most importantly, I already knew how to write a mathematical article when writing the second one, which helped me produce a version that I could submit to the *Bulletin AMQ* of the Mathematical Association of Quebec.

Dirk: You went on to study for a B. Sc. in Architecture at McGill University and more recently a Master in Architecture at Harvard University. How do you see the relation between creative work in architecture and in mathematics?

Juan: At McGill, my professors taught me to always observe and learn from my surroundings, and to draw them whenever I can. Just like with buildings, I love to observe and draw the geometry around us. So, I guess that observation is part of the creative process in both disciplines, since it can spark new ideas and/or stay in our subconscious. I also see some similarities between my approach to an architecture project and a mathematical discovery: they both start with an imaginative, dream-like phase, which is followed by a rational, rigorous phase. In addition, the great importance given to the process in architectural design, both at McGill and at Harvard, has made me value the process in mathematical discovery even more.

Dirk: What would you say are some differences between architecture and geometry?

Juan: I think that architecture cannot be separated from geometry, since it's about creating space! However, there are several differences, one of them being scale. An architecture drawing could represent something at the scale of an object, a person, a building, or a city. A geometrical doodle could represent anything from the scale of a subatomic particle to the scale of the universe, since dimensions don't always have to be specified. It's also interesting how the laws of physics apply to architecture but not to compass and straightedge constructions. You can make a straight line as long as you want, in any direction you want, and it will never bend or buckle! Another important difference is the relationship with people. I think that one of the most beautiful and meaningful aspects of architecture is that it is designed for people. On the other hand, geometry may seem to be completely independent from people.

Dirk: Are you working on any new mathematical ideas right now? If so, how did they start?

Juan: Yes! I've been working on an article about a new type of numerical sequences. It's related to compass and straightedge constructions, ellipses and prime numbers on a number line. This discovery started by trying to minimize the number of circles needed to make certain geometric constructions. I'm also trying to further explore the Midpoint Path Construction and other mathematical structures it relates to. Another recent project, which I worked on with my friend and classmate from McGill, Ankit Gongal, deals with math, COVID-19, and urban planning in informal settlements. Entitled "Unidirectional pedestrian circulation: physical distancing in informal settlements", it was published in the journal *Buildings and Cities*. This research started by reading an article about the Dharavi informal settlement in Mumbai during the pandemic. At school, I have been very interested in the applications of geometry to buildings and structures, especially when they minimize material use and are thus more sustainable, or when they contribute to universal accessibility.

Dirk: If you were asked for an advice for mathematicians or mathematics enthusiasts to come up with new ideas, what would it be?

Juan: It would probably be this, which I believe applies to math discoveries or any type of creative activity: You should keep your eyes open, because inspiration can come from anywhere. However, you should close your eyes from time to time! Your subconscious can give you an answer when you least expect it, maybe while falling asleep.... In addition, I think that it is very important to appreciate every part of the process of discovery, just as much as the results. Sometimes I patiently observe a geometrical doodle or an equation for long periods of time, just to appreciate it. I would also recommend drawing and, finally, trying to meet the people you admire for inspiration, help, and friendship.

Dirk: Thank you for this conversation, Juan!

Juan: It has been my pleasure, Dirk. Thank you!

3. Discussion

While the main point of this paper is the presentation of a case study of the contexts and processes of discovery and justification of a mathematical theorem, let us conclude with briefly highlighting some general themes that emerge from Juan's account.

Juan's first-hand comments, together with his careful documentation of the creative process in his notebooks, enables us a glimpse into the richness of mathematical thought that lies behind the sleek presentation of published papers. This account illustrates very clearly the difference between what Hersh called the "front" and "back" of mathematics [13]. He introduced this terminology to distinguish between mathematics in the making and mathematics as it is presented in academic publications. The difference between the front and back is particularly striking in the present case, because it brings out the difference between a gifted amateur and a professional mathematician in a clear way, both in the way they approached the problem and how they presented their results (Section 2.4). The move from the back to the front, facilitated by an experienced researcher and two anonymous referees, involved a radical transformation: the initial order of presentation, the appeal to examples, the use of concepts and results from other areas of mathematics, the terminology for new concepts, and even the title changed. Similar observations are made in a detailed study of the writing style in professional mathematics [1] and in Ashton's discussion of the role that intended audiences play in mathematical proofs [2].

With regard to the inversion of the order of presentation between the discovery phase and the final publication, Juan's remarks are telling.

During the creative process, the doodle leads to the theorem. In the publication, the theorem leads to the doodle. The nature of mathematical publications—from the theorem to the doodle—brings discoveries to the scientific public in the form of results, where the explanations often begin with general ideas that are made more particular during the presentation. Whether this convention, as opposed to explaining mathematical results in the order of the process of discovery (which Juan prefers with his interlocutors) is best suited for pedagogical purposes and for conveying understanding remains debatable. (For some recent discussions of mathematical explanation and understanding, see [20].)

When comparing Juan's account of the genesis of his theorem about homothetic paths with the published paper, we notice immediately that the latter contains only a few computer-drawn diagrams as illustrations, despite the fact that numerous hand-drawn doodles were an essential part of the formation of the original idea as well as further elaborations and generalizations. In particular, the first construction that sparked the project (shown in Figure 2) did not end up in the final presentation. These observations accord well with the reflections of Carter [3] and of Johansen and Misfeldt [15] on the use of diagrams and other external representations in the practice of research mathematicians.

In Juan's account we can identify several crucial steps in the development of the construction that makes it possible to multiply the length of a line segment by a rational number between 0 and 1 by constructing only midpoints and a straight line:

(i) Initial doodles and constructions with infinite series of midpoints ("midpoint paths").

(ii) Generalization of the midpoint paths. These were later called "homothetic paths," which allow the use of homotheties of ratios different than 1/2, thus constructing not only midpoints.

(iii) Discovery of the alignment of points and the "convergence line," making it possible to obtain the same results in a finite number of steps.

(iv) Discovery of the relation between variations of the original construction and their results; this was later formulated as the Homothetic Path Theorem, establishing a relationship between homothetic paths and their corresponding fixed points.

(v) Generalization to higher dimensions and discovery of a new way of constructing the center of gravity of a finite group of points with arbitrary weights in an arbitrary dimension.

(vi) Reconceptualization and connection to other areas of mathematics, such as linear algebra, binary representation, and modular arithmetic. This was achieved in Labelle's draft, which provided complete proofs and noticed the relationship between a midpoint path construction and the binary representation of the corresponding irreducible fraction.

(vii) Final formulation of the Midpoint Path Theorem and the Midpoint Path Construction, along with the completion of the article [5].

Even though Juan's doodles were essential to the first step, they accompanied him throughout the entire process, in particular in the exploration of constructions and their variations, and in showing patterns that could be exploited further (e. g., the convergence line).

Once the theorem is presented with its final proof, all necessary ingredients and their logical interrelations are exhibited, but certain avenues of exploration have also been closed.

Just as in Poincaré's account of his discovery, published in [10], and Pólya's discussion of "subconscious work" in mathematical discovery [23, 197–199], Juan also repeatedly referred to subconscious processes as well as the importance of an extended incubation time that is filled with extensive thinking about the problem at hand. However, with regard to the initial idea, this particular case study might be somewhat unusual as it did not originate in an attempt to solve a given problem, but in the recognition of a certain pattern in seemingly random doodles.

The creative attitude in mathematics is related by Juan to a similar attitude that he found in art and, in a university setting, in architecture. Note, that this connection is not about the use of mathematics in architecture, which has been widely discussed (see, e. g., [25] and [28]), but about the creative process itself in the two disciplines: the generation and exploration of ideas and their successive refinements through drawing. That not all explorations are immediately successful is part of the process, which Juan described as "trial and better." Its experimental character and the similarities of the process of discovery in different disciplines call for further investigations.

An interesting contrast that also comes out in Juan's answers is, on the one hand, that mathematics is experienced as "existing" and being independent of the individual researcher, which is not that uncommon among mathematicians [4]. On the other hand, Juan also mentions the freedom and power of thinking about mathematics with eyes closed, applying a sort of "manipulative imagination," to use an expression coined by Giardino [9].

Finally, the social and human aspect of mathematics is brought out in Juan's story by the numerous people (teachers, friends, and the Mathematical Association of Quebec) that fostered his mathematical development, encouraged and stimulated his creativity, and helped him shape and sharpen his ideas: from a doodle to a theorem.

A. An algorithm for the Midpoint Path Construction

The Midpoint Path Construction is explained in detail in "Chemins homothétiques" [5]. Two theorems are used to justify the construction: (1) The Midpoint Path Theorem establishes how a series of midpoints is associated to a fixed point P on a line segment $[UV]$, such that $\overrightarrow{UP} = \frac{r}{s}\overrightarrow{UV}$ with $0 < r/s < 1$. (2) The Homothetic Path Theorem establishes how the construction of a midpoint path starting at M_0 and ending at M_n, and the convergence line (M_0M_n), determine the point P at the intersection of the convergence line with $[UV]$. An algorithm for the construction of such a point P on a line segment $[UV]$ is presented below and illustrated with a concrete example. This algorithm does not appear in the original text, but it can be inferred from the definitions, theorems and examples.

Algorithm for the Midpoint Path Construction:

Given a line segment $[UV]$ and a rational number between 0 and 1, written as a reduced fraction r/s with s odd.

(i) Write r/s in binary notation as

$$\frac{r}{s} = 0.\overline{\beta_n\beta_{n-1}\dots\beta_1} \quad \text{(base 2)}.$$

(ii) Define a series $\mathcal{A} : (A_1, A_2, \dots, A_n)$ such that $A_i = U$, if $\beta_i = 0$, and $A_i = V$, if $\beta_i = 1$, for $i = 1, 2, \dots, n$. This is the *midpoint series* $\mathcal{A}$.

(iii) Choose a point M_0 not on the straight line (UV).

(iv) Construct a series of midpoints $M_1, M_2, \ldots, M_n$, such that $M_i = \frac{1}{2}M_{i-1} + \frac{1}{2}A_i$, for $i = 1, 2, \ldots, n$. The points $M_0, M_1, \ldots, M_n$ form the *midpoint path* $[M_0 : M_n \twoheadrightarrow \mathcal{A}]$.

(v) Draw the straight line (M_0M_n). This is the *convergence line.*

(vi) The intersection of (M_0M_n) and $[UV]$ is the point P, such that $\overrightarrow{UP} = \frac{r}{s}\overrightarrow{UV}$.

For the case that the denominator in r/s is even, divide the line segment $[UV]$ as many times as needed, by constructing midpoints between its endpoints, to arrive at a case where s odd. (For example, if $r/s = 1/6$, divide the initial line segment by two and use its first half as $[UV]$, where $\overrightarrow{UP} = \frac{1}{3}UV$.)

Example:

Given a line segment $[UV]$ and a rational number $r/s = 4/15$.

(i) Write r/s in binary notation as

$$\frac{4}{15} = 0.010001000100\ldots \text{ (base 2)} = 0.\overline{0100} \text{ (base 2)}.$$

(ii) Define the midpoint series $\mathcal{A} : (U, U, V, U)$.

(iii) Choose a point M_0 not on the straight line (UV). (See Figure 9).

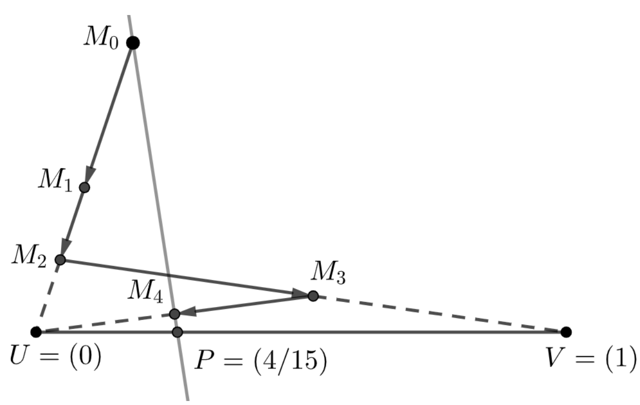

Figure 9: Example of the Midpoint Path Construction for $r/s = 4/15$.

(iv) Construct all the remaining points of the midpoint path: $M_1 = \frac{1}{2}M_0 + \frac{1}{2}U$, $M_2 = \frac{1}{2}M_1 + \frac{1}{2}U$, $M_3 = \frac{1}{2}M_2 + \frac{1}{2}V$, and $M_4 = \frac{1}{2}M_3 + \frac{1}{2}U$.

(v) Draw the convergence line (M_0M_4).

(vi) The intersection of (M_0M_4) with $[UV]$ defines the point P, such that $\overrightarrow{UP} = \frac{4}{15}\overrightarrow{UV}$.

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