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Weiqing Gu
Harvey Mudd College

Christopher Pries '03
Harvey Mudd College

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EXAMPLES OF CAYLEY 4-MANIFOLDS

WEIQING GU AND CHRISTOPHER PRIES

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Abstract. We determine several families of so-called Cayley 4-dimensional manifolds in the real Euclidean 8-space. Such manifolds are of interest because Cayley 4-manifolds and Cayley 4-cycles in Calabi-Yau 4-folds and Spin(7) holonomy manifolds are supersymmetric cycles that are candidates for representations of fundamental particles in String Theory. Moreover, some of the examples of Cayley manifolds discovered in this paper may be modified to construct explicit examples in our current search for new holomorphic invariants for Calabi-Yau 4-folds and for the further development of mirror symmetry.

We apply the classic results of Harvey and Lawson to find Cayley manifolds which are graphs of functions from the set of quaternions to itself. We consider graphs which are invariant under the action of three dimensional subgroups of Spin(7) which fix the quaternions as a subgroup of the Cayley numbers. Spin(7) is a subgroup of SO(8) which preserves the Cayley form. Systems of ODEs and PDEs are derived and solved, some special cases of a classic theorem of Harvey and Lawson are investigated, and theorems aiding in the classification of all such manifolds described here are proven. Several families of interesting Cayley 4-dimensional manifolds are discovered. Some of them are novel.

Key words and phrases. Cayley 4-form, calibration, calibrated 4-plane, Cayley manifold, $Spin_7$, $SO(8)$, symmetry group, symmetry action, Monge-Ampere operator, Dirac operator, Quaternions, Cayley numbers, and triple cross product.

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1. Introduction and Background

1.1. **Overview.** The objective of this paper is to use the method of calibration to identify several families of Cayley submanifolds in $\mathbb{R}^8$ which are graphs of functions $f : \mathbb{H} \to \mathbb{H}$ and which are invariant under certain three dimensional subgroups of Spin_7.

The 3-dimensional subgroups of Spin_7 we consider are those which leave $\mathbb{H}$ and $\mathbb{H}\mathbf{e}$ fixed. It is shown in section 7 that the Cayley manifolds found using these subgroups can be reduced to a few key examples. These key examples are examined individually. One of them coincides with the classic example of a coassociative manifold in Harvey and Lawson's well known paper [24]. Many of these subgroups result in trivial examples and linear examples. However, two new interesting examples of families of Cayley manifolds are found. The second one is novel.

Theorem 1.1. *For any $c \in \mathbb{R}$ fixed, the graph*

(1)
$$M_c = \left\{ s\frac{x}{|x|} + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, |x|^3 s - s^3|x| = c \right\}$$

is a Cayley manifold in $\mathbb{R}^8$.

Theorem 1.2. *For $c, k, s \in \mathbb{R}$ constants, and $\varphi : \mathbb{H} \to \mathbb{R}$, the graph*

(2)
$$M_{c,k,s} = \left\{ x\varphi(x) + s\varphi(x) + k + x\mathbf{e} \;\middle|\; \forall x \in \mathbb{H}, \varphi(x) - \varphi^3(x) = \frac{c}{(|\mathrm{Im}\, x|^2 + (s + \mathrm{Re}\, x)^2)^2} \right\}$$

is a Cayley Manifold in $\mathbb{R}^8$.

The first Cayley manifold was found using a similar technique that Harvey and Lawson used to find a coassociative manifold in [24]. The second one was found using a slightly different technique. Both are examples of new Cayley manifolds never discovered before.

We will organize this paper as follows.

- Section 1 starts with the problem and main results of the paper, presents the background of the problem and why Cayley manifolds are interesting, gives an introduction on Cayley Calibration, discusses its related PDE of Cayley manifolds, and ends up with some techniques on how to simplify the PDE.
- Section 2 carefully shows the techniques used throughout this paper on how to derive a Cayley manifold.
- Section 3 shows another action which results in a more general form of the same family of manifolds as derived in section 2.

- Section 4 derives a family of Cayley manifolds which includes the classic example of coassociative manifolds presented in [24].
- Section 5 discusses a collection of subgroups which all result in trivial and linear Cayley manifolds.
- Section 6 identifies the second interesting new family of Cayley manifolds mentioned above.
- Section 7 presents other related subgroups of Spin_7, showing why the selected subgroups are the key subgroups, and concludes with a brief discussion of future work.

1.2. **Background.** The method of calibrated geometries aims to solve optimization problems in geometry by the clever, nonstandard use of differential forms. It turns out that such optimization solutions (i.e. calibrated submanifolds or calibrated cycles) in Calabi-Yau manifolds and exceptional holonomy manifolds play fundamental roles in the development of string theory.

The field of calibrated geometry began with the work of Wirtinger [33] in the 1930s, de Rham [9] in the 1950s, and Federer [10] in the 1960s, which uses Kähler forms and their powers to prove that compact *complex submanifolds* of Kähler manifolds are volume-minimizing in their homology classes. In the early 1970s, Berger [1] extended this approach to quaternionic forms. In the early 1980s, Harvey and Lawson wrote a monumental work [24] on this subject. They exhibited and studied several beautiful geometries of minimal subvarieties other than complex submanifolds which include associative geometry, coassociative geometry and Cayley geometry. They also proposed the problem of determining the "characteristic class geometries". It was on some of these problems that the first author carried out in her previous research work. See [17], [15], [16], [18], [19] and [20]. Preliminary results in the direction of this research have been obtained by a number of mathematicians, including Harvey and Lawson [24], Morgan [13], [11], [27], Dadok [7], Lawlor [26], McLean [28], Bryant [3], Gluck [11], [8], and Ziller [13].

Calibrated geometry received renewed attention in 1996 when the role of the special Lagrangian in mirror symmetry was discovered by Strominger, Yau, and Zaslow. (See [30].) The reader might also consult [22], [6], [14], [5], [25], [21],[4] and [29].

There has been new interest recently in the geometry of Cayley cycles. Following [30], the roles of exceptional geometries in mirror symmetry were first investigated in [2], "Supersymmetric cycles in exceptional holonomy manifolds and Calabi-Yau 4-folds". From the physics point of view, the authors showed that the Cayley cycles in $Spin_7$ holonomy eight-manifolds and the associative and coassociative cycles in G_2 holonomy seven-manifolds preserve half of the space-time supersymmetry. They discovered that while the holomorphic and special Lagrangian cycles in Calabi-Yau 4-folds preserve half of the space-time supersymmetry, the Cayley submanifolds are novel as they preserve only one quarter of it. They also conjectured what kind of roles Cayley cycles will play in mirror symmetry for Calabi-Yau 4-folds (in contrast to the roles of holomorphic and special Lagrangian cycles in the mirror symmetry of Calabi-Yau 3-folds) and proposed the problem of *finding explicit examples of Cayley cycles to demonstrate the above conjectured phenomenon.*

Cayley submanifolds and Cayley cycles have another important applications in the further development of gauge theory as discussed by Gang Tian in his paper [31] on *Gauge Theory and Calibrated Geometry*. Identifying Cayley cycles in the complex 4-dimensional torus (a Calabi-Yau 4-fold) is a a problem suggested to the first author by Professor Gang Tian. This author has studied the linear cases of Cayley cycles in $T^8_{\mathbb{R}}$. For the nonlinear cases, one may study the Cayley manifolds in $\mathbb{R}^8$ first. Recall a flat torus T^8 can be identified as $T^8 \cong \mathbb{R}^8/\Lambda$, where Λ is a lattice in $\mathbb{R}^8$. Any Cayley cycle in $\mathbb{R}^8$ lifts to a Λ-periodic Cayley cycles in $\mathbb{R}^8$. Unfortunately, few non trivial Cayley submanifolds or Cayley cycles are known even in $\mathbb{R}^8$. Thus, as a first step in the direction of this research, we are going to identify Cayley manifolds in $\mathbb{R}^8$.

1.3. **Method of Calibrations.** Let M be a Riemannian manifold, a *calibration* on M is a closed p-form ϕ such that

$$\phi(\mathbf{e}_1, \ldots, \mathbf{e}_p) \leq 1 \tag{3}$$

on all orthonormal p-tuples of tangent vectors at all points of M, i.e. on all tangent p-planes $e_1 \wedge e_2 \wedge \cdots \wedge e_p$ with $|e_1 \wedge e_2 \wedge \cdots \wedge e_p| = 1$. A tangent plane is *calibrated* if ϕ achieves the maximal value 1 on it. A p-dimensional submanifold of M is called calibrated if all of its oriented tangent planes are calibrated. The crucial result is that any calibrated closed oriented p-dimensional cycle $N \subset M$ is of absolutely minimal volume in its homology class.

1.4. **Cayley Calibration on $\mathbb{R}^8$.** The Cayley Calibration is closely related to associative and coassociative calibrations. The following discussion on these calibrations mainly follows from [24]. Let $\mathbb{O} \cong \mathbb{R}^8$ denote the set of Cayley numbers, $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ the purely imaginary subset.

In the *associative geometry*, we use the associative calibration (i.e. a 3-form) $\phi(x, y, z) =< x, yz >$ on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. It is easy to check that ϕ is trilinear and alternating. So $\phi \in \wedge^3(\mathrm{Im}\,\mathbb{O})^*$. It is called associative because the local system of differential equations for this geometry is essentially deduced from the vanishing of the *associator* $[x, y, z] \equiv (xy)z - x(yz)$.

In the *coassociative geometry*, we use the coassociative form ψ on $\mathrm{Im}\,\mathbb{O}$. It is just $\star\phi$, where $\star : \wedge^3\mathbb{R}^7 \to \wedge^4\mathbb{R}^7$ is the usual map of Euclidean spaces, mapping simple forms to simple forms. It is called coassociative because it is the dual geometry of the associative geometry.

In the *Cayley geometry*, we use the Cayley calibration $\Phi \in \wedge^4\mathbb{O}^*$ on $\mathbb{O} \cong \mathbb{R}^8$. It is just $1^* \wedge \phi + \psi$. Cayley geometry is the most complex and fascinating geometry discussed in [24].

In local coordinates, Φ can be written as

$$\Phi(x, y, z, w) = \langle x, y \times z \times w\rangle \tag{4}$$

where $x, y, z, w \in \mathbb{O}$, here we make use of the triple cross product of Cayley numbers. (See [24], def. B.3).

One can verify that Φ is alternating, closed, and has comass one.

Theorem 1.3. *The form Φ has comass one. In fact, $\Phi(\zeta) \leq 1$ for all $\zeta \in G(4, 8) \subset \wedge^4\mathbf{O}$, with equality if and only if ζ is a Cayley 4–plane (i.e. ζ or $-\zeta$ is a complex 2–plane with respect to one of the complex structures determined by a two-plane contained in ζ.)*

Please see [24] for further details. We call a 4-plane a Cayley plane if Φ achieves the comass 1 on it. We call a 4-manifold a Cayley manifold if all its tangent planes are Cayley planes. We use $G(\Phi)$ to denote the set of Cayley planes.

Recall that $Spin_7$ is the subgroup of $SO(8)$ generated by $S^6 \equiv \{R_u : u \in \mathrm{Im}\,\mathbb{O}$ and $|u| = 1\}$, where R_u is the right Cayley multiplication. As shown in [24], there are several alternate definitions of Spin_7 which are particularly useful,

$$Spin_7 = \{g \in SO(8) : g^*\Phi = \Phi\}. \tag{5}$$

$$\mathrm{Spin}_7 = \left\{ g \in SO_8 \middle| g(uv) = g(u)\overline{\chi}_g(v) \right\} \qquad \forall u, v \in \mathbb{O} \tag{6}$$

where $\overline{\chi}_g : \mathrm{Spin}_7 \to SO(\mathrm{Im}\,\mathbb{O}) \cong SO_7$ is defined by $\overline{\chi}_g(v) = g(g^{-1}(1) \cdot v)$ for all $v \in \mathbb{O}$, which is the standard double cover of SO_7 by Spin_7. See [24] for details.

Theorem 1.4. *The action of $Spin_7$ on $G(\Phi)$ is transitive with isotropy subgroup $K = SU(2) \times SU(2) \times SU(2)/\mathbf{Z}_2$. Thus $G(\Phi) \cong Spin_7/K$.*

Remark 1.5. *The geometry of Cayley submanifolds includes several other geometries.*

(1) *A submanifold M which lies in* $\mathrm{Im}\,\mathbb{O} \subset \mathbb{O}$ *is Cayley if and only if M is coassociative.*
(2) *A submanifold M of $\mathbb{O}$ of the form $\mathbb{R} \times N$, where N is a submanifold of* $\mathrm{Im}\,\mathbb{O}$*, is Cayley if and only if N is associative.*
(3) *Fix a unit imaginary quaternion $u \in S^6 \subset \mathrm{Im}\,\mathbb{O}$. Consider the complex structure J_u and let $\mathbb{O} \cong \mathbb{C}^4$. Each complex surface in $\mathbb{O}$, with the reverse orientation, is a Cayley submanifold.*
(4) *In addition to choosing one of the distinguished complex structures J_u (as in (3)) choose a quaternion subalgebra $\tilde{\mathbb{H}}$ of $\mathbb{O}$ orthogonal to u and identify $\mathbb{R}^4 \subset \mathbb{C}^4$ with $\tilde{\mathbb{H}} \subset \mathbb{O}$. Each special Lagrangian submanifold of $\mathbb{C}^4 \cong \mathbb{O}$ is a Cayley submanifold.*

There are few known Cayley submanifolds which are not holomorphic and special Lagrangian.

1.5. **Partial Differential Equations of Cayley Manifolds.** One special type of Cayley submanifold we can look for is the graph of a function $f : \Omega \subset \mathbb{H} \to \mathbb{H}$. That is, manifolds parametrized as $(x, f(x)) \in \mathbb{H} \oplus \mathbb{H} = \mathbb{O}$. The local system of partial differential equations for this case is deduced in [24].

We denote a point in $\mathbb{H}$ by $x = x_1 + x_2\hat{\boldsymbol{\imath}} + x_3\hat{\boldsymbol{\jmath}} + x_4\hat{\boldsymbol{k}}$.

Definition 1. The *Dirac* operator D is defined on f as

$$Df = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\hat{\boldsymbol{\imath}} - \frac{\partial f}{\partial x_3}\hat{\boldsymbol{\jmath}} - \frac{\partial f}{\partial x_4}\hat{\boldsymbol{k}} \tag{7}$$

The *first order Monge-Ampere operator* on f is defined as

$$\sigma f = \left(\frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right) + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right)\hat{\boldsymbol{\imath}} -$$

$$\left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_4}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3}\right)\hat{\boldsymbol{k}} \tag{8}$$

and a third operator is defined by

$$\delta f = \mathrm{Im}\left[\left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}\right)\hat{\boldsymbol{\imath}}\right] +$$

$$\mathrm{Im}\left[\left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_3} + \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_4}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_4} - \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3}\right)\hat{\boldsymbol{k}}\right] \tag{9}$$

Theorem 1.6. *Suppose $f : \Omega \subset \mathbb{H} \to \mathbb{H}$ is C^1. The graph of f is a Cayley manifold if and only if f satisfies the differential equations*

$$Df = \sigma f \tag{10}$$

$$\delta f = 0 \tag{11}$$

Note that the resulting PDEs are only first order, however, the σf term is highly non-linear. No one knows how to solve this system in general. In order to simplify these PDEs, we will impose certain symmetry restrictions on our manifolds as the reader will see in next section.

1.6. **Symmetry Restrictions Used for Simplifying the PDEs.** How are we going to impose symmetry restrictions on our manifolds to simplify the above PDEs? We are going to fix a nonzero vector ϵ in $\mathbb{H}$ and seek a curve in the plane $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}\epsilon \oplus \mathbb{R} \subset \mathbb{O}$ which is going to sweep out a candidate Cayley manifold under the action of some 3-dimensional subgroup of Spin_7. Then the PDE will be simplified to an ODE or to one with less variables. We will refer to such a 3-dimensional subgroup as a *symmetry group* and to such an action as a *symmetry action* in this paper.

By (5), any 3-dimensional subgroups of Spin_7 will preserve the Cayley form (4). Since we are interested in finding Cayley graphs from $\mathbb{H}$ to $\mathbb{H}$, we wish to determine what subgroups of $\mathrm{Spin}_7 \subset SO_8$ fix $\mathbb{H}$ and $\mathbb{H}\mathbf{e}$.

First, notice that the only oriented Cayley 4-plane lying entirely in $\mathbb{H} \subset \mathbb{O}$ is $\xi = 1 \wedge \hat{\boldsymbol{\imath}} \wedge \hat{\boldsymbol{\jmath}} \wedge \hat{\boldsymbol{k}}$. Any subgroup of SO_8 that fixes $\mathbb{H}$ will necessarily send ξ to another 4-plane in $\mathbb{H}$. Since Spin_7 preserves the Cayley calibration, any $g \in \mathrm{Spin}_7$ that also fixes $\mathbb{H}$, must send ξ to another Cayley 4-plane in $\mathbb{H} \subset \mathbb{O}$. Since ξ is the only such 4-plane, we have $g(\xi) = \xi$, and hence $g \in K$, the isotropy subgroup of Spin_7 at ξ. Conversely, since $\mathrm{Spin}_7 \subset SO_8$, if $g \in K$ we are ensured that g fixes $\mathbb{H}$ and $\mathbb{H}\mathbf{e}$.

Thus $K \cong \frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$ is the subgroup of Spin_7 that we will use to search for symmetry actions. Here the action of $(q_1, q_2, q_3) \in K$ is as follows,

$$a + b\mathbf{e} \mapsto q_3 a \overline{q_1} + (q_2 b \overline{q_1})\mathbf{e} \tag{12}$$

for all $a, b \in \mathbb{H}$.

Fortunately K gives a wealth of 3-dimensional subgroups with which to use as symmetry groups. It will be shown that the simplified PDE's that result from the vast majority of these subgroups can be reduced to the PDE's resulting from just a few key subgroups. The next few sections examine these key subgroups in detail and the discussion of the relationship of these key subgroups to the remaining subgroups is taken up in Section 7.

Throughout this paper we use a particular notation to describe the key subgroups of K. For example,

$$(1,1,q) = \left\{(1,1,q) \;\middle|\; q \in Sp_1\right\} \subset \frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2} = K \tag{13}$$

Similarly we denote the other key subgroups we are going to consider as (1,q,1), (1,q,q), (q,1,1), (q,1,q), (q,q,1), and (q,q,q).

1.7. **Further Simplification of the PDEs.** In what follows we will search for Cayley 4-manifolds that are graphs over $\mathbb{H}\mathbf{e} \subset \mathbb{O}$. i.e. We will find manifolds of the form $M = \{f(x) + x\mathbf{e} \mid x \in \mathbb{H}\}$, where $f : \mathbb{H} \to \mathbb{H}$. We will employ 3-dimensional subgroups of the group K, defined above. However, it will be useful to examine what simplifications we can derive under such a subgroup action in general.

Let M be a 4-submanifold of $\mathbb{R}^8 \cong \mathbb{O}$. M is symmetric under $g \in \mathrm{Spin}_7$ if for all $\alpha \in M$, we have $g(\alpha) \in M$. We will prove a useful theorem concerning manifolds symmetric under the actions of subgroups of Spin_7.

Theorem 1.7. *Let M be a 4-submanifold of $\mathbb{O} \cong \mathbb{R}^8$, symmetric under the action of $< g > \subset Spin_7$, the subgroup generated by g, where the action is defined above. Let $\alpha \in M$. Let ξ_α be an oriented tangent 4-plane of M at α in $\mathbb{O}$. Similarly let $\xi_{g(\alpha)}$ be the oriented tangent 4-plane of M at $g(\alpha) \in M$ in $\mathbb{O}$ with orientation inherited from ξ_α via g. Then,*

$$\Phi(\xi_{g(\alpha)}) = \Phi(\xi_\alpha) \tag{14}$$

where Φ is the Cayley calibration.

There is an immediate corollary which will be of much use in this paper.

Corollary 1.8. *Let $M = \{f(x) + x\mathbf{e} \mid x \in \mathbb{H}\}$ be a graph over $\mathbb{H}\mathbf{e}$ that is symmetric under the action of the group generated by $(q_1, q_2, q_3) \in K \cong \frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$. Then $(Df - \sigma f)(q_2 x \overline{q_1}) = 0$ and $\delta f(q_2 x \overline{q_1}) = 0$ if and only if $(Df - \sigma f)(x) = 0$ and $\delta f(x) = 0$.*

Proof of Theorem 1.7. As above, let $\alpha \in M$. Let $h : \Omega \subset \mathbb{R}^4 \to M \subset \mathbb{O}$ be a local parametrization of M at α. Since $h(\Omega) \subset M$ we have $g \circ h(\Omega) \subset M$, and $g \circ h : \Omega \subset \mathbb{R}^4 \to M$ is a local parametrization of M at $g(\alpha)$.

Let $\{x_1, x_2, x_3, x_4\}$ denote the coordinates of $\mathbb{R}^4$, since g is a linear map, we get

$$\frac{\partial}{\partial x_i}(g \circ h) = g \circ \frac{\partial h}{\partial x_i} \tag{15}$$

Thus, if $p \in \Omega$ is the pre-image of α under h (and hence the pre-image of $g(\alpha)$ under $g \circ h$), we have

$$\xi_\alpha = \frac{\partial h}{\partial x_1}(p) \wedge \frac{\partial h}{\partial x_2}(p) \wedge \frac{\partial h}{\partial x_3}(p) \wedge \frac{\partial h}{\partial x_4}(p) \tag{16}$$

$$\xi_{g(\alpha)} = [g \circ \frac{\partial h}{\partial x_1}(p)] \wedge [g \circ \frac{\partial h}{\partial x_2}(p)] \wedge [g \circ \frac{\partial h}{\partial x_3}(p)] \wedge [g \circ \frac{\partial h}{\partial x_4}(p)]. \tag{17}$$

Or equivalently, $\xi_{g(\alpha)} = g(\xi_\alpha)$. Now by the definition of Spin_7 given in equation (5) and the fact that $g \in \mathrm{Spin}_7$ the theorem is proven. □

Before proving Corollary 1.8, we will state the following Proposition which is useful in simplifying our task in this paper.

Proposition 1.9. *Given $f : \Omega \subset \mathbb{H} \to \mathbb{H}$ of class C^1 with Ω an open subset of $\mathbb{H}$, the graph of f over $\Omega \subset \mathbb{H} \subset \mathbb{O}$ is Cayley if and only if the graph of f over $\Omega\mathbf{e} \subset \mathbb{H}\mathbf{e} \subset \mathbb{O}$ is Cayley.*

For the proof of this Proposition, see [24].

Proof of Corollary 1.8. First we must recognize the conditions that the symmetry imposes on the function $f : \mathbb{H} \to \mathbb{H}$. Let M be a graph over $\mathbb{H}\mathbf{e}$ (i.e. $M = \{f(x) + x\mathbf{e} \mid x \in \mathbb{H}\}$), that is symmetric under $(q_1, q_2, q_3) \in K \cong \frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$. Since

$$f(x) + x\mathbf{e} \mapsto q_3\, f(x)\, \overline{q_1} + (q_2\, x\, \overline{q_1})\mathbf{e} \tag{18}$$

must be contained in M, we immediately derive that

$$f(q_2\, x\, \overline{q_1}) = q_3\, f(x)\, \overline{q_1}. \tag{19}$$

Let $z = q_2\, x\, \overline{q_1}$ so that $f(x)+x\mathbf{e} \mapsto f(z)+z\mathbf{e} \in M$. Suppose that $(Df - \sigma f)(z) = 0$ and $\delta f(z) = 0$. By Theorem 1.6 we have that the tangent 4-plane to M at $f(z)+z\mathbf{e}$ is Cayley. Thus by Theorem 1.7 we know that the tangent 4-plane to M at $f(x)+x\mathbf{e}$ is also Cayley, and again by Theorem 1.6 we have $(Df - \sigma f)(x) = 0$ and $\delta f(x) = 0$. Thus the first half of the Corollary is proven. Since M is symmetric under $< g >$, it is symmetric under g^{-1} and the proof of the second half of the Corollary is identical to the first. □

2. A Detailed Example of a Family of Cayley Manifolds

2.1. **Overview.** As we mentioned in the introduction, in this section we will first provide a detailed example of two families of Cayley manifolds. Then we will demonstrate a technique similar to the one Harvey and Lawson used to derive coassociative manifolds. Finally, we will conclude this section with some analysis of these particular examples.

2.2. **A Family of Cayley Manifolds.** Here we will identify a family of Cayley manifolds in $\mathbb{R}^8 \cong \mathbb{O}$ that are symmetric under the group $Sp_1 = S^3 \subset \mathbb{H}$, acting on $\mathbb{O}$ as follows

$$a + b\mathbf{e} \mapsto qa + (qb)\mathbf{e} \tag{20}$$

for each $q \in Sp_1$. In Theorem 2.1 below, we obtain a linear graph, while in Theorem 2.2, we obtain an interesting, nonlinear one. This action was chosen as an example, because it demonstrates many of the subtleties of the technique we used in this paper, without being overly complicated.

Theorem 2.1. *For any $\varepsilon \in \operatorname{Im}\mathbb{H}$ fixed, for any $c \in \mathbb{R}$ fixed, the graph*

$$M_{\varepsilon,c} = \left\{ cx\varepsilon + x\mathbf{e} \;\middle|\; x \in \mathbb{H} \right\} \tag{21}$$

is a Cayley manifold in $\mathbb{R}^8$.

Theorem 2.2. *For any $k \in \mathbb{R}$ fixed, for any $c \in \mathbb{R}$ fixed, the graph*

$$M_{k,c} = \left\{ ks\frac{x}{|x|} + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, |x|^3 s - k^2 s^3 |x| = c \right\} \tag{22}$$

is a Cayley manifold in $\mathbb{R}^8$.

The first three figures depict slices of one member of the second family of Cayley Manifolds. The specific Cayley Manifold is $M = \{s\frac{x}{|x|} + x\mathbf{e} \mid x \in \mathbb{H}, |x|^3 s - s^3|x| = 5\}$. These figures begin to show the intricate structure of this family of manifolds.

2.3. **The Proof of Theorems 2.1 and 2.2.** Before proving Theorems 2.1 and 2.2, we will introduce two lemmas.

Lemma 2.3. *Assume that $M = \left\{ f(x) + x\mathbf{e} \;\middle|\; x \in \mathbb{H} \right\} \subset \mathbb{O}$ is the graph of a function $f : \mathbb{H} \to \mathbb{H}$. Then M is invariant under the action (20) of Sp_1 if and only if*

$$f(x) = \frac{x}{|x|} f(|x|) \tag{23}$$

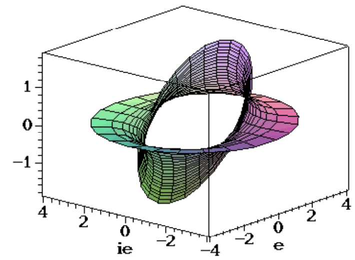

FIGURE 1. A slice of the Cayley manifold of Theorem 2.2 in $\{e, ie, real\}$ space. Here $x_3 = x_4 = 0$.

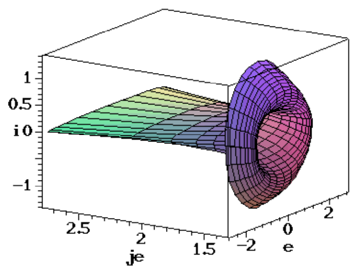

FIGURE 2. A slice of the Cayley manifold of Theorem 2.2 in $\{e, je, i\}$ space. Here $x_4 = 0$, and $x_1^2 + x_2^2 = x_3^2$.

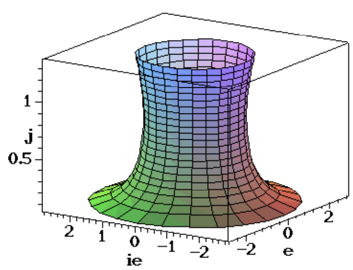

FIGURE 3. A slice of the Cayley manifold of Theorem 2.2 in $\{e, ie, j\}$ space. Here again $x_4 = 0$ and $x_1^2 + x_2^2 = x_3^2$.

PROOF. If M is invariant under the action (20) above, then for each $q \in Sp_1 \subset \mathbb{H}$ and each $x \in \mathbb{H}$, the point $qf(x) + (qx)\mathbf{e}$ also belongs to M. Thus

$$f(qx) = qf(x) \tag{24}$$

for all $q \in Sp_1$ and all $x \in \mathbb{H}$. Now replacing x by $|x|$ and q by $\frac{x}{|x|}$ in equation (24) recovers equation (23). Now consider a function characterized by equation (23). Plugging in qx yields

$$f(qx) = \frac{qx}{|qx|} f(|qx|) = q\frac{x}{|x|} f(|x|) = qf(x) \tag{25}$$

for all $q \in Sp_1$ and all $x \in \mathbb{H}$. Thus we obtain equation (24), and the graph of f is invariant under action (20) $\square$

By Theorem 1.8, for f symmetric under the action (20), we have $(Df - \sigma f)(qr) = 0$ if and only if $(Df - \sigma f)(r) = 0$. Thus it is enough to compute $Df - \sigma f$ at $x = |x|$.

In order to further simplify the non-linear σf portion of the PDEs (10) and (11), we choose a special case where $f(|x|) = \varepsilon\varphi(|x|)$, $\varphi : \mathbb{R}^+ \to \mathbb{R}$, and $\varepsilon \in \mathbb{H}$ is fixed.

Lemma 2.4. *Suppose that $\varepsilon \in \mathbb{H}$ is a fixed vector, $\varepsilon = \varepsilon_1 + \varepsilon_2\hat{\boldsymbol{\imath}} + \varepsilon_3\hat{\boldsymbol{\jmath}} + \varepsilon_4\hat{\boldsymbol{k}}$, and $\varphi : \mathbb{R}^+ \to \mathbb{R}$, is a given function. Let $f : \mathbb{H} \to \mathbb{H}$*

$$f(x) = x\varepsilon\frac{\varphi(|x|)}{|x|} \tag{26}$$

So that M, the graph of f, is invariant under the action (20). Then M is Cayley if and only if

$$\varepsilon\left(\varphi^{'}(r) - \frac{\varphi(r)}{r}\right)\left(1 + \frac{\varphi^2(r)}{r^2}|\varepsilon|^2\right) + 4\varepsilon_1\frac{\varphi(r)}{r} - 4\varepsilon_1\frac{\varphi^2(r)}{r^2}|\varepsilon|^2\varphi^{'}(r) = 0 \tag{27}$$

$$\left(\varphi^{'}(r)\frac{\varphi(r)}{r} + \frac{\varphi^2(r)}{r^2}\right)4(\mathrm{Re}\,\varepsilon)(\mathrm{Im}\,\overline{\varepsilon}) = 0 \tag{28}$$

PROOF. We must compute $Df - \sigma f$ and δf. As previously discussed, it is enough to compute $Df - \sigma f$ and δf at $x = |x| = r \in \mathbb{R}^+$. By direct calculation, we have that

$$\frac{\partial f}{\partial x_1} = \varepsilon\varphi^{'}(r), \qquad \frac{\partial f}{\partial x_2} = \hat{\boldsymbol{\imath}}\varepsilon\frac{\varphi(r)}{r}, \qquad \frac{\partial f}{\partial x_3} = \hat{\boldsymbol{\jmath}}\varepsilon\frac{\varphi(r)}{r}, \qquad \frac{\partial f}{\partial x_4} = \hat{\boldsymbol{k}}\varepsilon\frac{\varphi(r)}{r} \tag{29}$$

Thus, computing from the definitions, we have

$$Df(r) = \varepsilon\left(\varphi^{'}(r) - \frac{\varphi(r)}{r}\right) + 4\varepsilon_1\frac{\varphi(r)}{r} \tag{30}$$

$$\sigma f(r) = \varepsilon\left(-\varphi^{'}(r) + \frac{\varphi(r)}{r}\right)\frac{\varphi^2(r)}{r^2}|\varepsilon|^2 + 4\varepsilon_1\varphi^{'}(r)\frac{\varphi^2(r)}{r^2}|\varepsilon|^2 \tag{31}$$

$$\delta f(r) = \left(\varphi^{'}(r)\frac{\varphi(r)}{r} + \frac{\varphi^2(r)}{r^2}\right)4(\mathrm{Re}\,\varepsilon)(\mathrm{Im}\,\overline{\varepsilon}) \tag{32}$$

Thus, requiring $Df - \sigma f = 0$ and $\delta f = 0$ is equivalent to condition (27). □

By examining $\delta f = 0$ and noting that $\mathbb{H}$ is a division ring, we see there are now three cases to examine in detail. ε can be solely imaginary, solely real or $\varphi^{'}\frac{\varphi}{r} + \frac{\varphi^2}{r^2} = 0$. Let us first examine the case where $\varepsilon \in \mathrm{Im}\,\mathbb{H}$. Thus $\mathrm{Re}\varepsilon = 0 = \varepsilon_1$. Applying this to the differential condition (27) we get

$$\left(\varphi^{'}(r) - \frac{\varphi(r)}{r}\right)\left(1 + \frac{\varphi^2(r)}{r^2}|\varepsilon|^2\right) = 0 \tag{33}$$

Thus, we have one of two conditions. Either

$$1 + \frac{\varphi^2(r)}{r^2}|\varepsilon|^2 = 0 \tag{34}$$

or

$$\varphi^{'} - \frac{\varphi(r)}{r} = 0 \tag{35}$$

Since $\varphi : \mathbb{R} \to \mathbb{R}$ there is no solution to equation (34). Thus equation (35) must hold. It results in the simple solution

$$\varphi(r) = Cr \tag{36}$$

for a constant $C \in \mathbb{R}$. Plugging this back into equation (26) results in the simple linear graph for Theorem 2.1.

Now consider the case where $\varepsilon \in \mathbb{R}, |\varepsilon| = k$. Now the differential condition (27) simplifies to

$$(\varphi^{'}(r) + 3\frac{\varphi(r)}{r}) - k^2\frac{\varphi^2(r)}{r^2}(3\varphi^{'}(r) + \frac{\varphi(r)}{r}) = 0 \tag{37}$$

which results in the equivalent condition

$$\frac{\partial}{\partial r}\left[r^3\varphi - k^2\varphi^3 r\right] + \frac{\partial}{\partial\varphi}\left[r^3\varphi - k^2\varphi^3 r\right]\varphi'(r) = 0 \tag{38}$$

This yields an implicit definition of φ, as is often the case in other examples. Here φ is a solution provided

$$r^3\varphi(r) - k^2\varphi^3(r)r = C \tag{39}$$

for constants $C \in \mathbb{R}$ and $k \in \mathbb{R}$. Plugging this back into equation (26) results in the graph for Theorem 2.2.

Now consider the final case, where $\varphi' \frac{\varphi}{r} + \frac{\varphi^2}{r^2} = 0$. This differential equation has two solutions. The first is the trivial solution $\varphi = 0$, the other is

$$\varphi(r) = \frac{C}{r} \tag{40}$$

for some constant $C \in \mathbb{R}$. However, only the trivial solution is compatible with $Df - \sigma f = 0$. Hence no new families of Cayley manifolds arise.

We note that the manifolds in Theorems 2.1 and 2.2 are not in their simplest forms. By a change of variables, we can simplify the equations for these manifolds, resulting in the following equivalent families,

$$M_\varepsilon = \left\{ x\varepsilon + x\mathbf{e} \;\middle|\; x \in \mathbb{H} \right\} \tag{41}$$

equivalent to the first family of manifolds, and

$$M_c = \left\{ s\frac{x}{|x|} + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, |x|^3 s - s^3|x| = c \right\} \tag{42}$$

equivalent to the second family of manifolds. Similar simplifications are possible throughout this paper.

3. Cayley Manifolds Symmetric Under the Action $(q,1,1)$

3.1. **Overview.** In this section we will discuss Cayley Manifolds that are symmetric under the action $(q,1,1)$.

$$(a + b\mathbf{e}) \mapsto a\overline{q} + (b\overline{q})\mathbf{e} \tag{43}$$

It turns out that the family of Cayley manifolds resulting from this action is a more general family which includes the family of Cayley manifolds just obtained in Section 2.

3.2. The $(q,1,1)$-Cayley Manifolds.

Theorem 3.1. *The following family of manifolds are graphs from $\mathbb{H}$ to $\mathbb{H}$ that are symmetric under the action (43).*

$$M_{\varepsilon,c} = \left\{ \varepsilon\frac{x}{|x|}s + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, |x|^3 s - s^3|x||\varepsilon|^2 = c \right\} \tag{44}$$

for $\varepsilon \in \mathbb{H}$ constant, and $c \in \mathbb{R}$ constant. Further, these manifolds are Cayley manifolds.

Notice that this family of manifolds contains the family of manifolds in Theorem 2.2. More specifically $M_c = M_{1,c}$ where M_c is defined in Theorem 2.2. Also note that this family of manifolds was obtained using a different key symmetry group, hence it is stated as a separate family.

3.3. **The Proof of Theorem 3.1.**

Proof. Before proving the theorem, we will first introduce a supporting lemma.

Lemma 3.2. *If M is the graph of a function $f : \mathbb{H} \to \mathbb{H}$, then M is symmetric under the action (43), if and only if*

$$f(x) = f(|x|)\frac{x}{|x|} \qquad \forall x \in \mathbb{H} \tag{45}$$

Proof. Suppose M is symmetric under this action. Then $f(x)\bar{q} + (x\bar{q})\mathbf{e}$ must belong to M. Thus we have

$$f(x\bar{q}) = f(x)\bar{q} \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1 \tag{46}$$

Setting x to $|x|$ and q to $\frac{\bar{x}}{|x|}$ proves the first half of the lemma.

Now suppose that $f(x) = f(|x|)\frac{x}{|x|}$ for all $x \in \mathbb{H}$. We wish to show that f is symmetric under the action (43), or equivalently that $f(x\bar{q}) = f(x)\bar{q}$ for all $x \in \mathbb{H}$ and all $q \in Sp_1$. Thus we have

$$f(x\bar{q}) = f(|x\bar{q}|)\frac{x\bar{q}}{|x\bar{q}|} = f(|x|)\frac{x}{|x|}\bar{q} = f(x)\bar{q} \tag{47}$$

and the the lemma is proven. □

By Theorem 1.8, for f symmetric under the action (43), $Df(r\bar{q}) - \sigma f(r\bar{q}) = 0$ and $\delta f(r\bar{q}) = 0$ if and only if $Df(r) - \sigma f(r) = 0$ and $\delta f(r) = 0$, and it is enough to consider just the cases where $x = |x| = x_1 = r$.

Again, we let $f(|x|) = \varepsilon\varphi(r)$, $\varphi : \mathbb{R}^+ \to \mathbb{R}$, and $\varepsilon \in \mathbb{H}$, so as to simplify the nonlinear portion of the PDE. It is then straight forward to compute the partial derivatives of f.

$$\frac{\partial f}{\partial x_1} = \varepsilon\varphi'(r) \qquad \frac{\partial f}{\partial x_2} = \varepsilon\hat{\boldsymbol{\imath}}\frac{\varphi(r)}{r} \tag{48}$$

$$\frac{\partial f}{\partial x_3} = \varepsilon\hat{\boldsymbol{\jmath}}\frac{\varphi(r)}{r} \qquad \frac{\partial f}{\partial x_4} = \varepsilon\hat{\boldsymbol{k}}\frac{\varphi(r)}{r} \tag{49}$$

From this it is again straight forward to compute Df, σf and δf and from that we have $Df - \sigma f = 0$ if and only if

$$\varepsilon\left(\left(1 - 3\frac{\varphi^2(r)}{r^2}\right)\varphi'(r) + \left(3\frac{\varphi(r)}{r} - |\varepsilon|^2\frac{\varphi^3(r)}{r^3}\right)\right) = 0 \tag{50}$$

Further,

$$\delta f = \mathrm{Im}\left[3\varphi'\frac{\varphi}{r} + 3\frac{\varphi^2}{r^2}\right] = 0 \tag{51}$$

So $\delta f = 0$ is always satisfied. This results in a similar solution to the second case of the detailed example in Section 2. Note that ε can now be any quaternion, not only real. Hence by an analogous proof, see Section 2, we derive the manifolds of Theorem 3.1, and the Theorem is proven. □

4. Cayley Manifolds Symmetric Under the Action $(q, 1, q)$

4.1. **Overview.** In this section we will discuss Cayley manifolds that are symmetric under the action $(q, 1, q)$.

$$(a + b\mathbf{e}) \mapsto qa\overline{q} + (b\overline{q})\mathbf{e} \tag{52}$$

We will prove that they must belong to a specific family of manifolds, and conclude with some analysis of this family of manifolds. We will find that one of the families of Cayley manifolds discovered is just Harvey and Lawson's classic example of a family of coassociative manifolds as described in their classic paper [24]. This is important because here this family of manifolds was obtained via the Cayley differential equations. This provides verification and an example of Remark 1.5 (1), i.e. that a manifold that lies in $\mathrm{Im}\,\mathbb{O}$ is Cayley if and only if it is a coassociative manifold.

4.2. **The $(q, 1, q)$-Cayley Manifolds.**

Theorem 4.1. *The Cayley Manifolds that are graphs from $\mathbb{H}$ to $\mathbb{H}$ that are symmetric under the action (52) are of the following two families*

$$M_{\varepsilon,c} = \left\{\overline{x}\varepsilon x\frac{s}{|x|^2} + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, s(4|\varepsilon|^2 s^2 - 5|x|^2)^2 = c\right\} \tag{53}$$

for $\varepsilon \in \mathrm{Im}\,\mathbb{H}$ constant; and

$$M_c = \left\{c + x\mathbf{e} \;\middle|\; x \in \mathbb{H},\right\} \tag{54}$$

for $c \in \mathbb{R}$ constant.

4.3. **The Proof of Theorem 4.1.**

PROOF. Before proving the theorem, we will first introduce a supporting lemma.

Lemma 4.2. *If M is the graph of a function $f : \mathbb{H} \to \mathbb{H}$, then M is symmetric under the action (52), if and only if*

$$f(x) = \frac{\overline{x}}{|x|} f(|x|) \frac{x}{|x|} \qquad \forall x \in \mathbb{H} \tag{55}$$

PROOF. Suppose M is symmetric under this action. Then $qf(x)\overline{q} + (x\overline{q})\mathbf{e}$ must belong to M. Thus we have

$$f(x\overline{q}) = qf(x)\overline{q} \qquad \forall x \in \mathbb{H},\ \forall q \in Sp_1 \tag{56}$$

Setting x to $|x|$ and q to $\frac{\overline{x}}{|x|}$ proves the first half of the lemma.

Now suppose that $f(x) = \frac{\overline{x}}{|x|} f(|x|) \frac{x}{|x|}$ as above. We wish to show that f is symmetric under the action (52), or equivalently that $f(x\overline{q}) = qf(x)\overline{q}$ for all $x \in \mathbb{H}$ and all $q \in Sp_1$. Thus we have

$$f(x\bar{q}) = \frac{\overline{x\overline{q}}}{|x\bar{q}|} f(|x\bar{q}|) \frac{x\bar{q}}{|x\bar{q}|} = q\frac{\bar{x}}{|x|} f(|x|) \frac{x}{|x|}\bar{q} = qf(x)\bar{q} \tag{57}$$

and thus the lemma is proven. □

By Theorem 1.8, for f symmetric under the action (52), it is enough to consider just the cases where $x = |x| = x_1 = r$. Letting $f(|x|) = \varepsilon\varphi(r)$, $\varphi : \mathbb{R}^+ \to \mathbb{R}$, to simplify the nonlinear portion of the PDE, it is straight forward to compute the partial derivatives of f.

$$\frac{\partial f}{\partial x_1} = \varepsilon\varphi'(r) \qquad \frac{\partial f}{\partial x_2} = -\hat{\boldsymbol{\imath}}\varepsilon\frac{\varphi(r)}{r} + \varepsilon\hat{\boldsymbol{\imath}}\frac{\varphi(r)}{r} \tag{58}$$

$$\frac{\partial f}{\partial x_3} = -\hat{\boldsymbol{\jmath}}\varepsilon\frac{\varphi(r)}{r} + \varepsilon\hat{\boldsymbol{\jmath}}\frac{\varphi(r)}{r} \qquad \frac{\partial f}{\partial x_4} = -\hat{\boldsymbol{k}}\varepsilon\frac{\varphi(r)}{r} + \varepsilon\hat{\boldsymbol{k}}\frac{\varphi(r)}{r} \tag{59}$$

From this it is again straight forward to compute Df, σf, and δf and from that we have $Df - \sigma f = 0$ and $\delta f = 0$ if and only if

$$\varepsilon\left(\varphi'(r) + 4\frac{\varphi(r)}{r} - 4|\varepsilon|^2\frac{\varphi^2(r)}{r^2}\varphi'(r) + 4\varepsilon_1^2\frac{\varphi^2(r)}{r^2}\varphi'(r)\right) - 4\varepsilon_1\frac{\varphi(r)}{r} = 0 \tag{60}$$

$$\varphi^{'}(r)\frac{2\varphi(r)}{r}2(\mathrm{Re}\,\varepsilon)(\mathrm{Im}\,\varepsilon) = 0 \tag{61}$$

Examining this last equation (the $\delta f = 0$ equation) yields four cases: $\varepsilon \in \mathrm{Im}\,\mathbb{H}$, $\varepsilon \in \mathrm{Re}\,\mathbb{H}$, $\varphi^{'} = 0$, or $\varphi = 0$.

If $\varepsilon \in \operatorname{Im}\mathbb{H}$, then $\varepsilon_1 = 0$ and equation (60) simplifies to

$$\varphi'(r) + 4\frac{\varphi(r)}{r} - 4|\varepsilon|^2\frac{\varphi^2(r)}{r^2}\varphi'(r) = 0 \tag{62}$$

This equation has only implicit solutions of the form

$$\varphi(r)(4|\varepsilon|^2\varphi^2(r) - 5r^2)^2 = c \tag{63}$$

for $c \in \mathbb{R}$ constant. This is of course the classic example pioneered in Harvey and Lawson's paper [24], and recovers the first family of $(q, 1, q)$-Cayley Manifolds in Theorem 4.1.

However, it is important to note that in Harvey and Lawson's paper, this family of manifolds was derived in $\mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$, using the coassociative equations. Requiring this family of manifolds to live $\operatorname{Im}\mathbb{H}$ automatically places the resriction $\varepsilon \in \operatorname{Im}\mathbb{H}$. While it's clear that each of these coassociative manifolds is a Cayley manifold (see Remark 1.5 (1)), it was not clear how to generalize this family of coassociative manifolds to a family of Cayley manifolds in $\mathbb{R}^8 \cong \mathbb{O}$. Specifically, it was not clear what sort of restrictions were needed on ε, if any. From the above derivation we see that indeed the restriction $\varepsilon \in \operatorname{Im}\mathbb{H}$ is required for this family of Cayley manifolds, even though these manifolds live in $\mathbb{R}^8 \cong \mathbb{O}$.

If $\varepsilon = |\varepsilon| = k \in \mathbb{R}$, then equation (60) simplifies to

$$\varepsilon\varphi'(r) = 0 \qquad \Leftrightarrow \qquad \varepsilon\varphi(r) = c \tag{64}$$

for a constant $c \in \mathbb{R}$. Thus the second family of $(q, 1, q)$-Cayley Manifolds from Theorem 4.1 is recovered.

If $\varphi^{'}(r) = 0$, then then equation (60) simplifies and we get,

$$4\frac{\varphi(r)}{r} - 4\varepsilon_1\frac{\varphi(r)}{r} = 0 \tag{65}$$

Thus in this case $Df - \sigma f = 0$ if and only if $\varepsilon \in \operatorname{Re}\mathbb{H}$, i.e. this case is identical to the previous.

Finally, it is immediate that the trivial solution satisfies both $\delta f = 0$ and $Df - \sigma f = 0$.

□

5. Cayley Manifolds Symmetric Under the Actions $(1, 1, q)$, $(1, q, 1)$, and $(q, q, 1)$

5.1. **Overview.** In this section we will discuss Cayley Manifolds that are symmetric under three different actions. All three will result in trivial or linear Cayley manifolds and are thus discussed together. The first action, $(1,1,q) \cong Sp_1$,

$$(a+b\mathbf{e}) \mapsto qa + b\mathbf{e} \tag{66}$$

will yield only the trivial manifold, a graph of the function $f:\mathbb{H}\to\mathbb{H}$, $f(x)=0, \quad \forall x\in\mathbb{H}$. The second action $(1,q,1)$

$$(a+b\mathbf{e}) \mapsto a + (qb)\mathbf{e} \tag{67}$$

will prove only slightly more productive, resulting in only linear graphs. The third action $(q,q,1)$

$$(a+b\mathbf{e}) \mapsto a\bar{q} + (qb\bar{q})\mathbf{e} \tag{68}$$

will again result in the trivial zero-manifold.

In all of these cases the equation $\delta f = 0$ will immediately be satisfied by the resulting graphs and hence this equation will not be considered explicitly below.

5.2. **The $(1,1,q)$-Cayley Manifolds.** As mentioned, in this section we will prove that the only Cayley graph that is symmetric under the action (66) is the trivial graph.

Theorem 5.1. *The only Cayley Manifold that is a graph of a function $f:\mathbb{H}\to\mathbb{H}$, that is also symmetric under the action (66) is the trivial manifold*

$$M = \left\{ 0 + x\mathbf{e} \;\middle|\; x\in\mathbb{H} \right\} \tag{69}$$

Proof. We are looking for Manifolds that are graphs over the Quaternions, $M = \{f(x)+x\mathbf{e} | x\in\mathbb{H}\}$, $f:\mathbb{H}\to\mathbb{H}$. Since M is to be symmetric under the action (66), we have that $qf(x)+x\mathbf{e}$ must also belong to M. Thus

$$f(x) = qf(x) \qquad \forall x\in\mathbb{H} \quad \forall q\in Sp_1\subset\mathbb{H} \tag{70}$$

The only function that satisfies this is the trivial function, $f=0$. Hence Theorem (5.1) is proved.

□

5.3. **The $(1,q,1)$-Cayley Manifolds.**

Theorem 5.2. *The only Cayley Manifolds that are graphs from $\mathbb{H}$ to $\mathbb{H}$ that are symmetric under the action (67) are graphs of constant quaternion functions.*

$$f(x) = \beta \qquad \forall x\in\mathbb{H} \tag{71}$$

β constant in $\mathbb{H}$.

5.4. **The Proof of Theorem 5.2.**

Proof. Before proving the theorem, we will first introduce a supporting lemma.

Lemma 5.3. *If M is the graph of a function $f : \mathbb{H} \to \mathbb{H}$, then M is symmetric under the action (67), if and only if*

$$f(x) = f(|x|) \qquad \forall x \in \mathbb{H} \tag{72}$$

Proof. Suppose M is symmetric under the action, $f(x) + (qx)\mathbf{e}$ must belong to M. Thus we have

$$f(x) = f(qx) \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1 \tag{73}$$

Setting x to $|x|$ and q to $\frac{x}{|x|}$ proves the first half of the lemma.

Now suppose that $f(x) = f(|x|)$ for all $x \in \mathbb{H}$. We wish to show that M is symmetric under that action (67), or equivalently that $f(qx) = f(x)$ for all $x \in \mathbb{H}$ and all $q \in Sp_1$. Thus we have

$$f(qx) = f(|qx|) = f(|x|) = f(x) \tag{74}$$

and the lemma is proven. □

In other words the value of f depends only on the magnitude of x. Thus, it is enough to consider just the cases where $x = |x| = x_1 = r$. This is also assured by Theorem 1.8. Letting $f(|x|) = \varphi(r)$, $\varphi : \mathbb{R} \to \mathbb{H}$, it is straight forward to compute the partial derivatives of f.

$$\frac{\partial f}{\partial x_1} = \frac{d\varphi}{dr} \qquad \frac{\partial f}{\partial x_2} = 0 \qquad \frac{\partial f}{\partial x_3} = 0 \qquad \frac{\partial f}{\partial x_4} = 0 \tag{75}$$

From this we have that $Df = \frac{d\varphi}{dr}$ and $\sigma f = 0$. Thus $Df - \sigma f = 0$ if and only if

$$\frac{d\varphi}{dr} = 0 \tag{76}$$

Thus the only solutions are constant solutions, and the theorem is proven.

□

5.5. **The $(q, q, 1)$-Cayley Manifolds.** In this section we will prove that the only Cayley graph that is symmetric under the action (68) is the trivial zero graph.

Theorem 5.4. *The only Cayley Manifold that is a graph of a function $f : \mathbb{H} \to \mathbb{H}$, that is also symmetric under the action (68) is the trivial manifold*

$$M = \left\{ 0 + x\mathbf{e} \;\middle|\; x \in \mathbb{H} \right\} \tag{77}$$

Proof. Consider a Quaternion function $f : \mathbb{H} \to \mathbb{H}$ symmetric under the action (68). Applying the action to $f(x)+x\mathbf{e}$, we immediately have that $f(qx\bar{q}) = f(x)\bar{q}$ must hold for all $x \in \mathbb{H}$ and all $q \in Sp_1$. However, letting $q = -1$ yields $f(x) = -f(x)$, which is only true if f is the constant zero function. □

6. Cayley Manifolds Symmetric Under the Action (q, q, q)

6.1. **Overview.** In this section we will discuss Cayley Manifolds that are symmetric under the action (q, q, q).

$$(a + b\mathbf{e}) \mapsto qa\bar{q} + (qb\bar{q})\mathbf{e} \tag{78}$$

It turns out that it is more difficult to obtain the symmetry restrictions for this action. However we were able to simplify the required partial differential equations for these manifolds and solve them for a variety of cases. In particular, this resulted in a novel family of manifolds. In this section we will derive these simplifications of the partial differential equations and present the corresponding Cayley manifolds.

6.2. **The (q, q, q)-Cayley Manifolds.**

Theorem 6.1. *The following family of Cayley Manifolds are graphs from $\mathbb{H}$ to $\mathbb{H}$ that are symmetric under the action (78)*

$$M_{k,c} = \left\{ \frac{-4(k-k^3)}{1-3k^2}(\mathrm{Re}\,x) + c + kx + x\mathbf{e} \;\Big|\; \forall x \in \mathbb{H} \right\} \tag{79}$$

for arbitrary constants $k, c \in \mathbb{R}$ such that $k^2 \neq 1/3$.

Theorem 6.2. *The following family of Cayley Manifolds are graphs from $\mathbb{H}$ to $\mathbb{H}$ that are symmetric under the action (78)*

(80)

$$M_{c,k,\rho} = \left\{ x\varphi(x) + \rho\varphi(x) + k + x\mathbf{e} \;\Big|\; \forall x \in \mathbb{H}, \varphi(x) - \varphi^3(x) = \frac{c}{(|\mathrm{Im}\,x|^2 + (s + \mathrm{Re}\,x)^2)^2} \right\}$$

for $c, k, \rho \in \mathbb{R}$ constants, and $\varphi : \mathbb{H} \to \mathbb{R}$.

6.3. **The Proof of Theorems 6.1 and 6.2.**

Proof. Before proving these theorems, we will first introduce a supporting lemma and some supporting theorems.

Lemma 6.3. *Let $C_q : \mathbb{H} \to \mathbb{H}$, $C_q(x) = qx\bar{q}$, be an action of Sp_1 on $\mathbb{H}$. Then,*

(1) *C_q leaves the subspaces* $\mathrm{Re}\,\mathbb{H}$ *and* $\mathrm{Im}\,\mathbb{H}$ *invariant,*
(2) *C_q preserves* $\mathrm{Re}\,x$ *and* $|\mathrm{Im}\,x|$ *for each $x \in \mathbb{H}$, and*

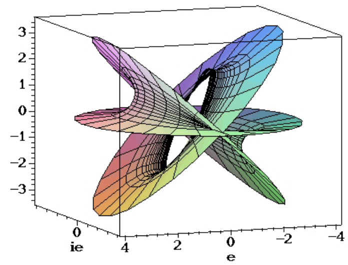

FIGURE 4. A slice of the Cayley manifold of Theorem 6.2 in $\{1, ie, e\}$ space

(3) *for non-real q the fixed point set of C_q is the subspace Span$\{1, q\}$ of $\mathbb{H}$.*

PROOF. Since both left and right multiplication by fixed non-zero unit quaternions correspond to elements of SO_4, we have that $C_q \in SO_4$. Clearly C_q fixes the real portion of $\mathbb{H}$. Thus C_q, leaves the orthogonal compliment of $\mathrm{Re}\,\mathbb{H}$, i.e. $\mathrm{Im}\,\mathbb{H}$, invariant. In fact, it is well known that $C_q|_{\mathrm{Im}\,\mathbb{H}} \in SO(3)$. Thus C_q preserves both $\mathrm{Re}\,x$ and $|\mathrm{Im}\,x|$ for all $x \in \mathbb{H}$.

Now suppose that q is non-real. Let $a + bq \in \mathrm{Span}\{1, q\}$, where a and b are real, then

$$C_q(a + bq) = q(a + bq)\overline{q} = a + bq. \tag{81}$$

So that $\mathrm{Span}\{1, q\} \subset Fix(q)$, where $Fix(q)$ denotes the fixed point set of C_q. Now Suppose that x is fixed by C_q. Thus we have

$$C_q(x) = qx\overline{q} = x \;\Leftrightarrow\; qx = xq \tag{82}$$

Now, the since q is non-real, the space generated by 1 and q is a complex subspace of $\mathbb{H}$. Hence, x commutes with q if and only if x is an element of this subspace, i.e. $x \in \mathrm{Span}\{1, q\}$. Thus we have $Fix(q) \subset \mathrm{Span}\{1, q\}$, proving the lemma.

□

For a more detailed discussion of the Geometry of Cayley multiplication please consult [32] or the appendix of [24].

Theorem 6.4. *If M is the graph of a function $f : \mathbb{H} \to \mathbb{H}$, then M is symmetric under the action (78), if and only if*

$$f(x) = x\varphi(\mathrm{Re}\,x, |\mathrm{Im}\,x|) + \nu(\mathrm{Re}\,x, |\mathrm{Im}\,x|) \qquad \forall x \in \mathbb{H} \tag{83}$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}$, and $\nu : \mathbb{R}^2 \to \mathbb{R}$.

PROOF. If M is symmetric under this action, then $qf(x)\overline{q} + (qx\overline{q})\mathbf{e}$ must belong to M. Thus we have

$$f(qx\overline{q}) = qf(x)\overline{q} \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1 \tag{84}$$

We have two cases. First suppose x is real. Then (84) becomes

$$f(x) = qf(x)\overline{q} \tag{85}$$

for all $q \in Sp_1$. Thus $f(x)$ must be real and hence in $\mathrm{Span}\{1, \frac{x}{|x|}\}$.

Now suppose x is non-real. Setting q to $\frac{x}{|x|}$ yields

$$f(x) = \frac{x}{|x|} f(x) \frac{\overline{x}}{|x|} = C_{\frac{x}{|x|}}(f(x)) \tag{86}$$

for all $x \in \mathbb{H}$. Thus $f(x)$ is fixed by $C_{\frac{x}{|x|}}$, and by Lemma 6.3 (3) we have that $f(x) \in \mathrm{Span}\{1, \frac{x}{|x|}\}$. Thus regardless of x we have that $f(x) \in \mathrm{Span}\{1, \frac{x}{|x|}\}$.

Thus we can rewrite f as follows. Let

$$f(x) = x\varphi(x) + \nu(x) \qquad \forall x \in \mathbb{H} \tag{87}$$

where $\varphi : \mathbb{H} \to \mathbb{R}$ and $\nu : \mathbb{H} \to \mathbb{R}$.

Returning now to equation (84), we have

$$f(qx\overline{q}) = qx\overline{q}\varphi(qx\overline{q}) + \nu(qx\overline{q}) \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1 \tag{88}$$

$$qf(x)\overline{q} = q(x\varphi(x) + \nu(x))\overline{q} = qx\overline{q}\varphi(x) + \nu(x) \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1 \tag{89}$$

Since we insist on equation (84) holding, these two equations must be equal. Therefore we have

$$qx\overline{q}(\varphi(qx\overline{q}) - \varphi(x)) + (\nu(qx\overline{q}) - \nu(x)) = 0 \tag{90}$$

Notice that if $x \in \mathrm{Re}\,\mathbb{H}$ then equation (90) is satisfied regardless of φ and ν. Thus we can split $x = \mathrm{Re}\,x + \mathrm{Im}\,x$, and since C_q sends $\mathrm{Im}\,\mathbb{H}$ to $\mathrm{Im}\,\mathbb{H}$ by Lemma 6.3 (1), we can separate the real and imaginary parts of (90)

$$q(\mathrm{Im}\,x)\overline{q}(\varphi(qx\overline{q}) - \varphi(x)) = 0 \tag{91}$$

$$q(\mathrm{Re}\,x)\overline{q}(\varphi(qx\overline{q}) - \varphi(x)) + (\nu(qx\overline{q}) - \nu(x)) = 0 \tag{92}$$

From the first of these equations and using the fact that $\mathbb{H}$ is a division ring, we get one of two possibilities. Either $\mathrm{Im}\,x = 0$ or $\varphi(qx\overline{q}) - \varphi(x) = 0$. The first case was commented on above, and in either case we get

$$\varphi \circ C_q(x) = \varphi(qx\overline{q}) = \varphi(x) \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1$$

Plugging this into the real part of equation (90) yields

$$\nu \circ C_q(x) = \nu(qx\overline{q}) = \nu(x) \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1 \tag{93}$$

By Lemma 6.3 (2), we know that $\mathrm{Re}\,x$ and $|\mathrm{Im}\,x|$ are fixed by C_q, however by choosing q appropriately, we can always rotate that imaginary portion of x onto the $\hat{\imath}$ axis, i.e. $x \mapsto \mathrm{Re}\,x + \hat{\imath}|\mathrm{Im}\,x|$. Thus, φ and ν only depend on $\mathrm{Re}\,x$ and $|\mathrm{Im}\,x|$, and may be replaced with functions $\varphi, \nu : \mathbb{R}^2 \to \mathbb{R}$ and equation (83) is recovered.

Now consider the graph of a function f, such that it obeys equation (83). We wish to show that such a graph is symmetric under action (78). Applying the action and Lemma 6.3 gives us

$$f(qx\overline{q}) = qx\overline{q}\varphi(\mathrm{Re}\,(qx\overline{q}), |\mathrm{Im}\,(qx\overline{q})|) + \nu(\mathrm{Re}\,(qx\overline{q}), |\mathrm{Im}\,(qx\overline{q})|)$$

$$= qx\overline{q}\varphi(\mathrm{Re}\,x, |\mathrm{Im}\,x|) + q\overline{q}\nu(\mathrm{Re}\,x, |\mathrm{Im}\,x|) = qf(x)\overline{q}, \qquad \forall x \in \mathbb{H}, \forall q \in Sp_1 \tag{94}$$

Thus this graph must be symmetric under the action, and Theorem 6.4 is proven. □

Theorem 6.5. *Let M be the graph of a function, $f : \mathbb{H} \to \mathbb{H}$, symmetric under the action (78), so that f obeys Theorem 6.4, i.e. $f(x) = x\varphi(\mathrm{Re}\,x, |\mathrm{Im}\,x|) + \nu(\mathrm{Re}\,x, |\mathrm{Im}\,x|)$ for $x \in \mathbb{H}$. Then M is Cayley if and only if*

$$\left(1 - \varphi^2(r,s)\right)\left(s\frac{\partial\varphi}{\partial r}(r,s) - r\frac{\partial\varphi}{\partial s}(r,s) - \frac{\partial\nu}{\partial s}(r,s)\right) = 0 \tag{95}$$

$$\left(r\frac{\partial\varphi}{\partial s}(r,s) + \frac{\partial\nu}{\partial s}(r,s) - s\frac{\partial\varphi}{\partial r}(r,s)\right)\varphi(r,s) = 0 \tag{96}$$

and

$$\begin{aligned} &4\varphi(r,s) - 4\varphi^3(r,s) + \frac{\partial\nu}{\partial r}(r,s) + r\frac{\partial\varphi}{\partial r}(r,s) + s\frac{\partial\varphi}{\partial s}(r,s) \\ &-3\varphi^2(r,s)\frac{\partial\nu}{\partial r}(r,s) - 3r\varphi^2(r,s)\frac{\partial\varphi}{\partial r}(r,s) - 3s\varphi^2(r,s)\frac{\partial\varphi}{\partial s}(r,s) \\ &+2s\varphi(r,s)\frac{\partial\nu}{\partial r}(r,s)\frac{\partial\varphi}{\partial s}(r,s) - 2s\varphi(r,s)\frac{\partial\varphi}{\partial r}(r,s)\frac{\partial\nu}{\partial s}(r,s) = 0 \end{aligned} \tag{97}$$

for $r, s \in \mathbb{R}^+, r = \mathrm{Re}\,x, s = |\mathrm{Im}\,x|$.

Proof. Given $w = \mathrm{Re}\,w + \mathrm{Im}\,w \in \mathbb{H}$, pick $q \in Sp_1$ such that $q\hat{\imath}\overline{q} = \frac{\mathrm{Im}\,w}{|\mathrm{Im}\,w|}$. Let $x = x_1 + x_2\hat{\imath} = \mathrm{Re}\,w + |\mathrm{Im}\,w|\hat{\imath}$, then $qx\overline{q} = q(\mathrm{Re}\,w)\overline{q} + q(|\mathrm{Im}\,w|\hat{\imath})\overline{q} = \mathrm{Re}\,w + \mathrm{Im}\,w = w$. By Theorem 1.8 we have $Df(qx\overline{q}) - \sigma f(qx\overline{q}) = 0$ and $\delta f(qx\overline{q}) = 0$ if and only if $Df(x) - \sigma f(x) = 0$ and $\delta f(x) = 0$, and thus it is enough to consider just the cases where $x = x_1 + \hat{\imath}x_2$.

We wish to compute $Df - \sigma f$ and δf. Computing directly from the definitions we have

$$Df = 4\varphi + \frac{\partial \nu}{\partial x_1} + x_1 \frac{\partial \varphi}{\partial x_1} + x_2 \frac{\partial \varphi}{\partial x_2} + \hat{\boldsymbol{\imath}} \left(x_2 \frac{\partial \varphi}{\partial x_1} - x_1 \frac{\partial \varphi}{\partial x_2} - \frac{\partial \nu}{\partial x_2} \right), \tag{98}$$

$$\sigma f = 4\varphi^3 + 3x_2\varphi^2 \frac{\partial \varphi}{\partial x_2} + 3x_1\varphi^2 \frac{\partial \varphi}{\partial x_1} + 3\varphi^2 \frac{\partial \nu}{\partial x_1}$$

$$+2x_2\varphi \frac{\partial \nu}{\partial x_1} \frac{\partial \varphi}{\partial x_2} - 2x_2\varphi \frac{\partial \varphi}{\partial x_1} \frac{\partial \nu}{\partial x_2} + \hat{\boldsymbol{\imath}} \left(x_2\varphi^2 \frac{\partial \varphi}{\partial x_1} - x_1\varphi^2 \frac{\partial \varphi}{\partial x_2} - \varphi^2 \frac{\partial \nu}{\partial x_2} \right), \tag{99}$$

and

$$\delta f = \left(x_1 \frac{\partial \varphi}{\partial x_2} + \frac{\partial \nu}{\partial x_2} - x_2 \frac{\partial \varphi}{\partial x_1} \right) \varphi \tag{100}$$

Combining these results in the three differential equations in Theorem 6.5, and hence proves the theorem.

□

Consider the case where $\varphi(x) = \pm 1$. Clearly equation (95) is then satisfied. Further, $\frac{\partial \varphi}{\partial x_1} = \frac{\partial \varphi}{\partial x_2} = 0$. Applying this to equation (97) yields

$$4\varphi(1 - \varphi^2) + \frac{\partial \nu}{\partial x_1}(1 - 3\varphi^2) = 0 \tag{101}$$

which reduces to

$$\frac{\partial \nu}{\partial x_1} = 0 \tag{102}$$

Applying (96) yields

$$\frac{\partial \nu}{\partial x_2} = 0 \tag{103}$$

Thus if ν is any constant function the cayley equations are satisfied. Notice however that this is simply a subcase of Theorem 6.1.

Similarly, if $\varphi = k$ is a constant, then (96) and (95) are satisfied if and only if

$$\frac{\partial \nu}{\partial x_2} = 0. \tag{104}$$

This again reduces (97) to,

$$4(k - k^3) + (1 - 3k^2) \frac{\partial \nu}{\partial x_1} = 0 \tag{105}$$

This seemingly yields two possibilities. First, if $k^2 \neq 1/3$, then we can solve the the last equation, yielding,

$$\nu(x_1, x_2) = \frac{-4k(1-k^2)}{1-3k^2} x_1 + c \tag{106}$$

for arbitrary constant $c \in \mathbb{R}$. Thus we recover Theorem 6.1. On the other hand, if we consider the case $k^2 = 1/3$, then we ge the following unsatisfiable condition,

$$4k(1-k^2) = 0 \tag{107}$$

Thus a constant φ only yields the manifolds of Theorem 6.1.

Naturally, we can consider the case where ν is a constant. We have that $\frac{\partial \nu}{\partial x_1} = \frac{\partial \nu}{\partial x_2} = 0$. Applying this to equation (96) if $\varphi \neq 0$ or to (95) if $\varphi \neq \pm 1$ yields

$$x_2 \frac{\partial \varphi}{\partial x_1} = x_1 \frac{\partial \varphi}{\partial x_2} \tag{108}$$

Applying this to equation (97) results in

$$4x_2(\varphi - \varphi^3) + (1-3\varphi^2)(x_1^2 + x_2^2)\frac{\partial \varphi}{\partial x_2} = 0 \tag{109}$$

and

$$4x_1(\varphi - \varphi^3) + (1-3\varphi^2)(x_1^2 + x_2^2)\frac{\partial \varphi}{\partial x_1} = 0 \tag{110}$$

By a simple u substitution, $u = \varphi - \varphi^3$, we can solve these equations and find that

$$\varphi(x) - \varphi^3(x) = \frac{c}{(x_1^2 + x_2^2)^2} = \frac{c}{|x|^4} \tag{111}$$

for $c \in \mathbb{R}$ constant.

However, it turns out that this is merely a subcase of a more general family of solutions. We begin by realizing that equation (97) can be rewritten as

$$4(\varphi - \varphi^3) + (1-3\varphi^2)(x_1 \frac{\partial \varphi}{\partial x_1} + x_2 \frac{\partial \varphi}{\partial x_2} + \frac{\partial \nu}{\partial x_1}) + 2x_2\varphi \frac{\partial(\nu, \varphi)}{\partial(x_1, x_2)} = 0 \tag{112}$$

Suppose that $\frac{\partial(\nu,\varphi)}{\partial(x_1,x_2)} = 0$. This is equivalent to

$$\begin{pmatrix} \frac{\partial \nu}{\partial x_1} \\ \frac{\partial \nu}{\partial x_2} \end{pmatrix} = \rho(x_1, x_2) \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_2} \end{pmatrix} \tag{113}$$

where $\rho : \mathbb{R}^2 \to \mathbb{R}$. Supposing that ρ is a constant function yields the following simultaneous equations:

$$x_2 \frac{\partial \varphi}{\partial x_1} - x_1 \frac{\partial \varphi}{\partial x_2} - \rho \frac{\partial \varphi}{\partial x_2} = 0 \tag{114}$$

$$x_1 \frac{\partial \varphi}{\partial x_1} + x_2 \frac{\partial \varphi}{\partial x_2} + \rho \frac{\partial \varphi}{\partial x_1} = -4 \frac{\varphi - \varphi^3}{1 - 3\varphi^2} \tag{115}$$

These equations can be simplified further by placing them in matrix form. From there, we can solve for φ implicitly, yielding the following solution:

$$\varphi - \varphi^3 = \frac{c}{(x_2^2 + (x_1 + \rho)^2)^2} \tag{116}$$

where c is an arbitrary real constant. Further, by our previous assumptions we get

$$\nu = \rho\varphi + k \tag{117}$$

where k is an another arbitrary real constant. Thus proving Theorem 6.2. □

7. Other subgroups of $\frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$

7.1. **Overview.** In this section we will discuss Cayley Manifolds that are symmetric under other subgroups of $\frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$. We will show that these can be reduced to considering the key subgroups as outlined in Section 1, Subsection 1.6. We will also discuss one last remaining family of subgroups that result in only the trivial zero manifold. Finally, we will conclude with a discussion of future work.

7.2. **Other subgroups of $\frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$.** Now we will consider 3-dimensional subgroups of $\frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2}$ and how they are related to the key subgroups already discussed.

First we will consider subgroups of the form $G_1(p) = \{(1, q, pq\bar{p}) \mid \forall q \in Sp_1\} \cong S^3$ where $p \in Sp_1$ is fixed. Suppose M is a graph of a function $f : \mathbb{H} \to \mathbb{H}$. Then M is symmetric under a particular group $G_1(p)$ if and only if

$$pq\bar{p}f(x) = f(qx) \tag{118}$$

for all $q \in Sp_1$ and all $x \in \mathbb{H}$. This in turn leads to a new condition.

Lemma 7.1. *If M is the graph of a function $f : \mathbb{H} \to \mathbb{H}$, then M is symmetric under $G_1(p)$ if and only if*

$$f(x) = p\frac{x}{|x|}\bar{p}f(|x|) \tag{119}$$

Proof. Suppose M is symmetric under the action so that $pq\bar{p}f(x) = f(qx)$. Letting q go to $\frac{x}{|x|}$ and x go to $|x|$ proves the first half of the this lemma. Now suppose that f is as in the lemma. Plugging in qx gives us

$$f(qx) = p\frac{qx}{|qx|}\bar{p}f(|qx|) = pq\bar{p}p\frac{x}{|x|}\bar{p}f(|x|) = pq\bar{p}f(x) \tag{120}$$

since $p\bar{p} = 1$. Thus the lemma is proven. □

Making the same assumptions about f, for the purpose of simplifying the PDE, we suppose that

$$f(x) = px\bar{p}\varepsilon\frac{\varphi(|x|)}{|x|} \tag{121}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}$. Now we introduce the following definition

$$g(x) = x(\bar{p}\varepsilon)\frac{\varphi(|x|)}{|x|} \tag{122}$$

and we have that

$$\frac{\partial f}{\partial x_n} = p\frac{\partial g}{\partial x_n} \tag{123}$$

for $n = 1, \ldots, 4$. Hence, we have $Df - \sigma f = 0$ and $\delta f = 0$ if and only if $Dg - \sigma g = 0$ and $\delta g = 0$ by Lemma 2.23 in [24]. Note that the condition on g is the same as that derived from the related key subgroup $(1, q, q)$ in Section 2. Thus, we can obtain conditions on φ and two new families of manifolds

Theorem 7.2. *Given the action $G_1(p)$, for any $\varepsilon \in \mathbb{H}$ fixed such that $\bar{p}\varepsilon \in \operatorname{Im}\mathbb{H}$, and for any $c \in \mathbb{R}$ fixed, the graph*

$$M_{\varepsilon,c} = \left\{ cpx\bar{p}\varepsilon + x\mathbf{e} \;\middle|\; x \in \mathbb{H} \right\} \tag{124}$$

is a Cayley manifold symmetric under the action $G_1(p)$.

and also

Theorem 7.3. *Given the action $G_1(p)$, for any $\varepsilon \in \mathbb{H}$ fixed such that $\bar{p}\varepsilon \in \mathbb{R}$, and for any $c \in \mathbb{R}$ fixed, the graph*

$$M_{\varepsilon,c} = \left\{ sp\frac{x}{|x|}\bar{p}\varepsilon + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, |x|^3 s - |\varepsilon|^2 s^3 |x| = c \right\} \tag{125}$$

is a Cayley manifold symmetric under $G_1(p)$.

The details of the proofs of these theorems are similar to the proofs of the detailed example in Section 2 and are thus omitted.

Similarly, we can consider subgroups of the form $G_2(p) = \{(q, 1, pq\bar{p}) \mid q \in Sp_1\}$. If M is the graph of a function, f, symmetric under $G_2(p)$, we again can obtain a condition on the function f:

$$f(x\bar{q}) = pq\bar{p}f(x)\bar{q} \tag{126}$$

which in turn leads to the following equivalent restriction on f

Lemma 7.4. *If M is a graph of a function $f : \mathbb{H} \to \mathbb{H}$, then M is symmetric under $G_2(p)$ if and only if*

$$f(x) = p\frac{\bar{x}}{|x|}\bar{p}f(|x|)\frac{x}{|x|} \tag{127}$$

for all $x \in \mathbb{H}$.

Proof. Suppose M is symmetric under $G_2(p)$. Letting x go to $|x|$ and q go to $\frac{\bar{x}}{|x|}$ proves the first half of the lemma. Now supposing that f is as in the lemma, plugging in $x\bar{q}$ yields

$$f(x\bar{q}) = p\frac{\overline{x\bar{q}}}{|x\bar{q}|}\bar{p}f(|x\bar{q}|)\frac{x\bar{q}}{|x\bar{q}|} = pq\bar{p}p\frac{\bar{x}}{|x|}\bar{p}f(|x|)\frac{x}{|x|}\bar{q} = pq\bar{p}f(x)\bar{q} \tag{128}$$

and the lemma is proven. ☐

Again making a similar assumption about the form of f to simplify the PDE, we have

$$f(x) = p\bar{x}\bar{p}\varepsilon x\frac{\varphi(|x|)}{|x|^2} \tag{129}$$

and using a similar technique as above we can again define a useful related function

$$g(x) = \bar{x}\bar{p}\varepsilon x\frac{\varphi(|x|)}{|x|^2} \tag{130}$$

and we again have that $Df - \sigma f = 0$ and $\delta f = 0$ if and only if $Dg - \sigma g = 0$ and $\delta g = 0$. Note, however, that the graph of g is symmetric under the key group $(q, 1, q)$, and thus we can get restrictions on $\varphi : \mathbb{R}^+ \to \mathbb{R}$ and we have two new families of manifolds:

Theorem 7.5. *The Cayley Manifolds that are graphs from $\mathbb{H}$ to $\mathbb{H}$ that are symmetric under the action $G_2(p)$ are of the following two families*

$$M_{\varepsilon,c} = \left\{p\bar{x}\bar{p}\varepsilon x\frac{s}{|x|^2} + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, s \in \mathbb{R}, s(4|\varepsilon|^2s^2 - 5|x|^2)^2 = c\right\} \tag{131}$$

for $\varepsilon \in \mathbb{H}$ *fixed such that* $\bar{p}\varepsilon \in \operatorname{Im}\mathbb{H}$ *constant; and*

$$M_c = \left\{ p\bar{x}\bar{p}\varepsilon x + x\mathbf{e} \;\middle|\; x \in \mathbb{H}, \right\} \tag{132}$$

for ε *fixed such that* $\bar{p}\varepsilon \in \mathbb{R}$.

Similarly, we can define the family of actions $G_3(p) = \{(q, pq\bar{p}, 1) \mid q \in Sp_1\}$. This action family is related to the key subgroup $(q, q, 1)$ discussed in Section 5. The proofs of the statements in this section are very similar to those just outlined above and so are omitted. Here it is easy to derive a condition on the graph of a function, f, that is equivalent to that graph being symmetric under $G_3(p)$. We have

$$f(x)\bar{q} = f(pq\bar{p}x\bar{q}) \tag{133}$$

for all $q \in Sp_1$. However, just as with $(q, q, 1)$ we can set q to -1 and get $-f(x) = f(x)$, which implies that f must be the trivial zero graph. Thus no new families of manifolds are obtained for this class.

7.3. **Additional Discrete Symmetry Subgroups.** Recall that the circle group, S^1, is contained in Sp_1 as a subgroup in a natural way. Hence each of the discrete cyclic groups $\mathbf{Z}_n$ is contained as a finite subgroup of Sp_1. Notice further that there yet other discrete groups contained in Sp_1. For example the quaternion group, $\mathbf{Q}_8 = \{\pm 1, \pm\hat{\boldsymbol{\imath}}, \pm\hat{\boldsymbol{\jmath}}, \pm\hat{\boldsymbol{k}}\}$, has a very natural inclusion. Since each of these groups is discrete we can incorporate them into our already discussed 3-dimensional subgroups without changing the dimension. Perhaps an example is most illuminating.

We can form the subgroup $\mathbf{Q}_8 \times Sp_1$ as follows,

$$\mathbf{Q}_8 \times Sp_1 \cong \left\{ (r, q, q) \in \frac{Sp_1 \times Sp_1 \times Sp_1}{\mathbf{Z}_2} \quad \middle| \quad q \in Sp_1, \quad r \in \mathbf{Q}_8 \right\} \tag{134}$$

Naturally we can ask what sort of Cayley manifolds result from this and similar 3-dimensional subgroups. First notice that this example contains the key subgroup $(1, q, q)$. Thus all the conditions derived in Section 2 for the group $(1, q, q)$ must still be satisfied. In addition, other discrete symmetry conditions will also be required. Continuing with this example the symmetry condition on a graph $f : \mathbb{H} \to \mathbb{H}$ is,

$$f(x) = \frac{x}{|x|} r f(|x|)\bar{r} \tag{135}$$

for all $x \in \mathbb{H}$ and all $r \in \mathbf{Q}_8$. Hence $f(|x|)$ must be real. This added restriction means that only the second family of manifolds (see Theorem 2.2) is symmetric under this new symmetry group.

In summary, the addition of a discrete group to one of the previously discussed 3-dimensional symmetry subgroups does not yield any new families of manifolds. Such an addition merely adds additional symmetry conditions which must be satisfied.

7.4. **The $S^1 \times S^1 \times S^1$-Actions.** Finally, there is a third class of three dimensional subgroups of the action $(e^{q_1\theta}, e^{q_2\phi}, e^{q_3\psi}) \in S^1 \times S^1 \times S^1 \subset K$

$$(a + b\mathbf{e}) \mapsto e^{q_3\psi} a e^{-q_1\theta} + (e^{q_2\phi} b e^{-q_1\theta})\mathbf{e} \tag{136}$$

Here $\theta, \phi, \psi \in [0, 2\pi)$ and $q_1, q_2, q_3 \in Sp_1 \cap \operatorname{Im}\mathbb{H}$ are fixed constants, and $e^{q\theta} = \cos\theta + q\sin\theta$.

Theorem 7.6. *The following Cayley Manifold is the only graph from $\mathbb{H}$ to $\mathbb{H}$ that is symmetric under the action (136)*

$$M = \left\{0 + x\mathbf{e} \;\middle|\; \forall x \in \mathbb{H}\right\}. \tag{137}$$

Proof. Consider a function f, whose graph is a Cayley manifold symmetric under the action (136). Applying this action we find that $f(e^{q_2\phi} x e^{-q_1\theta}) = e^{q_3\psi} f(x) e^{-q_1\theta}$ must hold for all $x \in \mathbb{H}$ and all $\theta, \phi, \psi \in [0, 2\pi)$. Setting $\phi = \theta = 0$ yields

$$f(x) = e^{q_3\psi} f(x), \qquad \forall x \in \mathbb{H}\ \forall \psi \in [0, 2\pi). \tag{138}$$

which is only satisfied when f is the constant zero function. □

7.5. **Future Work.** It is irresistible to ask the following questions:

(1) Are all the Cayley manifolds obtained in this paper complete?
(2) Is an obtained Cayley manifold a holomorphic surface, a special Lagrangian, $\mathbb{R}$ crossed with an associative manifold, or a coassociative manifold?
(3) Are the 3-dimensional projections of an obtained Cayley manifold associative manifolds?
(4) Are the 2-dimensional projections of an obtained Cayley manifold holomorphic curves?
(5) Is every 2-dimensional projection of a Cayley manifold a minimal surface?
(6) Which of these Cayley manifolds are periodic?

These questions are the subject of current investigation.

In the future, we expect to use some of these Cayley manifolds to construct Cayley cycles in Spin_7 and Calabi-Yau manifolds, especially those obtained by discontinuous action of a group.

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(Weiqing Gu) 1250 N. Dartmouth Ave., Claremont, CA 91711
E-mail address: `gu@math.hmc.edu`

(Christopher Pries) 340 E. FootHill Blvd., Claremont, CA 91711
E-mail address: `cpries@hmc.edu`