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Leonardo da Vinci in Raphael's School of Athens

Cover Page Footnote

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Leonardo da Vinci in Raphael's *School of Athens*

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Abstract

At the center of the *School of Athens*, Raphael painted Plato with a face similar to that of Leonardo da Vinci. In this article I argue that the likeness was intentional, and that Raphael's fresco contains a set of references to the book *De Divina Proportione*, to which Leonardo contributed with drawings of polyhedrons. *De Divina Proportione* was written by Leonardo's friend and teacher of mathematics, Luca Pacioli, and contains arguments for raising the status of the art of painting, similar to the arguments found in Leonardo's *Paragone*. Pacioli and Leonardo thought painting should be regarded as a liberal art, due to the painters' use of mathematical principles. In this article, I show how Plato with the face of Leonardo is part of a set of allusions to these arguments found in *The School of Athens*.

Keywords: Raphael, Leonardo da Vinci, Luca Pacioli, *The School of Athens*, Plato, *Timaeus*, quadrivium

It has often been commented that Raphael (1483-1520) in his fresco *The School of Athens* (1510-1511) (see Figure 1) painted Plato with a face similar to that of Leonardo da Vinci (1452-1519).¹ Although Leonardo and Raphael were probably in Florence at the same time in periods during the first decade of the sixteenth century, and we know that Raphael had a particular interest in Leonardo and his paintings, we do not know whether Raphael

¹ Scholars mentioning the likeness include Ross King [30], Daniel Arasse [3], Joseph Manca [35], Laurie Schneider Adams [1], A. Richard Turner [55] and Bette Talvacchia [52]. Arguments against identifying the face as that of Leonardo can be found in William E. Wallace: "Leonardo as Plato" [57]. My article can be regarded as one long argument in support of the idea that Raphael intentionally did paint Plato with the face of Leonardo.

Figure 1: Raphael, *The School of Athens* (1510-1511). Public domain image from Wikipedia, available at https://en.wikipedia.org/wiki/The_School_of_Athens, last accessed on July 3, 2021.

ever met Leonardo.² If Raphael didn't get a glimpse of Leonardo while they both stayed in the same city, his friend, the painter and architect Donato Bramante (1444-1514) could have shown him what Leonardo looked like.

² Vasari wrote about Raphael's admiration for Leonardo in his *Lives of the Artists*:

[...] when [Raphael] saw the works of Leonardo da Vinci, who had no equal in rendering expressions in the heads of men as well as women, and who in giving grace to his figures and their movements surpassed all other painters, Raphael was left wholly astonished and amazed; in short, since the style of Leonardo pleased him more than any other he had ever seen, he set himself to studying it, and gradually leaving behind Pietro's style albeit with great effort, he sought to imitate the style of Leonardo to the best of his knowledge and ability [56, page 331].

See also [60] and [12].

Bramante worked together with Leonardo at the court in Milan for almost twenty years, and was employed by Pope Julius II in Rome when Raphael painted his *School of Athens* in the Vatican Palace.³

As seen in Figure 2, the face of Plato is similar to portraits thought to be of Leonardo. The one in the middle is believed by many to be a self-portrait from around 1512, while the drawing to the right is a somewhat idealized portrait of Leonardo by his pupil Francesco Melzi from approximately 1515.

Figure 2: Left: Plato/Leonardo in Raphael's *The School of Athens*, detail from Figure 1. Middle: Leonardo Da Vinci's presumed self-portrait, circa 1512, public domain image from Wikipedia (https://commons.wikimedia.org/wiki/File:Leonardo_da_Vinci_-_presumed_self-portrait_-_WGA12798.jpg), last accessed on July 3, 2021. Right: Francesco Melzi, *Portrait of Leonardo*, 1515-1517, public domain image from Wikipedia (https://commons.wikimedia.org/wiki/File:Francesco_Melzi_-_Portrait_of_Leonardo.png), last accessed on July 3, 2021.

In the first part of this article I will argue that the face of Leonardo in Raphael's fresco belongs to a set of references found in *The School of Athens*, connecting the fresco with the mathematician Luca Pacioli's book *De Divina Proportione*. Luca Pacioli (1445-1517) was Leonardo's friend and teacher of mathematics, and Leonardo contributed to *De Divina Proportione* with drawings of geometric solids. The manuscript of *De Divina Proportione* was

³ According to Vasari, Bramante was a relative of Raphael, and the one who brought Raphael to Rome [56, page 312].

finished in 1498, and three manuscript copies were made. The first printed version was published in Venice in 1509, which coincides with when Raphael begun his work on *The School of Athens*.

In the second and third parts of the article, I will argue, based on aspects of Raphael's fresco, that the references to Pacioli's *De Divina Proportione* should be interpreted in light of Leonardo da Vinci's ambition of establishing painting as a science based on mathematics, elevating the status of painting from that of a mere craft. Leonardo's thoughts on this issue had not been published when Raphael painted *The School of Athens*, but in *De Divina Proportione* Luca Pacioli presents arguments for raising the status of painting, similar to those of Leonardo. The theory developed in this article is that the references to Leonardo da Vinci and to Pacioli's *De Divina Proportione* found in *The School of Athens* form part of a message telling us that the painter is an intellectual on a par with the philosophers, scientists, and mathematicians depicted in Raphael's fresco.

1. Luca Pacioli, Plato, and Euclid

I want to begin by taking a closer look at a famous portrait of Luca Pacioli (Figure 3) since it presents us with many of the features which later show up in his *De Divina Proportione*, in particular a focus on the so-called Platonic solids and on Euclid, but also because the painting once hung on a wall in the Ducal Palace in Raphael's hometown of Urbino.⁴ Raphael frequented the court of Urbino, where his father had been a court painter, and Raphael himself painted portraits of several men and women attached to the court. Since the portrait of Pacioli, commonly attributed to Jacopo de' Barbari, with its focus on mathematics, contains several of the elements we shall see connects *The School of Athens* with Pacioli's *De Divina Proportione*,

⁴ Pacioli had tutored Guidobaldo da Montefeltro (the Duke of Urbino from 1482 to 1508), in mathematics in his youth. The portrait was painted soon after Pacioli published his *Summa de arithmetica* in 1494, which was dedicated to Guidobaldo and is also depicted in the painting. The inscription "Divo Principi Guido" earlier found along the bottom of the portrait has been interpreted as either stating that the portrait was dedicated to Guidobaldo or that Guidobaldo is the young man standing to the right in the painting. An inventory documents that the painting was in the Ducal Palace in 1631 [5].

it could possibly have been an inspiration for Raphael.⁵

Figure 3: Jacopo de' Barbari, *Portrait of Luca Pacioli*, 1495. Public domain image from Wikipedia (<https://commons.wikimedia.org/wiki/File:Pacioli.jpg>), last accessed on July 3, 2021.

The man dressed in friar robes, pointing at a geometric construction is Luca Pacioli (Figure 3). A regular dodecahedron, consisting of twelve regular pentagons, is placed on the book with red covers to the right (Pacioli's own *Summa de arithmetica*). This is one of the five Platonic solids, called "Platonic" because they were described by Plato in his *Timaeus*. On the frame of the tablet to the left we can read EVCLIDES. The open book Pacioli has placed his left hand on is Euclid's main mathematical work, *The Elements*.

⁵ At the time when Raphael begun his work on *The School of Athens*, in 1509 or 1510, Luca Pacioli had made a name for himself by first holding a public lecture about Euclid's *Elements* in Venice in 1508, where 500 people attended. The following year (1509) both *De Divina Proportione* and a Pacioli's Latin edition of Euclid's *Elements* were published.

Studying the painting more closely than what the resolution of the image above provides for, one can see that Pacioli is pointing to a proposition from book XIII of *The Elements* [5]. This presents us with one more connection to the Platonic solids, since book XIII of *The Elements* is where Euclid proves the geometric constructions of the Platonic solids.⁶ The portrait is assumed to have been painted in 1495, just before Pacioli went to Milan, where he collaborated with Leonardo Da Vinci from 1496 to 1499 [18]. It was in Milan *De Divina Proportione* was completed in 1498, with drawings by Leonardo.

The title of *De Divina Proportione* refers to its treatment of what is also called the golden section or golden ratio.⁷ Published in Venice in 1509, the book consists of three separate manuscripts, of which the first, *Compendio divina proportione* (*Compendium on the Divine Proportion*) has given the book its name. The main focus of this manuscript is on connections between the golden ratio and polyhedrons, in particular the regular polyhedrons Plato had described in the *Timaeus*. The manuscript consists of 72 short chapters. In chapters 10 to 22, Pacioli describes thirteen properties of the divine proportion (called ‘effects’ (*effecto*)). The first nine of these properties Pacioli connects to propositions from book XIII of the *Elements*, which, as noted above, is the book where Euclid treats the Platonic solids. Chapter 24 to 52 are explicitly about Plato’s regular solids, with frequent references to *The Elements*, in particular to Book XIII. The manuscript also contains explicit references to Plato’s treatment of the regular solids in the *Timaeus*. We find this in chapter 5, 22, 24, and extensively in chapter 55.

The third manuscript in the book is an Italian translation of Piero della Francesca’s treatise *Short Book on the Five Regular Solids* (*Libellus de quinque corporibus regularibus*), which, as the title makes clear, is also about the five Platonic solids. At the end of *De Divina Proportione* we find 60 woodcuts based on Leonardo da Vinci’s original drawings of polyhedrons, including ten drawings of the five Platonic solids, two of each (Figure 4).

⁶ Up to the left in the painting we see what Kepler named a rhombicuboctahedron, half filled with water. The rhombicuboctahedron is described in chapter 53 of *Divina Proportione*, and it is one of the polyhedrons drawn by Leonardo in the book.

⁷ A point C divides a line segment AB according to the golden section if $AB/AC = AC/CB$.

Figure 4: Leonardo's drawings of Platonic solids. Images from Pennsylvania State University Libraries' facsimile of the 1498 manuscript of Pacioli's *De Divina Proportione* in Biblioteca Ambrosiana, available at <https://www.maa.org/book/export/html/116816>, last accessed on July 3, 2021.

Let us now look at some of the correspondences between *The School of Athens* and *De Divina Proportione*. We have seen that the Platonic solids described in Plato's *Timaeus* were a main focus of Pacioli's book, to which Leonardo contributed drawings of the Platonic solids. At the center of *The School of Athens*, we find Plato standing next to Aristotle. Raphael chose both to paint Plato with the face of Leonardo and to place the *Timaeus* in his left hand. The *Timaeus* was in the renaissance considered to be one of Plato's most significant dialogues, and we can imagine many different reasons for placing that particular dialogue in Plato/Leonardo's hand. In *De Divina Proportione*, the treatment of the Platonic solids is closely connected to Euclid's *Elements*.

If the face of Leonardo and the *Timaeus* placed in Leonardo's left hand were meant as allusions to *De Divina Proportione*, we should therefore expect a pictorial connection with Euclid, and that is what we find in *The School of Athens*.

Figure 5: Detail from *The School of Athens* (Figure 1).

Euclid is depicted in the foreground to the right in Raphael's fresco, as he is explaining a geometric construction to a group of pupils (Figure 5). Comparing Plato and Euclid, we find that Raphael has chosen to dress them in almost identical clothes. Both Plato and Euclid are wearing a red himation on top of a tunic. The himation is covering their left shoulders and not their right shoulders. Looking at the back of Euclid, his tunic is yellow, and therefore of a different color from Plato's tunic, but the shadow on Euclid's breast and right upper arm gives those parts of Euclid's tunic a color very similar to the color of Plato's tunic. Both Plato and Euclid's tunics cover their upper right arms, while their right forearms are bare. And they are both painted with the index finger on their right hands stretched out.

Dressing Euclid and Plato in similar clothes may indicate that Raphael wants us to see a connection between them. By placing the *Timaeus* in Plato's hand, Raphael could be telling us that the connection concerns the Platonic solids described both in the *Timaeus* and in Euclid's *Elements*. And by giving Plato the face of Leonardo, Raphael might be referring to Pacioli's book with its focus on the Platonic solids and Euclid's *Elements*, where Leonardo drew models of the Platonic solids.

Strengthening the idea that Raphael in *The School of Athens* has intended a reference to *De Divina Proportione*, we should notice that there is also a correspondence between the depiction of Plato/Leonardo in the fresco and the beginning of Pacioli's book. Raphael has painted Leonardo in the center of the fresco as he engages in a public debate with groups of listeners on each side (Figure 6).

Figure 6: Detail from *The School of Athens* (Figure 1).

Corresponding to this, *De Divina Proportione* begins by describing a public debate taking place at the court of the Duke of Milan in February of 1498, at a time when Leonardo was employed by the Duke.

Today, Eminent Duke, the 9th day of February, the year of our Lord 1498, we are gathered in the impregnable citadel of your

illustrious city of Milan, the most worthy place of your usual residence, in the presence of your majesty, and an assembly [dedicated to] praiseworthy scientific debate, composed of people of all ranks, famous and most wise, both religious and secular, with whom your magnificent court constantly abounds. [41, page 1].

Pacioli then lists several of the participants in the debate, before he points out that one of the participants was Leonardo da Vinci. Corresponding to how Leonardo is placed at the center of the debate in *The School of Athens*, Pacioli allots more space to his laudatory presentation of Leonardo than to the other participants in the debate taken together.⁸ As Monica Azzolini notes in her article “Anatomy of a Dispute: Leonardo, Pacioli and Scientific Courtly Entertainment in Renaissance Milan”:

Remarkably, the praise of Leonardo’s skills as a sculptor and painter takes up almost two pages of the manuscript. No other single figure receives nearly the same attention as Leonardo. [4, page 119]

To regard Plato painted with the face of Leonardo, the *Timaeus* placed in Plato/Leonardo’s hand, the similar clothes of Plato and Euclid, and the correspondence with the opening of *De Divina Proportione* as allusions to Pacioli’s book, requires an amount of interpretation. In the second and third parts of this article I will present several other features of *The School of Athens* which similarly will be interpreted as referring to *Divina Proportione*. But before I turn to part two, I want to point out that *Stanza della Segnatura*, the room in the Vatican palace where Raphael painted *The School of Athens* on one of the walls, contains one very direct and unmistakable reference to Pacioli’s *Divina Proportione*.⁹

⁸ It is worth noticing that the court poet Bellincioni, in a poem published in his *Rime* in 1493, calls Milan the new Athens, and compares Leonardo to the renowned classical Greek painter Apelles: “Milan is the new Athens! Here Lodovico holds his Parnassus; here rare and excellent artists flock as bees to seek honey from the flowers; here, chief among them all, is the new Apelles whom he has brought from Florence.” [14, page 119].

⁹ *Stanza della Segnatura* is thought to once have been the private library of Julius II. [53].

In Figure 7, we see *The School of Athens* on the wall to the right. Marked with a red circle, we find a Trompe-l'œil, a painting giving the illusion of being a cabinet with doors halfway open.¹⁰ Inside the cabinet two of the Platonic solids are painted, the icosahedron and the dodecahedron.

Figure 7: Stanza della Segnatura (Room of the Signatura), photograph / image from Wikipedia ([https://commons.wikimedia.org/wiki/File:1_Estancia_del_Sello_\(Vista_general_I\).jpg](https://commons.wikimedia.org/wiki/File:1_Estancia_del_Sello_(Vista_general_I).jpg)) covered under the Creative Commons Attribution-Share Alike 3.0 license, last accessed on July 3, 2021.

Comparing the Platonic solids inside the cabinet with the Platonic solids of Leonardo from *De Divina Proportione*, we can see that the painter of the Platonic solids in the cabinet has not only copied Leonardo's inventive skeleton design and painted the polyhedrons as they are hanging from a string, as in Leonardo's originals from 1498, but he has also rather closely copied how the Platonic solids are positioned and the perspective from which they are seen (Figure 8).¹¹

¹⁰ A good view of the positioning of the painted cabinet in the room can be found by going to the Vatican Museum's Visual tour: <https://www.museivaticani.va/content/museivaticani/en/collezioni/musei/stanze-di-raffaello/tour-virtuale.html>. Click on the double arrow down to the right to get to Stanza della Segnatura.

¹¹ Martin Kemp claims that the skeleton design was Leonardo's invention [29, page 88].

Figure 8: Left: Two of Leonardo's Platonic solids from the 1498 manuscript of Pacioli's *De Divina Proportione* in Biblioteca Ambrosiana (Figure 4). Right: Platonic solids in Stanza della Segnatura. (Photo by Robert Jacobson, available at <https://www.robertjacobson.dev/platonic-solids-and-the-school-of-athens>).

It is therefore quite clear that the Platonic solids inside the painted cabinet in Stanza della Segnatura are copied from Leonardo's original illustrations in Luca Pacioli's *De Divina Proportione*.

I have not been able to ascertain who copied Leonardo's Platonic solids or when they were painted.¹² If they were painted on the wall of Stanza della Segnatura to confirm the allusions to *De Divina Proportione* I argue are to be found in *The School of Athens*, the following interpretation suggests itself: The Platonic solids from Pacioli's *De Divina Proportione* are painted as partly hidden in a cabinet corresponding to how references to these Platonic solids are partly hidden in Raphael's fresco.

¹² The monochrome frescos beneath the *School of Athens* were painted by Raphael's pupil and assistant Perino del Vaga, but these do not resemble the painting of the platonic solids. [26, page 135]

I turn now to the second part of this article, where I will present a possible explanation of why *The School of Athens* contains these references to *De Divina Proportione*.

2. The Quadrivium

The educated cultural and intellectual elites at the courts, universities, and cathedral chapters in Italy had traditionally regarded the painter as a mere craftsman whose manual labor required practical skill and not intellectual knowledge.¹³ From Leonardo da Vinci's notebooks, we know that Leonardo was preoccupied with raising the status of painting. The most well-known examples of this preoccupation are found in the manuscript referred to as the *Codex Urbinas Latinus 1270*. The manuscript is thought to have been compiled from Leonardo's notebooks by his assistant Francesco Melzi [43, page 95]. In the first part of the manuscript, often referred to as the *Paragone*, Leonardo argues that painting is a science based on principles of mathematics, and that it therefore should be classified as belonging to the liberal arts.¹⁴

¹³ See [18, chapter 4]. See also Kristeller's influential article "The Modern System of the Arts", where he writes:

"From the end of the fourteenth century through the sixteenth, the writings of the artists and of authors sympathetic to the visual arts repeat the claim that painting should be considered as one of the liberal, not of the mechanical arts. [...] the claim of Renaissance writers on painting to have their art recognized as liberal, however weakly supported by classical authority, was significant as an attempt to enhance the social and cultural position of painting and of the other visual arts, and to obtain for them the same prestige that music, rhetoric, and poetry had long enjoyed. And since it was still apparent that the liberal arts were primarily sciences or teachable knowledge, we may well understand why Leonardo tried to define painting as a science and to emphasize its close relationship with mathematics. [33, pages 513–514].

¹⁴ Leonardo was here continuing a tradition begun by Leon Battista Alberti, whose book *De Pictura* Leonardo clearly had studied ([18, page 61], [16, page 128]). This is from Robert Zwijnenberg's summary of Alberti's project in *De Pictura*:

"Leon Battista Alberti's treatise *De Pictura* (1435–36) is best understood as an attempt to elevate painting from its lowly position as a craft, which it still had in Italy at the beginning of the fifteenth century, and endow it with the

Although Leonardo's arguments were not made available in print until long after his death, Clair Farago has, in her seminal commentary and translation of Leonardo's *Paragone*, documented how Leonardo's ideas influenced his contemporaries. In particular, she points out that one of the earliest published books containing Leonardo's arguments was *De Divina Proportione* by his friend Luca Pacioli [18, pages 361 and 367].

Before I go into details about how Raphael in *The School of Athens* has included references to Leonardo's arguments as they appear in Pacioli's *De Divina Proportione*, I want to mention that Farago also points out another early book containing Leonardo's arguments. The book is Baldassare Castiglione's *The Book of the Courtier* (*Il libro del Cortegiano*), written between 1508 and 1528 [51]. It is of interest here since it involves several connections between Leonardo's arguments about painting and Raphael. The author of the book, Baldassare Castiglione (1478-1529), was a close friend of Raphael and is today perhaps equally well-known as the model for one of Raphael's most famous portraits as he is as the author of *The Book of the Courtier*. According to Farago, Castiglione's book played a significant role in making Leonardo's views on painting as a mathematically-based science known to a wider audience.¹⁵

Leonardo's defense of painting may have been known in other forms as well, but his views definitely were disseminated widely when Baldassare Castiglione's *Il libro del Cortegiano* was first published in 1528. This fictionalized account of courtly conduct, in a polite debate led by Signora Emilia Pia, repeats several of Leonardo's arguments on the rivalry of the arts almost verbatim,

status of a liberal art. [...] To accomplish this goal, it was necessary to invent a theoretical foundation for it, including a specific technical vocabulary for discussing painting as a liberal art. The liberal arts differ from the mechanical arts precisely in that they are based on general theoretical principles. [...] Alberti tried to achieve his aim by concentrating his discourse in *De Pictura* on linear perspective and its use of optical and geometrical principles; [...] [61, page 167].

¹⁵ Farago suggests that Castiglione may have heard these arguments directly from Leonardo while they both were at the court in Milan (see footnote 42 in [18, page 18]).

including his defense of painting as a mathematical science based on perspective. The publication of Castiglione's book may account for the dissemination of Leonardo's polemical comparisons of the arts. [18, page 18]¹⁶

The fictional dialogue comprising *The Courtier* is presented as taking place at the court of Raphael's hometown of Urbino in 1507. It was around this time that Raphael made friends with Castiglione in Urbino, and several other friends of Raphael participate in the dialogue, such as Pietro Bembo, Bernardo Dovizi, and Ludovico da Canossa. Already in the dedicatory letter addressed to Don Michel de Silva, Castiglione makes a reference to Raphael and to the art of perspectival painting.

I send you this book as a picture of the Court of Urbino, not by the hand of Raphael or Michelangelo, but of a humble painter, who knows only how to trace the chief lines, and cannot adorn truth with bright colouring, or by perspective art make that which is not seem to be. [15, pages 2–3]

The arguments Farago ascribes to Leonardo are presented by Ludovico da Canossa in the dialogue, and they are explicitly connected with Raphael, since Canossa in presenting them is accused of primarily being motivated by his friendship with Raphael:

I really think that you are speaking against your convictions and that you are doing so solely for the sake of your friend Raphael; and perhaps too the excellence you find in his painting seems to you so consummate that sculpture cannot rival it: but consider that this is praise of an artist and not of his art. [15, page 66]

Although Castiglione's dialogue is fictional, the presence of Leonardo's arguments in *The Book of the Courtier* shows that they were known in the milieu surrounding Raphael.

¹⁶ The term 'perspective' ('prospectiva') was used in two meanings by renaissance theorists: "By perspective [the fifteenth-century painters] sometimes meant the whole science of vision, the means by which a visual impression is received in the retina; more frequently they limited the word to the scientific representation of receding figures in space." [16, page 128]

Given how Leonardo, as seen also in Castiglione’s dialogue, based his argument for heightening the status of painting in part on the mathematical basis of the newly invented technique of linear perspective, several immediately perceivable features of *The School of Athens* suggest that Raphael’s fresco should be interpreted in light of Leonardo’s ideas.¹⁷ In addition to displaying one of the most striking demonstrations of linear perspective in renaissance art, Raphael has, according to widespread interpretations, placed the three greatest living painters of his time (himself, Leonardo, and Michelangelo) among the groups of philosophers, mathematicians, and scientists found in the fresco, indicating that painters should also be regarded as intellectuals.¹⁸ The specific depictions of these painters strengthens that idea. Painting Leonardo as Plato associates him with one of the foremost thinkers in the history of philosophy. Michelangelo is portrayed in deep contemplation while he is writing (the figure is frequently referred to as *Il Pensieroso* — the thinker).

¹⁷ We do not know for certain how freely Raphael chose the content of the fresco. Ingrid D. Rowland thinks Pope Julius II gave Raphael much freedom: “Julius seems to have given artists comparative freedom, not for lack of interest [...] but because he seems to have known how to delegate responsibility. [...] For the intricate details of the Stanza della Segnatura, the pope may have given significant responsibility for their actual design to the artists themselves; this certainly seems to be the case for Raphael, whose drawings for some of the Stanze frescoes show major changes as he thought through the room’s design.” [50, pages 97–98]. It is also uncertain who might have helped Raphael with the design. It is generally assumed that Raphael was assisted by someone more well-read than him. On the basis of what is presented in this article, Raphael must have had support from someone who had studied classical texts in both Greek and Latin. For the sake of simplicity, I write as if the fresco is the work of just Raphael.

¹⁸ The identification of the figure leaning on a marble block in the center of the fresco as Michelangelo is supported by many scholars. In addition to a likeness with portraits of Michelangelo, Raphael’s imitation of Michelangelo’s painting style in the painting of this figure supports this idea. The fact that this figure is dressed in contemporary clothes, including a large pair of boots, strengthens the hypothesis that this is a portrayal of a contemporary of Raphael. Maria L. Loh discusses the history of this identification in “Renaissance Facility” [34]. She is skeptical, although she admits that the figure is painted in Michelangelo’s style. Based in particular on the likeness with the self-portrait now in the Uffizi Gallery, there is unanimous agreement among scholars that Raphael has depicted himself in the far right of the fresco.

Figure 9: Left: Daniele da Volterra, *Michelangelo*, circa 1544. Public domain image from Wikipedia ([https://commons.wikimedia.org/wiki/File:Miguel_%C3%81ngel,_por_Daniele_da_Volterra_\(detalle\).jpg](https://commons.wikimedia.org/wiki/File:Miguel_%C3%81ngel,_por_Daniele_da_Volterra_(detalle).jpg)), last accessed on July 3, 2021. Middle: Detail from *The School of Athens* (Figure 1). Right: Michelangelo, *Prophet Jeremiah* (detail from the Sistine Chapel ceiling), 1508-1512. Public domain image from Wikipedia (https://commons.wikimedia.org/wiki/File:Michelangelo_Buonarroti_027.jpg), last accessed on July 30, 2021.

The images above in Figure 9 show that the figure thought to be Michelangelo has similarities both with the looks of Michelangelo and with Michelangelo's depiction of Jeremiah from the Sistine Chapel, by some thought to be a self-portrait. As for Raphael himself, several scholars have pointed out that, by placing himself close to Euclid in the fresco, and signing the fresco by the letters "R.V.S.M." ("Raphael Vrbinas Sua Manu", "Raphael of Urbino, His Hand" in English) on the neck of Euclid's tunic, he seems to want to bring attention to the mathematical basis of painting (Figure 10). One example is Cristian K. Kleinbub in his *Vision and the Visionary in Raphael*:

To the right of Ptolemy and Zoroaster, Raphael presents himself with another painter, possibly Giovanni Bazzi (II Sodoma). Raphael looks out from inside the painting, calling our attention to himself as a representative of the science of perspective. He and his companion serve to show how perspective — and thus painting — derives from Euclid's teachings. This idea is further memorialized in the four letters embroidered on Euclid's collar, "R.V.S.M" (or "Raphael Vrbinas Sua Manu"), which bear witness to Raphael's hand. [31, page 51].

Figure 10: Detail from *The School of Athens* (Figure 1): Raphael to the right, looking at us.

Another scholar expressing a similar interpretation of Raphael's placement of himself in the group around Euclid is Ingrid Alexander-Skipnes. She also thinks that Raphael, by that choice of composition, is expressing a message about the status of painting similar to that of Leonardo:

That Raphael included himself in this group indicates that the painter views his art as a mathematical activity governed by principles of geometry. [...] Raphael wants the viewer to see that the artist is not merely a craftsman but an inspired intellectual with knowledge of mathematical principles. [2, page 161].

The ideas that Kleinbub and Alexander-Skipnes attribute to Raphael would give painting the theoretical underpinning necessary to be regarded as a liberal art. A classification scheme dominant through the middle ages had restricted that status to only seven disciplines, commonly referred to as *the seven liberal arts*. These seven liberal arts were the three disciplines forming the *trivium*; grammar, rhetoric, and logic, and the four mathematical or mathematically-based disciplines of the *quadrivium*; arithmetic, music, geometry, and astronomy.

Farago points out in [18] that the arguments articulated both by Leonardo in *Paragone* and by Pacioli in *De Divina Proportione* concern the quadrivium, implying that the mathematical basis of perspective qualifies it for a place among the mathematical disciplines of the quadrivium:

Around 1497 Luca Pacioli recorded an argument in his treatise *De divina proportione* for the liberal status of perspective used by painters by comparing it to music. The same argument in nearly identical language is made by Leonardo, who claimed that perspective should be included among the sciences of the Quadrivium along with music. [18, page 361]

Leonardo, in his *Paragone*, writes about including painting among the liberal arts, but since vital parts of the argument concern the mathematical basis of painting, Leonardo is essentially writing about the inclusion of painting among the arts of the quadrivium. Here is an example from chapter 31 of his *Paragone*:

Therefore, since you have put music among the liberal arts, either you should put painting there or else take music away. [...] If you would say that the sciences are not mechanical but mental, I will tell you that painting is mental and that, just like music and geometry, it considers the proportions of continuous quantities [while] arithmetic considers discontinuous quantities, so painting considers all continuous quantities, the qualities of the proportions of shadow and lights, and distances through its [science of] perspective. [18, page 247].

Pacioli explicitly couched the argument in terms of including painting among the quadrivial arts. In *De Divina Proportione* this argument is tightly connected with Leonardo, since Pacioli ends the argument with an extensive praise of Leonardo's *The Last Supper* as a paramount example of the art of painting:

Nonetheless, scholars commonly acknowledge only the first four, that is Arithmetic, Geometry, Astronomy and Music, and call the others subordinate, that is dependent upon these four. And this is said by Plato and Aristotle, and Isodore in his *Etymology*, and Severinus Boethius in his *Arithmetica*. But in our judgment, imbecilic and low as it might be, we must confine it to either three, or five: that is Arithmetic, Geometry and Astronomy, and exclude Music from them for the very reasons that other scholars omit Perspective from the five primary disciplines; or five, by adding Perspective to the said four disciplines, for just as good reasons as the others gave for adding Music to the three we indicated.

If these scholars say that music satisfies the hearing, one of the senses given us by nature, I say that perspective satisfies the sight, which is much more worthy, as it is the first door of the intellect; if they say that music applies to the sonorous number, and to the implied measure of its extension in time, I say for my part that perspective observes the natural numbers according to all its definitions, such as the measure of the visual line. If the former is recreation for the soul through harmony, then the latter does this by the necessary distance and the very delectable variety of many colors; if the former respects its harmonic proportions, then the latter, respects the arithmetic and geometric proportions. In brief, excellent Duke, for many years and until the present moment, this question has tormented my spirit; nothing makes clear to me why four should be better than three or five, even if we should say that so many scholars could not err; yet by what they say, my ignorance is not remedied. Who is it that sees a graceful figure [painted] with its necessary lineaments well disposed, where the only thing missing seems to be the breath, that does not judge it a thing divine rather than human?

Painting imitates nature as much as could possibly be done. The which appears clearly to our eyes, if we will look at that marvelous representation of the ardent desire of our redemption, in that painting it is not possible to imagine the apostles more alive to the sound of the voice of the infallible truth when he says: “Unus vestrum me traditurus est”, one of you shall betray me; where with acts and gestures to one another, with living and painful astonishment, it seems that they be speaking: so justly did Leonardo depict this moment for us with his skillful hand. [41, pages 8–9]

We shall now see how Raphael in *The School of Athens* refers to the arguments of Leonardo and Pacioli, by telling us that painting belongs together with the disciplines of the quadrivium. Towards the end of this article we shall see that the depiction of Plato with the face of Leonardo plays a significant part in expressing that message as well.

It has often been argued that Raphael, following a long tradition of iconography, represented the seven liberal arts in *The School of Athens*. There is a major problem with that idea though, since scholars don't seem to agree on which figures in the fresco represent which of the seven arts. In “Reading Raphael: *The School of Athens* and Its Pre-Text” [38], Glenn W. Most has reviewed that idea. After pointing out that we can identify geometry, astronomy, arithmetic, and music in the painting, he asks where the disciplines of the trivium are to be found:

But the other liberal arts cannot be satisfactorily assigned to the remaining figures. Where are grammar, dialectic, and rhetoric? Various suggestions have been made; all are artificial and arbitrary. [38, page 153]

Examining the contradictory suggestions of scholars such as A. Springer, J. Schlosser, E. H. Gombrich, H. Einem, and J. Klawns, Most concludes that it cannot have been Raphael's intention to represent the seven liberal arts:

In fact, though, it is an essential feature of that tradition that all seven liberal arts be exhaustively and unambiguously identifiable.

If Raphael's intention was to portray the seven liberal arts, he was evidently so incompetent that to this day his intention could not be understood clearly and unmistakably. Hence this could not have been his intention. [38, page 154]

I think Most is right in his analysis; Raphael did not intend a representation of the seven liberal arts in *The School of Athens*. But Most does not reflect on the fact that the four disciplines which he finds identifiable in the fresco are the four disciplines of the quadrivium (geometry, astronomy, arithmetic, and music), even as he remarks:¹⁹

Some of the figures shown in *The School of Athens* can easily be interpreted as practitioners of some of the seven liberal arts. A geometer and an astronomer are prominently displayed in the right foreground; [...] If we wish, we can also locate arithmetic and music in the tablet held up for Pythagoras in the left foreground — though it is odd that, in a painting with almost sixty figures and only seven liberal arts to distribute amongst them, one figure has to serve for two arts. [38, page 153].

It would be odd if Pythagoras had to serve for both arithmetic and music, as Most suggests here, but as I will show, that is not the case. We shall see that Raphael has represented the quadrivium by four distinct figures, and that he has placed a number of identifiers in his fresco telling us that arithmetic and music are represented by Pythagoras and the early medieval Neo-Platonist Boethius (480-524/525). Indeed, it is in the first chapter of Boethius's *De institutione arithmetica* that we find the first known use of the term *quadrivium*.

In the foreground of the fresco, to the right, we find Ptolemy and Euclid, representing astronomy and geometry, respectively.²⁰ Ptolemy is holding a terrestrial globe next to a figure holding a celestial globe. Conforming to convention, Raphael has placed a crown on Ptolemy's head.

¹⁹ Alexander-Skipnes makes a similar observation: "While Raphael has distinctly depicted the quadrivium disciplines in the foreground, the artist has not given the same clarity to the trivium. Instead, the trivium is informally distributed among the background figures." [2, page 156]

²⁰ A neat discussion of the geometric figures on Euclid's tablet can be found in [24].

Pythagoras, sitting in the foreground to the left, is identified by the Pythagorean diagrams on the tablet in front of him (Figure 11).

Figure 11: Detail from *The School of Athens* (Figure 1).

To the right of Pythagoras Raphael has painted a man who is clearly meant to be associated with Pythagoras. This man is turned towards Pythagoras while he is pointing inside an open book. Pythagoras and this man are also linked to each other in that they both are depicted holding an open book and have their left feet placed on a stone. I will return to what I think is symbolized by placing their feet on these stones. Marry D. Garrard argues that the two figures are Pythagoras and Boethius, and refers to other scholars who have suggested that Boethius is found in the group around Pythagoras.²¹

²¹ "Arithmetic is usually represented by Boethius, who formulated the quadrivium in the sixth century. This philosopher is not immediately recognizable in the *School of Athens* by appearance or attribute, although both Schlosser and d'Ancona suggested that he might be found in the group around Pythagoras in the left foreground." [21, page 370].

I will argue that Boethius is the figure standing to the right, by showing how Raphael has inserted a number of features in his composition confirming this interpretation. Boethius's qualification to figure in representations of the quadrivium was both his coining of the term *quadrivium* and the educational use of his textbooks in music and arithmetic.²² Raphael's use of him also accords well with a rising interest in Boethius in the decades around 1500.²³

The School of Athens contains many books (at least eight), and I have already presented reasons to believe that the fresco contains features referring to Pacioli's *De Divina Proportione*. The main reason to think that the figure standing to the right of Pythagoras is Boethius is what I will argue is a number of references to a combined edition of Boethius's *De institutione arithmetica* and *De institutione musica* with the title *Arithmetica geometria et musica* [10] (it also contains extracts from the first three books of Euclid's *Elements*).²⁴ This final portion of Boethius's *Opera* was first published in Venice in 1492, less than twenty years before Raphael begun to decorate the walls of Stanza della Segnatura. Identifying Boethius through references to this book fits well with how the figure of Boethius in the fresco is directing our attention to the open book he is holding.

The first indication that Raphael is directing us to the combined edition of *Arithmetica* and *Musica* of Boethius is that both diagrams on the tablet held up between Pythagoras and Boethius are found in this book. In the lower part of the tablet Raphael has painted an illustration of the fourth triangu-

²² Representing the quadrivium by Euclid, Ptolemy, Pythagoras, and Boethius accords with the representation of the liberal arts found in Gregor Reisch's contemporaneous *Margarita Philosophica* from 1503. Reisch's book, which came in over twenty editions during the 16th century, contains an illustration where arithmetic is represented by both Pythagoras and Boethius, but when all the liberal arts are represented on the title page, Pythagoras is the representative of music, while Boethius represents arithmetic (Euclid and Ptolemy represent geometry and astronomy). See [36, page 72].

²³ According to Ann E. Moyer, "Quadrivial studies enjoyed renewed attention in the late 15th and early 16th centuries. Scholars of astronomy and astrology, scholars of the studies newly inspired by Ficino and the Platonic corpus he made available, educational reformers, and theorists of the musical and visual arts, all relied on Boethian texts and traditions." [39, page 491]

²⁴ It is worth noting that to Boethius, the term *arithmetica* denotes theory of numbers, and not the discipline of calculation [11, page 11].

lar number that is quite similar to the illustrations of the fourth triangular number found twice in Boethius's arithmetic; see Figure 12 below.

Figure 12: Left: Slate in *The School of Athens* (Figure 1), detail from the photograph by Rafael Edwards, available at <https://www.flickr.com/photos/rafa2010/42172531755/in/album-72157698489392615/>, last accessed on July 3, 2021. Right: Fourth triangular number from Boethius, *Arithmetica geometria et musica* [10, page 165]. See <https://archive.org/details/arithmeticegeome00boet/page/n25/mode/2up>.

The fourth triangular number is ten, the sum $1 + 2 + 3 + 4$. We see that there are ten Roman ones in a triangular shape both on Raphael's tablet and in the illustration of the fourth triangular number from *Arithmetica*. The number ten is written as 10 in Boethius's illustration and by a Roman X below the triangle on Raphael's tablet. I will return to why Raphael chose that particular illustration, but one reason is that it also helps in identifying Pythagoras, since the Pythagoreans attached a special significance to the fourth triangular number, often referred to as the *tetractys*.

The diagram in the upper part of the tablet illustrates Pythagorean harmonics, and is essentially identical to the first diagram found in Boethius's *Musica* (see Figure 13). Both diagrams make use of the numbers 6, 8, 9, and 12 to illustrate the ratios corresponding to an octave (2:1), a perfect fourth (4:3), a perfect fifth (3:2) and a whole tone (9:8), by curved lines connecting the numbers. The corresponding Greek words "diapason", "diatessaron" and "diapente" in Boethius's diagram are written with Greek letters as ΔΙΑΠΑΣΩΝ, ΔΙΑΤΕΣΣΑΡΩΝ, and ΔΙΑΠΕΝΤΕ on Raphael's slate.

Figure 13: Left: Slate in *The School of Athens* (Figure 1), detail from the photograph by Rafael Edwards, available at <https://www.flickr.com/photos/rafa2010/42172531755/in/album-72157698489392615/>, last accessed on July 31, 2021. Right: Diagram from Boethius, *Arithmetica geometria et musica* [10, page 176]. Image can be seen at <https://archive.org/details/arithmeticageome00boet/page/n47/mode/2up>.

This connection between Raphael’s slate and Boethius’s book seems to be indicated in *The School of Athens* by how the slate is held up, oriented towards Boethius.

I think Raphael’s main interest is in the first chapter of Boethius’s *Arithmetica*, and one reason to place the two diagrams on the tablet between Pythagoras and Boethius is to direct our attention to that first chapter.²⁵ Comparing Raphael’s tablet with Boethius’s diagram as in Figure 13, we might notice one particular difference. The musical interval corresponding to the numerical relation between 8 and 9 is marked in Boethius’s diagram as “Tonus”, while this relation is not marked in the diagram on Raphael’s tablet. Instead, we find the word ΕΠΟΓΔΟΩΝ (Epogdoon), which denotes the numerical ratio 9:8 as a title above Raphael’s diagram.²⁶ The words “Diapason”, “Diapente”, “Diatessaron”, and “Tonus” are all used hundreds of times in Boethius’s *Musica*, but there are no occurrences of “Epogdoon”.

²⁵ Notice that Boethius and “his *Arithmetica*” are referred to in the quote from *De Divina Proportione*, where Pacioli presents his argument about painting and the quadrivium.

²⁶ As I will come back to, ΕΠΟΓΔΟΩΝ is misspelled as ΕΠΟΓΛΟΩΝ on the tablet.

However, this word (in the Latin form “Epogdous”) is found in two passages in the combined edition of *Arithmetica* and *Musica*, where it is found in the first and last of the 86 chapters of *Arithmetica*. In both of these passages, *Epogdous* occurs together with the other three terms on Raphael’s tablet; *Diapason*, *Diapente*, and *Diatessaron*.²⁷ The last chapter also contains a diagram containing these terms, but this diagram is visually less similar to Raphael’s diagram than the diagram in Figure 13. We shall soon see that the passage in the first chapter of *Arithmetica* containing the four terms on Raphael’s tablet seems to be of particular significance to Raphael.

One reason why the first chapter of Boethius’s *Arithmetica* should be of special interest to Raphael is that therein lies the first known use of the term “quadrivium”, and also Boethius’s explanation of this concept. The term is introduced already in the first line of the first chapter:

Here begins the First Book. 1. Proemium: the division of mathematics. Among all the men of ancient authority who, following the lead of Pythagoras, have flourished in the purer reasoning of the mind, it is clearly obvious that hardly anyone has been able to reach the highest perfection of the disciplines of philosophy unless the nobility of such wisdom was investigated by him in a certain four-part study, the quadrivium, which will hardly be hidden from those properly respectful of expertness. [11, page 71]

Two aspects of Raphael’s depiction of Pythagoras and Boethius fit well with this beginning. First, Boethius writes about “following the lead of Pythagoras”, while Raphael has depicted Boethius pointing inside an open book and turning towards Pythagoras. The second aspect concerns the etymological root meaning of Boethius’s metaphor “quadrivium”. *The Oxford Guide to Etymology* tells us that “In classical Latin quadrivium meant a crossroads, a place where four roads meet, [...]” [17, page 241].

²⁷ The term *Epogdoon* stands out in Raphael’s diagram also by belonging to a different category than *Diapason*, *Diapente*, and *Diatessaron*. The three latter words literally mean “through all”, “through five”, and “through four”, respectively, referring to the musical intervals “through all the notes”, “through five notes”, and “through four notes”. Meanwhile *Epogdoon*, which corresponds to the musical tone, means “an eight more” and refers to the numerical relation of being one eight more than another number, such as 9 is to 8.

Looking at the fresco we can see that Raphael has placed Boethius in the center of a cross on the ground floor, with four white paths leading out from the center (we can assume that one of the thick white lines is hid behind the stone Boethius has placed his left foot on).

Both figures on Raphael's tablet represent groups of four elements, a so-called tetrad (from the Greek word τετρας, meaning "four"). The fourth triangular number (10) was called a tetrad or tetractys since it is the sum of the first four natural numbers. The four numbers 6, 8, 9, and 12 are often referred to as the musical tetractys [6]. The choice of placing these figures on the slate indicates the significance of the tetrad. We shall now see that there is a particular connection between the tetractys and Boethius's arguments about the quadrivium (also a tetrad) in the first chapter of his *Arithmetica*.

In the short first chapter of *Arithmetica*, Boethius explains how the four disciplines of the quadrivium are ordered in a hierarchy, with arithmetic as the primordial discipline, followed by music, geometry, and astronomy. Boethius argues, for instance, that arithmetic is more fundamental than geometry since geometry depends on arithmetic in that you can have numbers without geometry, but not geometry without numbers

The same thing is seen to occur in geometry and arithmetic. If you take away numbers, in what will consist the triangle, quadrangle, or whatever else is treated in geometry? All of those things are in the domain of number. If you were to remove the triangle and the quadrangle and all of geometry, still "three" and "four" and the terminology of the other numbers would not perish. Again, when I name some geometrical form, in that term the numbers are implicit. But when I say numbers, I have not implied any geometrical form." [11, page 74]

It is in the subsequent argument that arithmetic is also more fundamental than music, that we find the first of the two occurrences of all the four terms on Raphael's tablet: *diapason*, *diatessaron*, *diapente*, and *epogdous*:

The same relationship which we remarked in geometry can be found in music. The names diatessaron, diapente, and diapason are derived from the names of antecedent numerical terms.

The proportion of their sounds is found only in these particular relationships and not in other number relationships. For the sound which is in a diapason harmony, the same sound is produced in the ratio of a number doubled. The interval of a diatessaron is found in an epitrita comparison; they call that harmony diapente which is joined by a hemiola interval. An epogdous in numbers is a tone in music. [11, page 75]

Thus, it seems that Raphael wants to bring our attention to this hierarchical structure of the quadrivial arts, which should prepare the student to the study of philosophy.²⁸ This can also be seen in his choice of letting the fourth triangular number represent arithmetics. The point here is that the same hierarchical structure Boethius attributes to the quadrivium is also exhibited by the fourth triangular number. This hierarchical structure is both described in Boethius's text, and made visible in the second illustration of the fourth triangular number, where the relations to the three preceding triangular numbers are pointed out (see Figure 14).

Figure 14: Triangular numbers from Boethius, *Arithmetica geometria et musica* [10, page 166]; see <https://archive.org/details/arithmeticageome00boet/page/n27/mode/2up>. Note that here the fourth triangular number has been erroneously printed with a fifth row of five ones. In the separate 1488 edition of Boethius's *Arithmetica* [9], the printing is correct; see <https://archive.org/details/deinstitutionear00boet/page/n57/mode/2up>.

²⁸ Michael Masi sums up the Boethian view on how the quadrivium leads to philosophy in this way: “Each discipline is logically prior to the following and each is dependent on the previous for its principles of procedure. The four arts of the quadrivium must be studied in a given order and they ultimately serve as preparation for a study of philosophy.” [36, pages 58–59]

Comparing the first, second, third, and fourth triangular numbers, we see that the smaller ones are contained within the larger ones. The second triangular number contains the first. The third triangular number contains the first and the second, and the fourth triangular number contains all the three previous. This is evident both in the geometric representation and in an arithmetic representation:

$$\begin{aligned} T_1 &= 1 \\ T_2 &= 1 + 2 = 3 \\ T_3 &= 1 + 2 + 3 = 6 \\ T_4 &= 1 + 2 + 3 + 4 = 10 \end{aligned}$$

$$T_4 = T_3 + 4 = T_2 + 3 + 4 = T_1 + 2 + 3 + 4$$

The dependences between these four triangular numbers are thus the same as that between the four quadrivial disciplines. For instance, take away the numeral one at the top of the illustration of the fourth triangular number and none of the triangular numbers are represented, corresponding to how both music, geometry, and astronomy depend on arithmetic. On the other hand, we can remove the row with four ones at the bottom of the illustration of the fourth triangular number, and the ones left will still represent the first, second, and third triangular numbers, analogous to how arithmetic, music, and geometry do not depend on astronomy.

A similar idea, explicitly connecting the hierarchical quadrivium and the tetrad of 1, 2, 3, and 4, is found in a text of the Pythagorean tradition Boethius based his *Arithmetica* on. Boethius's *Arithmetica* is a loose translation into Latin of Nicomachus's *Introduction to Arithmetic*, written in Greek. Nichomachus also wrote a book called *The Theology of Arithmetic*. That work is lost, but an anonymous book, also called *The Theology of Arithmetic* [58], is thought to be in part based on Nichomachus's work of the same name [23, pages 135–136]. In this anonymous work, we find the same fourfold division of mathematics into arithmetic, music, geometry, and astronomy as in Boethius, and similar arguments for the same hierarchy. Furthermore, in this work, the structural similarity between the hierarchical quadrivium and the tetractys (the tetrad of 1, 2, 3, and 4) is made explicit,

and even attributed to Pythagoras himself. Writing about the tetrad, the anonymous writer tells us:

If number is the form of things, and the terms up to the tetrad are the roots and elements, as it were, of number, then these terms would contain the aforementioned properties and the manifestations of the four mathematical sciences — the monad of arithmetic, the dyad of music, the triad of geometry and the tetrad of astronomy, just as in the text entitled *On the Gods* Pythagoras distinguishes them as follows: “Four are the foundations of wisdom — arithmetic, music, geometry, astronomy — ordered 1, 2, 3, 4.”

[...]

In the first place, the association of arithmetic with the monad is reasonable: for when arithmetic is abolished, so are the other sciences, and they are generated when it is generated, but not vice versa, with the result that it is more primal than them and is their mother, just as the monad evidently is as regards the numbers which follow it. [58, page 56]

The tetractys on Pythagoras's tablet in *The School of Athens* could therefore function as a visual expression of the hierarchical structure of the quadrivium described in the first chapter of Boethius's *Arithmetica*, and could therefore have been placed there also to highlight the significance of the quadrivium in Raphael's fresco.²⁹

In accordance with this hierarchical structure, Boethius adopts a metaphor of the quadrivium as a stairway leading up to philosophy, partly based on Nicomachus who likened mathematics to a ladder, as described here by Jean-Yves Guillaumin:

²⁹ In the edition of Boethius's *Arithmetica* from 1492, the fourth triangular number is represented by the Arabic numeral 10. Raphael on the tablet chose to use the Roman numeral X, and that is probably how Boethius would have written it in the original manuscript, but we can also notice that the X presents us with one more cross with four paths leading out from the center, fitting the etymological root meaning of “quadrivium”.

In the first pages of *De Institutione arithmetica*, after defining philosophy while invoking the patronage of Pythagoras, Boethius undertook in effect, to explain the paths that lead to it. These are, naturally, the “four sciences,” which alone are capable of leading to philosophy by knowledge of the truth. [...] Boethius here imitates Nicomachus (who, in 1.31, compared mathematics to a ladder (κλίμαξι τισι ἔοικε ταῦτα τὰ μαθήματα [mathematics seem like a ladder]): he employs a suggestive image, that of the steps (gradus) through which the ascent toward perfect knowledge is effected, the passage from what can be felt to what can be understood (*a nobiscum procreatis sensibus ad intellegentiae certiora*, 1.1.7). Mathematics thus constitutes a “step”, an indispensable stage in reaching philosophy, and within mathematics, the four sciences themselves represent four “steps”, ranked in an order to which we will return later. [23, page 138]

This fits well with several aspects of how the quadrivium is represented in *The School of Athens*. By placing both Pythagoras’s and Boethius’s left feet on a stone, Raphael seems to be telling us that each represents one step (gradus) in the quadrivium (see Figure 15).

Figure 15: Detail from *The School of Athens* (Figure 1).

The left foot of Pythagoras is located close to the tablet where arithmetic is represented at the bottom and music at the top, analogous to how arithmetic and music are the two first steps in the hierarchy of the quadrivium.

Widening the perspective, we should notice that all four representatives of the quadrivium (Pythagoras, Boethius, Euclid, and Ptolemy) are placed on the ground floor, and that there are four steps in the stairway, corresponding to the four disciplines of the quadrivium, leading up to Plato and Aristotle, the two preeminent philosophers of the classical era. The squares on the ground floor, with their four corners and four sides, are called *quadratum* in Latin and *quadrato* in Italian, and are probably meant to strengthen the association between the ground floor and the tetrad of the quadrivium. The idea that the stairway up to Plato and Aristotle represents a mathematical ascent to philosophy was proposed already in 1695 by Bellori, who writes that “Raphael was concerned with the ancient custom of the Greeks, who rose step by step from mathematics to the speculative sciences” [8, page 55].³⁰

Given all of the above, how could Raphael now communicate that painting belongs together with the disciplines of the quadrivium, as Leonardo and Pacioli argue? The most obvious thing to do would be to place painters together with the representatives of the quadrivium on the ground floor. And that is exactly what Raphael has done. Together with Pythagoras, Boethius, Euclid, and Ptolemy on the ground floor, Raphael has placed two of the three major living painters of his time, himself and Michelangelo (Figure 16).³¹ We should also notice that the figure next to Raphael on the ground floor (dressed in white) is held to be a painter, either Raphael's teacher Pietro Perugino or Sodoma, and that some scholars have identified Euclid as the painter and architect Bramante (see for instance [42, page 409] or [25, page 555]).

³⁰ Matthias Winner suggests a similar interpretation of the stairway in his article “The Mathematical Sciences in Raphael's School of Athens”. He thinks Raphael got this idea from Boethius through Ficino [59, page 306].

³¹ As already mentioned, there is not unanimous agreement among scholars that the figure leaning on the marble block in the center of the fresco represents Michelangelo. I regard that idea as being strengthened by how it would fit well together with other aspects of the interpretation of the fresco presented here. There is no disagreement that Raphael has depicted himself standing on the ground floor.

Figure 16: Painters and representatives of the quadrivium on the ground floor in *The School of Athens* (detail from Figure 1).

Given the fresco's references to Pacioli's *De Divina Proportione*, and Pacioli's explicit references to the concept of the quadrivium in his version of Leonardo's argument for raising the status of painting, I think we should regard the representation of the quadrivium in *The School of Athens*, the four steps up to Plato and Aristotle, and the positioning of painters on the ground floor as alluding to Pacioli's version of Leonardo's arguments. But Raphael's fresco contains one more feature that I will argue is meant to confirm this interpretation.

3. The Quadrivium and the Timaeus

In this last part of the article I show how Raphael in one more way signals that painting belongs together with the quadrivial arts. This involves both the portrait of Leonardo and the book he is holding in his hand, the *Timaeus*.

The character Timaeus in Plato's eponymous dialogue is a Pythagorean philosopher. In the dialogue he describes the structure of the world, and explains how the world was created. While doing this, he makes use of arithmetic, Pythagorean harmonics, geometry, and astronomy. Plato thus makes use of the four disciplines of the quadrivium in the dialogue, and for this reason, the *Timaeus* was a main inspiration for later references to this fourfold categorization of mathematical and mathematically-based disciplines.

One of the philosophers focusing on this particular categorization was the Neo-Platonist Proclus (410-485), who wrote a large and influential commentary on the *Timaeus*, and also a commentary on Euclid's *Elements*. Proclus's quadrivial interpretation of the *Timaeus* is perhaps most obvious in a passage from his commentary on Euclid's *Elements*, a commentary Bartolomeo Zamberti quotes extensively from in the introduction to his own 1505 edition of *The Elements* (see for example [22, page 125]). Like Boethius, Proclus attributes the fourfold division of mathematics to the Pythagoreans:

The Pythagoreans considered all mathematical science to be divided into four parts: [...] Arithmetic, then studies quantity as such, music the relations between quantities, geometry magnitude at rest, spherics magnitude inherently moving. [47, pages 29–30]

(In a footnote to the translation of σφαρικός as spherics, the translator writes “I.e. Astronomy.”). Proclus then turns to how this fourfold division is manifested in the *Timaeus*. In a dense and difficult passage starting with “Now since, as the *Timaeus* has taught us, [...]” Proclus explains how arithmetic, geometry, music, and astronomy are instantiated in the *Timaeus*, before he sums up the passage by telling us that “[t]his, then, is the doctrine of the Pythagoreans and their fourfold division of the mathematical sciences” [47, page 31].

This accords well with what Philip Merlan writes about the Neo-Platonic tradition both Boethius and Proclus belonged to in his book *From Platonism to Neoplatonism* [37]. In a chapter called “The Origin of the Quadrivium”, Merlan emphasizes the important role of the *Timaeus* in the development of the concept of the quadrivium, and seems somewhat puzzled by how closely the two are connected.

We saw that the interpretation of the *Timaeus* became combined with the problem of the division of mathematics. The meaning and importance of this latter division is well known: the quadriparted mathematics is the quadrivium. It is surprising to see how closely the quadrivium is connected with the interpretation of the *Timaeus*. [37, page 88]

The quadrivial interpretation of the *Timaeus* held sway also when Raphael was painting his fresco. Marsilio Ficino (1433-1499), leader of the Platonic academy in Florence and the most influential interpreter of Plato in Renaissance Italy, in particular in the decades preceding Raphael's painting of *The School of Athens*, was influenced by Proclus in his commentary on the *Timaeus*, and also by Calcidius, another influential early medieval commentator who interpreted the dialogue according to a quadrivial scheme.^{32,33} Thus, Ficino, too, looked at the *Timaeus* through a quadrivial lens. In the words of Jacomien Prins, writing about Ficino's commentary on the *Timaeus*,

A major part of Ficino's Compendium explains the Timaeian view of cosmic harmony in terms of the four mathematical disciplines of the quadrivium: arithmetic, geometry, music, and astronomy. [46, page 47]

Given Raphael's representation of the quadrivium in *The School of Athens*, we, therefore, have reasons to believe that the close link between the *Timaeus* and the quadrivium is one reason why Raphael placed this book in Leonardo's left hand. This hypothesis is strengthened by how several features of Raphael's representation of the quadrivium are connected with how the quadrivium is instantiated in the *Timaeus*. In the first part of this article I argued that Euclid, representing geometry, should be associated with the Platonic solids described in the *Timaeus*. Raphael's use of Pythagoras and, in particular, the diagram illustrating Pythagorean harmonics are also closely connected to the *Timaeus*.

The *Timaeus* is Plato's most Pythagorean dialogue. In a well-known passage in the *Timaeus*, Plato refers to the ratios exemplified by 6, 8, 9, and 12 on Raphael's tablet when he explains how the world's soul was constructed in

³² According to Denis J-J Robichaud, "Proclus's commentary on the *Timaeus* is probably the most important exegetical guide for Ficino's interpretation, but Calcidius and medieval traditions of interpreting the dialogue would also have been relevant for his orientation toward the text" [49, page 155].

³³ According to Magee, in his edition of Calcidius commentary on the *Timaeus*, Calcidius was "structuring the first part of the commentary along the lines of what after Boethius would become known as the Quadrivium, or the mathematical disciplines (arithmetic, geometry, music, astronomy), which [...] are described as the foundations of a liberal education and the first stages of a philosophical ascent" [13, pages xviii-xix].

accordance with the ratios of Pythagorean harmonics [44, page 65]. As part of this explanation, Plato presents a sequence of numbers, where the first two numbers are 1 and 2.³⁴ Placing the harmonic and arithmetic means between the numbers in the sequence, the beginning of the sequence becomes 1, $\frac{4}{3}$, $\frac{3}{2}$, and 2.³⁵ Both Proclus and Calcidius, in their commentaries on *Timaeus*, describe how these four numbers give us the numbers on Raphael's slate (6, 8, 9, and 12) when we multiply by 6 to get rid of the fractions.³⁶

We find the same idea also in Raphael's contemporary Marsilio Ficino's commentary on the *Timaeus*. He explains this in connection with a diagram frequently found in commentaries and translations of the *Timaeus*. The diagram is often referred to as the *Platonic lambda*, since it has the shape of the Greek uppercase letter lambda (Λ).³⁷ In Figure 17, we see versions of this diagram from the 1518 edition of Ficino's commentary and translation of the *Timaeus*.

³⁴ Plato is writing about the division of a length into portions, but it is common among interpreters to focus on the relative measures of these portions, expressed as numbers. The sequence in the original order is 1, 2, 3, 4, 9, 8, 27, which many interpreters split up into the doubles 1, 2, 4, and 8, and the triples 1, 3, 9, and 27. See note 37.

³⁵ Plato describes the harmonic and arithmetic means as "two means within each interval — a mean that exceeds one extreme and is exceeded by the other by the same *fractional part*, and another mean that exceeds one extreme by a *number* equal to the amount by which it is exceeded by the other extreme" [44, page 65].

³⁶ From Proclus's commentary: "[...] since [Plato] exhorts us to bind the double and triple intervals with harmonic and arithmetic middles, and it is not possible to discover these middles between 1 and 2, some first number must be assumed, which being the least, may have a half and a third part. For every number may have a double, and this must therefore be investigated. Let then 6 be assumed, and the double of it 12, the former having the same ratio to the latter as 1 to 2. Between these therefore, viz. 1 and 2 multiplied by 6, placing as media 8 and 9, we shall have the above-mentioned middles" [48, page 62]. From the commentary of Calcidius: "Given that the number 6 constitutes one term and 12 a second, and that the difference between them is in the ratio of the double, two means are intercalated, one of them being the number 8 and the other 9" [13, page 185]. This is followed by an elaborate argument showing that 8 and 9 are the harmonic and arithmetic means of 6 and 12.

³⁷ Plutarch, in his *Moralia*, attributes the lambda-diagram to the Greek Platonist Crantor: "[...] concerning the arrangement whether one is to set them out as Theodorus does all in a single row or rather as Crantor does in the figure of a lambda with the first placed at the apex and the double and triple numbers ranged separately from each other in two rows underneath" [45, page 265].

Figure 17: Diagrams of Platonic lambda in Ficino's commentary on *Timaeus* (1518) [20]. See https://archive.org/details/bub_gb_Q0kjv0BiQAUC/page/n599/mode/2up.

In his commentary on *Timaeus*, Ficino explains the motivation behind placing 6, 8, 9, and 12 along the left leg of the large lambda—the first figure referred to in the quote is the smallest lambda-diagram:

After Plato has explained the first figure, which has seven limits and in which the double and triple proportions are contained, he then asks for another figure to be pictured which includes larger numbers, so that when the doubles are filled with *sesquitertials* and *sesquialterals* these medials may also be filled with *sesquioc-tavals*, so that the spaces of the *sesquitertials* contain a tone. Let us therefore describe a figure similar to the first, but let us put six at the top, with its double [twelve] far beneath, filling the intervening space with eight and nine. [19, page 75]

The *Sesquitertials*, *sesquialterals* and *sesquioctavals* Ficino mentions here are the numerical ratios corresponding to the *diapente*, *diatessaron*, and the tone. As we have seen, Boethius used the same words in the illustration of Pythagorean harmonics presented earlier in this article (cf. Figure 13).³⁸

Given the close link between the *Timaeus*, the concept of the quadrivium, and Raphael's representation of the quadrivium, Leonardo carrying this book opens up the possibility that this is also meant as a reference to Leonardo's arguments about including painting in the quadrivium, especially since the technique of linear perspective is both central to Leonardo's argument and masterly applied by him in depicting the Platonic solids described in the *Timaeus*. As we shall now see, Raphael has, if I am right, included a specific feature in *The School of Athens* intended to confirm this interpretation.

The School of Athens was painted using the technique of linear perspective with one vanishing point. It is Raphael's positioning of the vanishing point in the fresco that I think is meant to confirm that Leonardo carrying the *Timaeus* is intended as a reference to Leonardo's arguments about painting and the quadrivium.

First, note that Leonardo da Vinci was famously left handed. In a manuscript titled *De Viribus Quantitatis*, Pacioli praises Leonardo's use of perspective in drawing the Platonic solids for *De Divina Proportione*, and tells us that they were drawn with Leonardo's "ineffable left hand".³⁹ In *The School of Athens*, it is Leonardo's left hand which is holding the *Timaeus*.

³⁸ In the original passage discussed here, Plato used the Greek word "epogdoon" (frequently translated as "9:8" in English translations of the *Timaeus*). Looking closer at "epogdoon" as it appears in the title above Raphael's diagram, we can see that it contains a curious misspelling. Instead of delta (Δ) as the fifth letter in ΕΠΟΓΔΟΩΝ, we find a lambda (Λ), and the title reads ΕΠΟΛΔΟΩΝ. Bellori corrected this error when he reproduced the content on Raphael's tablet in his *Descrizione delle Immaginidipinte da Rafaele d'Urbino* (1695). An image of this can be found at <https://archive.org/details/descrizione00bell/page/34/mode/2up>. My guess is that the misspelled lambda is intentional, and that it is hinting at the connection between Raphael's tablet and the Platonic lambda, signaling the link to the *Timaeus*.

³⁹ "[I]t is impossible doing better the Platonic solids and mathematical bodies using the drawing in perspective ... that were made by that ineffable left hand, talented in all disciplines of mathematics of the prince among mortals, our Florentine Leonardo da Vinci" [28, pages 207–208].

And as Nicholas Temple observes,⁴⁰ this left hand is located at the vanishing point of the linear perspective in the painting:

Given the importance of the *Timaeus* in the Renaissance, it is perhaps not surprising that Plato’s left hand should coincide with the vanishing point of the fresco. [54, pages 139–140]

I have already suggested that the squares on the ground floor (called “quadratum”/“quadrato” in Latin/Italian) should be associated with the quadrivium. We can now see (in Figure 18) that drawing perspective lines along the sides of the squares brings us to the vanishing point at Leonardo’s left hand.

Figure 18: Lines meeting at the vanishing point in *The School of Athens* (Figure 1).

My interpretation of this is that Raphael, by locating Leonardo’s “ineffable” left hand at the vanishing point of the linear perspective, is telling us that painting and perspective belong together with the mathematical disciplines of the quadrivium found in the *Timaeus* that Leonardo is holding in his left hand.

⁴⁰ Other authors have also noted this. See in particular Fred S. Kleiner who writes, “The vanishing point falls on Plato’s left hand, drawing attention to *Timaeus*” [32, page 256]. Also see David Heyd in [27, page 170].

If this interpretation is correct, Raphael has managed to concentrate several aspects of his message into one feature of his composition: Leonardo carrying the *Timaeus* refers to the Platonic solids drawn by Leonardo, demonstrating his skill in linear perspective, but this also refers to the quadrivium exemplified in the *Timaeus*, pivotal to Leonardo and Pacioli's argument for raising the status of painting. The mathematical basis of linear perspective is central to that argument, and we have now seen how Raphael brings it all together by locating the vanishing point of the linear perspective at the hand carrying the *Timaeus*.⁴¹

To further support the thesis that Raphael intentionally placed Plato/Leonardo's left hand at the vanishing point of his fresco, and that this is connected to Leonardo and Pacioli's arguments about painting and the quadrivium, we observe that Raphael seems to have copied Plato/Leonardo's left arm from a left arm at the center of Leonardo's *The Last Supper*. (Recall that Pacioli ended his argument about painting and the quadrivium with a praise of this masterpiece.) Comparing the left arm of Plato/Leonardo with the left arm of John from the *The Last Supper*, we can see that both are covered by a red cloak over the left shoulder, with darker clothes sticking out beneath, near the hand (see Figure 19). Next to these left arms, the right hand of Aristotle is held in a position similar to the right hand of Jesus. Moreover, Plato/Leonardo's pointing right hand is similar to the pointing right hand next to Jesus. Since *The Last Supper* (painted with Leonardo's "ineffable" left hand) is probably Leonardo's most famous application of the technique of linear perspective, I think these correspondences were inserted intentionally.

⁴¹ Raphael's choice of giving Plato the face of Leonardo da Vinci might also be motivated by the conflict between the views of Plato and Leonardo on the status of painting. While Leonardo argued for raising the status of painting due to its basis in mathematical principles, Plato placed painting at the bottom of a hierarchy of arts. Book 10 of Plato's *Republic* offers evidence for Plato's thoughts on this, in a comparison between a carpenter and a painter. While the carpenter imitates the ideal form of a bed when he is making a bed, the painter imitates the physical bed, and really only the appearance of the physical bed, and thus produces an imitation of an imitation, which Plato regards as being misleading and far from the true essence of a bed. The painting is twice removed from reality, as Plato puts it. Giving Plato the face of Leonardo can also be thought of as giving him the head of Leonardo, and by this Raphael could be expressing that Plato's ideas about the status of painting and the hierarchy of arts should be replaced by those of Leonardo.

Figure 19: Top: Detail from Giampietrino's 1520 copy of *The Last Supper*. Public domain image from Wikipedia (<https://commons.wikimedia.org/wiki/File:Giampietrino-Last-Supper-ca-1520.jpg>), last accessed on July 3, 2021.⁴² Bottom: Detail from *The School of Athens* (Figure 1).

According to the analysis of *The School of Athens* presented here, Raphael was engaged in a sort of self-promotion as a painter. Given the fresco's location on a wall in the Vatican palace, it is understandable that the self-promotion had to be subtle and only indicated. Like Leonardo, Raphael wanted to be seen and respected as an intellectual (although an artistic genius, he was only 27-28 when he painted the fresco). But Raphael is not only claiming that the painter is an intellectual; he is demonstrating it with inventive and subtle references to the history of ideas in addition to his mastery of pictorial composition.

⁴² I use Giampietrino's 1520 copy, the main source for the 1978–1998 restoration of the original, since the details are more distinct than in Leonardo's original.

4. Conclusion

In this article I have argued that Raphael's portrayal of Plato with the face of Leonardo should be understood as alluding to Leonardo's arguments that painting deserves to be counted as a liberal art due to its basis in mathematics. Leonardo's arguments had not been published when Raphael painted *The School of Athens*, but a version of his arguments are found in *De Divina Proportione*, a book written by Leonardo's friend, the mathematician Luca Pacioli. I argue that placing the *Timaeus* in Leonardo/Plato's hand serves several purposes. It alludes to Pacioli's *De Divina Proportione*, since the Platonic solids Leonardo drew for Pacioli's book are described in the *Timaeus*. Based on how the disciplines of the quadrivium are instantiated in the *Timaeus*, placing that book in Leonardo's hand also alludes to Pacioli's version of Leonardo's arguments, with its focus on including painting among the quadrivial arts. Raphael seems to confirm that interpretation by locating the hand holding the *Timaeus* at the vanishing point of the linear perspective in the fresco, and copying details from Leonardo's *The Last Supper*. The representation of the quadrivial disciplines, and how that is connected to the location of painters in the fresco, gives further support for this interpretation.⁴³

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