

A Non-Euclidean Story or: How to Persist When Your Geometry Doesn't

Rami Luisto

Digital Workforce Services, Finland

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A Non-Euclidean Story or: How to Persist When Your Geometry Doesn't

Rami Luisto

`rami.luisto@gmail.com`

Synopsis

Too little mathematics has been written in prose. Thus we prove here, via a fantasy novellette, that a locally L -bilipschitz mapping $f: X \rightarrow Y$ between uniformly Ahlfors q -regular, complete and locally compact path-metric spaces X and Y is an L -bilipschitz map when Y is simply connected. The motivation for such a result arises from studying the asymptotic values of BLD-mappings with an empty branch set; see e.g. [L17].

As far as the author is aware, the result is new, even though it would not be hard for specialists in the field to prove. The proof is essentially a modest extension of the ideas in [L17] in a more general setting when the branch set is empty.

This novellette is designed to be printed on A5-sheets.

DRAMATIS PERSONÆ:

Dwarves	Short, sturdy creatures fond of drink and industry.
Urist McMapmaker	A multidimensional dwarf.
Alnis Estuker	A student of Urist.
Erlin Osodshàmmándák.....	Another student of Urist.
Armok.....	The God of Blood.

DRAMATIS LOCATIONÆ

Geshud Kab	The Mountainhome
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DRUMS. Drums echo in the deep. The light is always dim here, even in the great halls furnished with everflowing falls of magma. Smaller burrows are lit with candles and strange rocks shed musty light in the dark, but outside the glowing hearts of the forges nothing is ever bright. Narrow bridges are banked by chasms falling to terrible depths and the sound of grinding rock never stops. This is a vision of comfort and happiness to the river of dwarfs inhabiting Geshud Kab, their beloved home beneath the surface.



Chapter 1

The Task of Urist

URIST had been recently promoted to the position of the royal mapmaker. Sitting in his new meager office, he studies the task appointed to him: construct a single complete map of their home. The need for such a map had been decreed to ease traffic planning within their home – spanning thousands of dwarves living and working on dozens of levels deep within the Mountain, Geshud Kab had grown into a proper dwarf fortress.

Urist's task is far from easy. Ever since their clash with the local necromancer a century ago, the standard three dimensions no longer limit the geometry of the world as a terrible curse shattered the shape of space. The extent of the distor-

tion is so far unknown; it continues as deep as they have dug, and no one has bothered to go outside to check the situation there. The space-altering curse might be reversible, but few are willing to go and ask since a necromancer tends to hold a grudge. Necromancers are up to all practical purposes immortal, but luckily the specific one here has been quiet since the latest altercation. This probably has something to do with being encased in a massive block of solid obsidian which is displayed as a major attraction near the main gates of the fort. For all outside appearances, it is just a cubical block of black rock rising 10 stories high, but it is the symbolism that counts.

Resilient as ever, the dwarves adapted and the newer generations are already at home in this multidimensional existence. Not that the concept of “dimension” makes much easy sense anymore; it took the scholars years to figure out that while the concept of “direction” was no longer present, distances and volumes still exist and the volume of any given ball of radius r seemed to be equal to r^q , up to some global multiplicative constants at least. The technical term for this is that *the fortress (X, d, μ) is a uniformly locally Ahlfors q -regular metric measure space, i.e., there exists constants $C \geq 1$ and $r_0 > 0$ such that*

$$C^{-1}r^q \leq \mu(B(x, r)) \leq Cr^q$$

for all $x \in X$, $0 < r < r_0$. This fact had immediate applications to the crucial brewing and storing industries, and the scholars were heralded as the saviors of ale and thus of dwarves.



Part of Urist's task is already done as local maps exist. All the workshops, forges, walkways and storage areas, along with their immediate surroundings, have been mapped to make work and logistics at least possible if not easy. It is standard practice for novice mapmakers to hone their skills first in making partial copies of existing maps and later on designing custom local maps for people wishing to understand the structure of their own neighbourhood. The chests and cabinets in Urist's office are filled with such local maps. Though vast, the collection is far from complete – only the gigantic central archive contains a copy of all the licenced maps manufactured since the first entrance was carved into the flesh of the Mountain – but his collection does give an adequate covering of the concurrent state of the fort. The fortress is of course continuously expanding as the miners delve greedily and deep (it is what dwarves do) but the expansions are happening in the deep mining pits away from the central hubs which are geometrically stable... more or less.

As the maps are made for dwarven logistics, the distances

between places were measured by walking distances. Conditional on the success of this project, there would be more maps commissioned, next one most likely being to model the advancement of fluids and gases in their fort, and such maps would have a different concept of distance. But for now, the natural thing to do was to measure the length of paths that a dwarf could walk. Furthermore all mining has been completed in the main hub, so any place you can get close to you can also get in to; again technically, *both the fort and the map(s) are complete path-metric spaces.*

Counting their blessings of Armok the mapmakers had at some point realized that while the geometry of the world had been heavily altered, mapmaking was still possible as any given small area could always be covered by finitely many maps, regardless of their size. For the technically oriented, their home was *a locally compact space*. Thus the major task reserved for Urist was to try and attach all these local maps together to form a single model of their home.



The benefits of 2-dimensional paper had been diminishing ever since the unexpected reality shattering incident. Besides a few of the older dwarves from the pre-necromancer days who were fond of the traditional methods, most of the literate and some of the illiterate dwarves had adapted to more

inventive forms of writing that took advantage of the reformed geometry of space. The inadequacy of 2-dimensional sheets was accentuated in the field of mapmaking. In the end, 2-dimensional piece of paper was severely limited in its possibilities to contain information by the fact that it had merely two dimensions. Thus all the mapmakers were exquisite craft-dwarves and would build their maps as small models using rock, bone, or some other preferred material. Maps on display in the meeting halls of diplomats and the royal chambers were naturally made of gold, obsidian, or precious gemstones, but the maps in Urist's office were mostly practical granite. After a swig of ale, he sets to work. Starting from the main surface gates, fused shut a century ago, he begins modifying and attaching smaller maps to an ever-increasing construction.

Deep into the night in the warm womb of the cold rock the dwarf works.



Chapter 2

The Lamentation of Urist

UERY few conscious species survived the space break heralded by the necromancer, but dwarves had been most resilient to the geometrical distress. In particular those who happened to be drunk had been even more malleable to reality shifts. As the event happened amidst a great battle during a siege, the drunken dwarves had consisted of literally all of the dwarves.

The subterranean caverns, once teeming with life and annoying pests, have since been quiet. Almost all of the animals and many of the plants had not been able to adapt to the phase shift in reality. Most mushrooms hadn't seemed to mind the new geometry for some reason, but then again, mushrooms

are rarely confused about anything – instead some of them were known to cause confusion when consumed by a careless dwarf. Thus the modern food farming focused on the several species of edible mushrooms and fungi still surviving, and thriving, with whatever little resources and geometry was left. It had been said that in the old days, certain varieties could make a dwarf feel like seeing new dimensions; in modern times the same species of mushroom was said to make you see less of them. A weird feeling for a modern dwarf, and it only went to show that even a fungus can have a sense of humor.

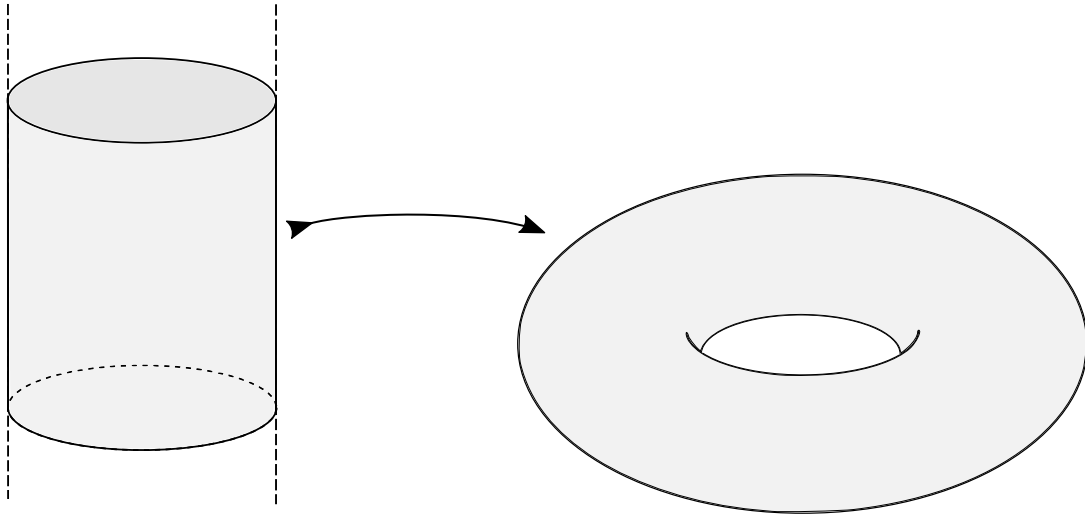


Despite outside appearances, being a mapmaker in the post-necromancer world was much more than just being a good crafts-dwarf. The modern training of a mapmaker was a hard but fascinating combination of crafts, mathematics, philosophy, and engineering. The crafts and engineering parts of Urist's work were mostly done now, but it was the maths-dwarf side of Urist that was currently under mental duress. His construction was complete and he called it *Y*. The name was onomatopoeic as prolonged visual exposure to this compact representation of the cursed geometry of their massive fort would cause most dwarves to start drooling glassy-eyed while producing a small whining sound close to “*yyyyyy...*”. The

dwarves were used to their new geometry, but it was one thing to observe the local twists of space, and another whole challenge to have the horrifying totality of the structure visible all at the same time. It was big, it was complex, and it brought dread to the minds of the restless.

Filling a huge space, but somehow seemingly containing even more volume, his creation occupied a previously vacant great hall built as a backup dining hall for a hundred dwarves. He had used every single one of the local maps given to him and, vice versa, every part of Y corresponded, at least locally, to one of the local maps. Yet Urist was not satisfied, as this was not enough to guarantee that the map was correct.

In its current geometry, the fort had some apparently endless tunnels, which were mostly harmless and often used as storage spaces, but they did require special attention and were often the first target of novice mapmakers' advanced studies. The standard fable in the classes of future mapmakers was the story of Urdim Othobottan who had constructed a hollow circular tube as a map of an endlessly long tunnel with constant width. Oh, how they always laughed at the story of Urdim. Mapping a long endless tunnel as a torus would seem fine locally since in any given point you would only observe rectangular tunnel wall around you, and still the whole map failed to represent the tunnel. Urist had to make sure he was not making the same mistake as Othobottan. The question



An endless tunnel being mapped onto the failed map of Urdim Othobottan.

was both critically important and very nontrivial.

It was a small relief that since the map was meant for large scale logistics planning, it did not need to have exact proportions. Instead it sufficed that given a distance is not distorted more than by a global multiplicative constant. The main question currently in Urist's mind therefore is: “*Is the mapping $U: X \rightarrow Y$ an L -bilipschitz mapping?*”, i.e. does it hold that for all $x, y \in X$ we have

$$L^{-1}d(x, y) \leq d(U(x), U(y)) \leq Ld(x, y)?$$

Urist knows that in small scales the answer is yes: by studying the records filed together with the local maps he knew that none of the local maps distort the distances more than by a fixed global multiplicative constant L . But this is also the

case in Othobottan's silly map. Urist needs to get hold of the global behaviour of his mapping of their home.



The nontriviality of the task is in the sheer complexity of their home, and thus of the map. Spanning in strange connections on various “directions”, the geometry of Y is too much to hold in anyone's head all at once. Thus Urist needs to deduct the global matching of the map with the terrain using the little data he has. The first thing he checked, which was possible by combining large and small scale measurements, is that just like their home the map Y has strict constraints on the volumes of balls; *the map (Y, d', μ') is a uniformly locally Ahlfors q -regular metric measure space*, i.e. there exists constants $C \geq 1$ and $r_0 > 0$ such that

$$C^{-1}r^q \leq \mu'(B(x, r)) \leq Cr^q$$

for all $x \in Y$, $0 < r < r_0$.

Another fact Urist does know is that even if the map is failing, the reason cannot be as simple as in Othobottan's case. This is because their fortress has areas which have been built to be *tether-safe*. After some unfortunate mishaps with overly powerful winches attached to safety tethers and important support pillars, it was decreed that the structure of the fort in certain critical areas must not allow in any case that a closed

loop of tether is wound around any fixed structure. In the current multidimensional environment figuring out whether or not this happens is a challenging problem, but due to its importance and the resources appointed to the study of tether-yank theory there are several methods that can be used to deduce when given structure is tether-safe. Using these techniques Urist has been able to ensure that his map is also tether-safe; i.e. *the space Y is simply connected*. All in all, in the technical lingo of mapmakers, the problem of Urist is the following:

Let $U: X \rightarrow Y$ be a locally L -bilipschitz mapping between locally compact and complete path metric spaces both equipped with a uniformly locally Ahlfors q -regular Borel measure. Furthermore suppose Y is simply connected. Is U then an L -bilipschitz mapping?

The major part of the problem can be condensed to this simple point: do there exist two disjoint locations in the fort which are represented by a single point in the map? This is exactly what goes wrong in Othobottan's project or when you try to use a circle as a map to an infinite line – locally the map looks okay but the large scale information is broken due to overlaps. At this point Urist has no choice but assume the worst: maybe there are several locations in the fort X represented by a single point in the map Y . The challenge of Urist is then to deduce additional properties of the structure

of the map Y and see if these properties lead to something impossible. All other things holding this would mean that the assumption of the overlap was, in fact, false and the map is indeed functional.

“As the bottom of a pit is there only to be excavated, the purpose of difficulties is to be overcome,” sighs a dwarf in the night.



Chapter 3

The Students of Urist

WHEN living and working in a place where deaths are often sudden drops instead of lingering diseases, most projects are worked on by groups of people who are encouraged not to walk on the same narrow bridges at the same time. This ensures that projects are rarely set back by months or years when the only person with some critical information is hospitalized by a falling rock. As the royal mapmaker, Urist was allowed to have not only one but two apprentices at the same time. For his new project these were selected among the novice mapmakers who succeeded in losing neither their composure nor their breakfast when exposed to *Y*. Alnis and Erlin are two such stout creatures fond of drink and industry, young as dwarves go and eager to work

for the prestige and excitement offered by the map project.



An old type of dwarven meditation was to find a quiet place, sit down, and try to hear amongst all the noises the quiet echoes of bygone battles and celebrations, since loud noises and songs would echo endlessly through the halls of a dwarvenhome. Thus in Geshud Kab most of talking happened in what a human might call whispers, not so much to preserve ancient echoes but to stop the pandemonium of sound from accumulating shouts. This didn't hinder communication in any degree. A quiet raspy word can contain all the colors of feeling just as you don't need a shout to convey your anger.

"Good morning, my dear Erlin. Are you prepared for another day of headache, toiling away your waking hours in front of the construct?" The jolly whisper-shout carried in the stairway leading towards Urist's office. Erlin had never been a morning dwarf, at least when compared to Alnis, and he was happy to meet his colleague as her company would surely wake him up on the long walk down to Urist's office.

"I am prepared, good Alnis, but I'm hoping my preparations are for naught. I seem to be getting more resilient for the head-wrenching effects of *the monster*." Erlin replied.

Alnis smirked, pretending to look shocked and failing. "You

do know he gets angry when you call his great creation a monster, don't you?"

Erlin was not awake enough to actually physically smile, but there was an amused spark in his eyes. "This I have indeed learned in the hardest of ways my good friend," Erlin replied with mock seriousness. "But he is not here, is he? And I can't help if that is what I think of Y in my heart of hearts."

They passed into yet another descending stairway as Erlin continued. "I do not support the ideas of those who say that Y should not be, and that anything describing the Home which is not the Home itself is an abomination, but I can sometimes sympathize with them. At least in the mornings. Or during the first week when Urist decided that we should do all our work in the same room with the map."

"That was an intense week. I could almost feel Y staring and judging us." Alnis replied.

Alnis had an almost serious expression her face for a while. "As unreasonable it is, they do have some interesting ideas why presenting something complicated within itself will create a fractal-like paradoxical loop. Based on this they claim that Y cannot be yet complete but it must not be allowed to gain completion. Even though the ideas are wrong, the pseudo-mathematical arguments might make them sound like plausible problems. Some people might even get worried."

“I think you give ‘them’ too much credit by using the word ‘they’,” Erlin said while yawning. “We are talking about two old madmen who have been chained to a wall for everyone’s safety since even before the necromancer event. I do agree that their ideas are mathematically entertaining, but I believe they are not actual *reasons* but *constructs* to have a more academical alternative to saying ‘I’m afraid of the new thing since it makes my head hurt’.”

They nodded to some acquaintances passing by before Alnis responded. “I think most would agree with you. All these clever arguments about the mathematical dangers of *Y* did seem to appear only after people saw it and got scared, didn’t they? Luckily only young children and the broken ones give much credit to people who do not make advance predictions but only make up plausible sounding excuses after the fact.”

There was a moment of silence before Alnis continued, a bit more quietly. “Talking about hindsight, though, showing the mind-wrangling construction to people with pre-existing mental conditions was quite stupid.”

“I agree. I think everyone was just too excited about it at the time.” Erlin felt a small cringe sneak into his face from feeling a bit ashamed. He had not been responsible for the viewing of *Y* to *everyone*, but he also had not realized to raise any kind of protest or given it proper thought instead of blindly

accepting a cool-sounding idea.

He pushed the guilt aside in an act of mental self-preservation and continued, “All in all, after actually studying it for a while, I’m starting to believe that there is nothing supernatural in it. The adverse effects just stem from the fact that even though we have grown accustomed to the new geometry on a local basis, seeing it all at once is literally mind boggling.”

Alnis snortled before controlling herself. “I’m sorry, I do agree with your sentiment but how do you literally boggle a mind?”

Erlin responded with a grin while Alnis continued. “In all seriousness, though, if you read the old descriptions of the necromancer event, the sickness and symptoms that most dwarves experienced are just what you would expect to happen from an extended exposure to Y .”

“Any comparisons of Y to reality-breaking cataclysms seem apt,” said Erlin, feeling more and more awake by the moment as they continued their long descent towards their master’s chambers with playful debates and bickering.

Deep into the depths the two students descend, but feel no fear as their home is solid and well-guarded.



Chapter 4

The Lifts of Urist

THE culture of dwarves is easily misunderstood. Outside the dwarven community, dwarves are often seen as stoic miners or tenacious warriors, rarely expressing any warm emotions for anything but the spilled blood of an elf. This is not an accidental misunderstanding but a deliberate public relations strategy by the eight dwarven kingdoms. As much as possible, any outside contact should encourage the image of every dwarf as a smaller copy of their fortress: a heavily armored and tough construction filled with barely contained drunken fury towards elves and drunken indifference for everything else on the topside.

This charade is easy to keep up since, besides some trusted

traders, very few living creatures are invited within a fort, exposed even then only to the stoic tradesdwarves in the upper layers of the fort, bridged and gated away from the fortress proper. And those entering uninvited may never leave, so that no one outside might know what they have seen. Thus most people are biased in knowing only the stories of the warrior dwarves, surfacing on the occasions when tensions with the elven race grow too strong, marching in the familiar dark of the nights, leaving a trail of destroyed forest and mangled elves in their wake.

The hatred between the dwarven kingdoms and the elves might also seem like a fixed part of reality, but the average dwarf who will never don non-ceremonial armor rarely feels any specific animosity towards the pointy-eared topsiders. Elves mostly appear as semi-mythical boogeymen in old stories or as evil cowards in the tall tales of old axelords, having very little effect on the everyday life the fort. But, as much as a dwarf might hate to admit it, a living tree breaks the strongest rock. Once a root appears to the ceiling of a cavern, there is potential for an uncontrolled flow of water to enter from the surface, and destroying a root only when it emerges is too late. Thus the locations of larger dwarf fortresses can be inferred from a mountainous terrain where a large area of forest has been chopped down, burned, and the whole area covered with either salt, sulfur, or preferably

both. The tensions between the two races arise when a fort is threatened by a tree with very deep roots, as tree with deep roots tends to be old, and old trees are precious to the elves.



Urist is feeling content and happy. Their mapping project was ahead of schedule and he had allowed himself some free time for personal projects. Thus he is enjoying the little things in life, spending his evening crafting a small toy forge for a newborn dwarf. The baby was born laughing and sprouting a full beard, which is considered a good omen. She might even be related to Urist, he hadn't checked. Dwarves live long lives and their resources are limited even as they are increasing and thus births are few and far apart. Partly because of this, the group of dwarves usually considered as family is less dependent on blood relations than in some other societies. The new occupant of Geshud Kab will be given gifts also by friends of her parents, but it is customary for the master craftdwarfs to congratulate the family as well.

Despite the focus on his work on the toy forge, Urist can feel his mind wandering towards his great project. He is the maker, but not the master, of *Y* and he is getting unsure even about the first part – the legends did talk about gods taking direct control of dwarfs to produce legendary items like the old iron gates of their fortress that will never rust. Even as its

possible creator Urist cannot contain all of Y in his head, but he may memorize fleeting images and instructions of routes planned within its cryptic structures. The long trips between disjoint hubs in Geshud Kab are usually travelled via routes that were discovered by trial and error in the first years after the cataclysm of geometry. They are taught by one hauling dwarf to another and people diverging from the known routes usually end up in a wrong place.

Urist and a few other mentally resilient dwarves able to withstand the effect of Y have discovered several new interhubal routes by studying Y in the past few months. Yet another nontrivial task has been to figure out if all routes so far planned this way are actually implementable in the fort as the functionality of the map is yet to be verified. But even partial functionality of the global map demonstrated by the discovery of more efficient routes has given credit to Urist and convinced the useless nobles that Urist was the right choice for the job.

Seeing as the whole point of the eldritch map Y is to enable logistics planning, the critical question is now if *any* path planned on the map would actually correspond to a route in the fort. In small scales there are no problems – at least as long as moving within a single local map. But all the interesting problems begin when we ask how to extend these paths further and outside single local maps. The question is

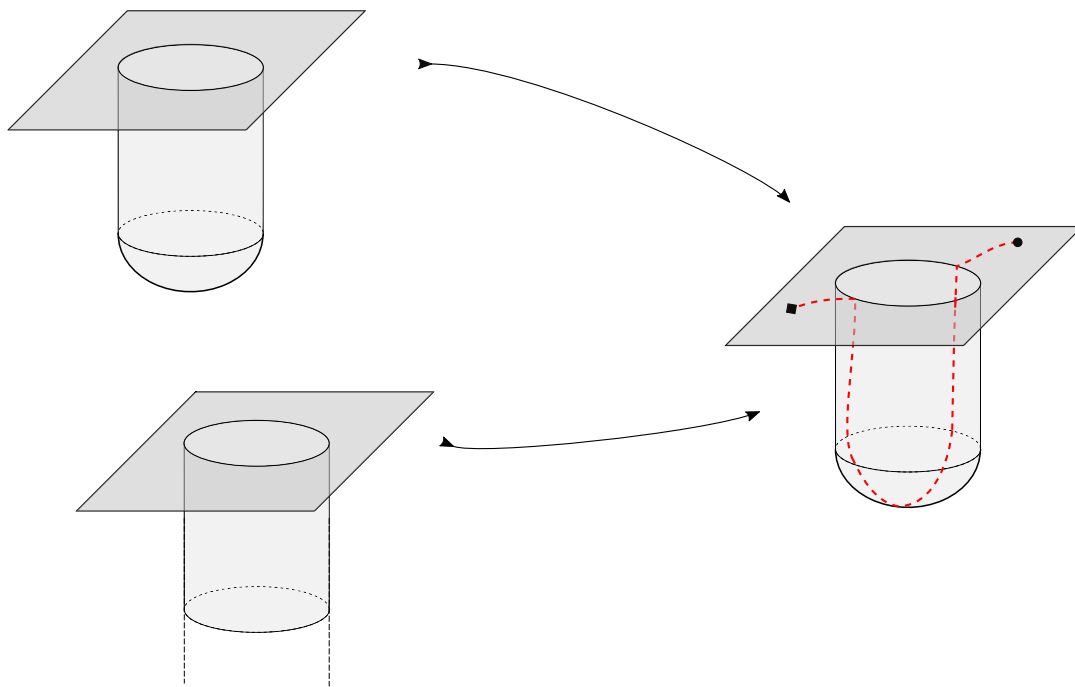
far from easy and the difficulties subtle.



The deep is always calling. Put a dwarf to the edge of a chasm with no bottom in sight and they will feel not dread but a macabre yearning. They know that there are forgotten things in the deep, usually rediscovered only for the brief moment between a surprised gasp of an explorer and the gory eradication of their expedition, but that is just how the deep is – the stygian perils an indispensable part of its beauty.

Some of the deep places in Geshud Kab are more interesting than others. After reality broke, some pits have been deemed ‘almost deep enough’ even by the more conservative dwarfs, seeing as these pits have literally unreachable depths. Philosophical debates have arisen whether these pits have no bottom or have a bottom infinitely far away. So far the question remains undecided. Besides philosophical concerns, these pits also pose a problem for the correctness of our mapping. Not from a logistical perspective, as they are rarely traversed, but they need to be understood for the whole map to be adequately comprehended.

The problem that might emerge looks as follows. Suppose we have two deep pits, *A* and *B*, the first being just deep but the second one continuing without end. If the pits had similar mouths, it could be conceivable that the mouths of



Two pits, one merely deep and one bottomless, both of whose mouths are being mapped to the same pit mouth in the map. The route planned in the map cannot be ‘lifted’ or ‘realized’ in the domain when the mouth opens up to an bottomless pit.

both of them were represented in the map by a single place, call it C . In particular, if the mouth of the pit in the map is that of a finite pit, starting a path down the pit at C in the map would ‘lift’ well along one but not the other pit, since in the bottomless pit the bottom could not be reached. In particular, starting to ‘lift’ the path in the domain where the pit is endless would result in an endless route which would map only partially onto the planned route.

For the more technically oriented person, the question is the following:

Does the map $U: X \rightarrow Y$ have *asymptotic values*?

It was in these deep pits that a large part of the current problem lay, and near such a deep pit were Urist and his two students convening.

“So. Let us talk through it once more,” Urist says to Alnis who was walking next to him.

“Of course master mapmaker,” she replied. “Though I doubt we’re quite at the same level with this problem. Would you like me to play the part of a slightly confused but clever student again?”

Urist was in a serious mood, but there was unmistakable warmth in his voice when he spoke to his student. “Please. And in this play I shall be a student only somewhat your

senior, no fancy titles or stiff manners required.”

Ekast Midon, i.e. the practice of playing roles of students or teachers, had a long tradition in the dwarven educational system, but the method was good for much more than practicing teaching or empathetic skills. Many of the masters took the habit of using it a lot in their research, not only to teach their students but to understand tough ideas themselves.

“So tell me the idea of ‘lifting’ one more time, teacher,” Alnis said. “I want to go from x_0 to x_1 in our fort X , and so I have planned a route $\alpha: [0, 1] \rightarrow Y$ with finite length in Y connecting the corresponding points $U(x_0)$ and $U(x_1)$.”

“Correct,” replied Urist, as Alnis and Erlin sat down next to Urist’s desk at the chamber of Y . “So the aim of this lifting is to construct a route $\beta: [0, 1] \rightarrow X$ in our fort, starting from x_0 and having the property that $U \circ \beta = \alpha$.”

“Sounds good. The beginning I understand; since the correspondence U is locally bilipschitz, in some neighbourhood of x_0 the correspondence U is a homeomorphism, and we can define a subroute $\beta|_{[0, t_1)}$ as $U^{-1} \circ \alpha|_{[0, t_1)}$ for some small enough parameter t_1 . But this is not enough – we want our route to cover *all* of the planned route α ,” Alnis continued.

“Indeed. So how do we continue?” Urist asked while turning his gaze towards Erlin. In or out of a playing act, Urist was very fond of the ‘teach by questioning’ method, even though

he had vivid negative memories of being the subject of such teaching in his youth.

“Well, I guess the natural thing would be to look at the point $\lim_{t \rightarrow t_1} \beta(t)$ and try to repeat the argument above for the path $\alpha|_{[t_1, 1]} \dots$ ” Erlin said.

“Why the hesitation and conditional statements?” Urist asked.

“The problem is that since I don’t yet know if these lifts are possible, I have no guarantee if the limit $\lim_{t \rightarrow t_1} \beta(t)$ exists,” Erlin responded.

“Very good. And this is an actual problem *in general*, but...” Urist urged with an extending silence directed to both of his students.

“Ah! We haven’t used yet the fact that the designed path α has finite length!!” Alnis exclaimed. “And since the mapping U is locally L -bilipschitz, the local inverses are L -Lipschitz. Thus the distance of $\beta(t)$ from x_0 is bounded by L times the distance of $\alpha(t)$ to $\alpha(0)$, which is in turn bounded by L times the length of α .”

“That’s right. And now a simple compactness and continuity arguments show that $\lim_{t \rightarrow t_1} \beta(t)$ exists. What next?” A smile was beginning to rise in Urist’s granite eyes. They had been getting closer and closer to the students actually internalizing the argument.

“Well, since we were able to repeat the starting step once, we can start repeating it over and over again. At each point the finite length bound gives rise to a limit, even after infinitely many steps. The details need to be checked, but that should give rise to a complete lift, right?” Both Alnis and Erlin were looking hopefully towards their teacher.

“It does. For the details you can use transfinite induction or, if you want to be clever, look at the supremum of those t for which the above method yields a partial lift $\beta|_{[0,t]}$. It is not hard to see that that supremum needs to be 1.” Urist said.

“We’ll do that. So now we can lift paths! Is this not enough?”

“Not quite. We now know that any plan on the map can be realized, but what if the point x_1 of the map would have several pre-images in the fort? Then you couldn’t guarantee that the path α you lifted would end up in the correct pre-image.” Urist replied. It was a subtle point and had been far from obvious to him before.

“How do we fix this?” asked Erlin.

“We don’t quite know for certain yet, but a promising direction lies in the direction of finding lifts of non-rectifiable paths as well,” Urist said.

“It seems kind of counter-intuitive that we would need to do that for a map made for logistical purposes,” Alnis said.

“It is, but there is a surprising connection between the ideas. But that is a topic for another day. For now I wish you to go through the details of what we did today, and go through some reading material to prepare you for the study of the next steps.”

Deep into the sleepless night the students toil, content under their solid sky of rock.



Chapter 5

The Asymptotics of Urist

IT was only humans who used the term “magic” in any seriousness. Their arcane practitioners would make reality their plaything by carving strange symbols and muttering powerful words which you could not understand, or even afterwards remember, no matter how hard you tried. The proud human wizard would boast on their secret knowledge and terrible power.

Elven druids, on the other hand, connected with their trees and persuaded the usually phlegmatic vegetation to move at horrifying speeds. Their archers would guide their wooden arrows with uncanny precision and heal from fatal wounds with simple herbs of the forest. The condescending elf claims

that there is nothing supernatural in this, only nature, and the elves being in tune with their forests and the living wood. The dwarves had their magma forges and legendary smiths. The exceptional properties of the artifacts forged and imbued by them were not counted as magic by their makers. The uncanny outcomes of their work were thought to be merely the result of an unrivalled craftsman's skill of the grand masters. This was far from being implausible; the longevity of dwarfkind was legendary, second only to the elves, and any grandmaster had honed their skills singlemindedly for more than a hundred years.

What the practitioners of dark arts thought and what their work contained is not spoken of, since to know the art of a necromancer is to become a necromancer and this goal is only for the mad. For those fools who ask, in words like a feverish dream one might hear the whisper “ཀམ་ལྷན་ལྷན་གང་” which is best ignored and forgotten.

The common theme in all these views of the apparent supernatural seemed to be that when you paid enough attention to a singular idea, the universe might get curious and focus on your doings as well.



“So, what have you gathered from our previous discussion?”

asked Urist.

Alnis and Erlin shared a glance, and seemed to decide that Erlin would start to describe their efforts. “Well, we noted that even without rectifiability and locally bilipschitz mappings the argument goes through as long as the limit $\lim_{t \rightarrow t_0} \beta|_{[0, t_0)}$ exists for any $t_0 \in [0, 1]$.”

“Very good,” replied Urist. “Any further notions on when such limit might exist?”

“One of the things we noted was that as long as $\beta|_{[0, t_0)}$ is bounded, the local compactness implies that the sequence $\beta|_{[0, t_0)}((1 - n^{-1})t_0)$ has an accumulation point. This combined with the fact that U is a local homeomorphism implies that the limit itself actually exists.” Erlin was showing some calculations together with their explanation.

“So you noted that the argument fails only when the path $\beta|_{[0, t_0)}$ escapes to infinity. Well done,” Urist commented with a thin smile.

“We used the term ‘drops into a pit’ instead of ‘escapes to infinity’, but yes.” Alnis intervened.

“I like the term. Anyway, anything more?” Urist asked.

“Not really, no, it seemed implausible that such a thing would

happen, as then $\beta|_{[0,t_0)}$ would drop into a pit while

$$U \circ \beta|_{[0,t_0)} = \alpha|_{[0,t_0)}$$

converges to a point $\alpha(t_0)$ when $t \rightarrow t_0$, but we couldn't quite grasp the problem formally," Erlin finished, while looking at Alnis who was shaking her head.

"Neither could I, for a long while, but there is a natural approach," Urist started while grabbing a writing implement. "We will first need to define an auxiliary concept that has turned out to be most useful. We say that a mapping f is an *L-Lipschitz Quotient mapping*, or just *LQ* for short, if for any point x_0 and any radius $r > 0$ it satisfies the following:

$$B(f(x_0), L^{-1}r) \subset fB(x_0, r) \subset B(f(x_0), Lr)."$$

"So this is some kind of weakening on the local bilipschitz requirement?" asked Alnis.

"At first yes, and we note that the condition of a mapping being LQ is weaker than being locally bilipschitz, but the crucial note here is that the LQ-condition is naturally a global one," said Urist enthusiastically.

"I think I see it; for mappings between locally compact and complete path-metric spaces if this condition is satisfied at every point x for all radii small enough, then it will be satisfied for all points and all radii, right?" Alnis asked.

“Quite so. Every locally L -bilipschitz mapping is thus a globally an L -LQ mapping,” Urist replied with satisfied voice. “And now the crucial part is this. If we have a path α dropping into a pit for $t \rightarrow t_0$, but with an existing limit $\lim_{t \rightarrow t_0} f(\alpha(t))$, then we may take a ball $B(\alpha(t), r_0)$ together with k disjoint smaller balls

$$B(\alpha(s_{t,k}), r(k)) \subset B(\alpha(t), r_0)$$

and see what happens when $t \rightarrow t_0$.”

“Like some kind of pearl necklace...” said Erlin mostly to himself.

“Indeed.” Urist replied. “And now, why would this be useful? In particular in connection with the LQ-condition?”

“Oh! By the LQ-condition all of those smaller balls will map into sets each containing a small ball of controlled radius, and so for t close enough to t_0 all of the images will have mutual intersection point,” exclaimed Erlin.

“Yes. And?” asked Urist, turning towards Alnis.

“And thus we may find for any $r_0 > 0$ and any $k \in \mathbb{N}$ a point $x = \alpha(t_k^{r_0})$ in the domain such that the multiplicity of f in $B(x, r_0)$ is at least k . But this is absurd. Or at least it feels like it should not be possible, right?” Alnis replied after a moment’s thought.

“Very good,” Urist said smilingly. “And indeed this unbounded

multiplicity is not possible for us. It's been a while since we looked at the relevant techniques, but some bilipschitz ideas can be extended for LQ-mappings in a weaker form. In particular there is for any radius R a constant K such that the multiplicity of an L -LQ-map in $B(x, R)$ is below K for any $x \in X$."

The three dwarves sat in contemplative silence for a while before Urist continued. "So, any ideas how you would go about proving such a property?"

"Maybe some sort of covering argument?" said Alnis with a slightly hesitant voice. "Since the mapping is locally L -bilipschitz any point in $B(x, r)$ had a ball-neighbourhood that maps injectively into Y , and the measure is not altered much? This would take care of the local behaviour at least."

"That's about it on the local side. You need to connect it to the more large scale behavior of the map, but the LQ-condition is your friend here," replied Urist. "It was not completely obvious, and we'll go through it with more detail tomorrow, but for now let us pretend we know it holds."

"So thus we know that for a path in Y , the lifting procedure can be started, the lift cannot drop into a pit, and whenever it terminates it can be restarted if we are not finished already," Alnis stated.

"That we indeed now know, and so any path can now be

lifted,” Urist said, leaning back on his comfy granite chair with a content smile on his lips.

For a long time did their discussions go on, as did the night.



Chapter 6

The Injectivity of Urist

ANOTHER fortress had fallen silent in the eight dwarven kingdoms. Under heavy siege the last wounded defender had sealed the entrance and collapsed the tunnels. Demoralized with the traps and pitfalls the invading forces withdrew after some weeks, still thinking that the defenders were fighting them, when in reality none were alive anymore. The young fortress, suffering from heavy casualties and sealed from the rest of the dwarven society, did not survive. In nearby human villages the people still assumed that the fort was occupied, with the occupants locked in from dangers. Citizens in Geshud Kab knew better. They knew how to listen to the rock and had somberly heard the sounds of a society wane to nothingness.



“So,” Urist began, “have you been able to revive your thoughts on the multiplicity bound techniques?”

“It seems quite good at the moment, even if a tad technical,” Alnis replied in a relaxed tone and continued. “Start with a point x_0 and a radius $R > 0$. Since f is L -LQ, we note that its image, call it V , both contains and is contained in a ball of radius comparable to R . Thus due to the Ahlfors regularity of the spaces, we know that the volume of V is comparable to R^q and thus to the volume of $B(x_0, R)$.”

The master and the students were discussing in one of the classrooms this time, the location a necessity due to the amount of formula and pictures needed for the discussion. Alnis and Erlin were once more taking turns in going through the argument. Their extended collaboration had helped them find a natural rhythm in this regard.

“Then the critical thing is to note that we know how to lift rectifiable paths without distorting the length too much. This enables us to define a bijective correspondence with the pre-images of points. In particular, for any two points $z_1, z_2 \in V$ we may take a path of length at most R connecting these points and lift it from each of the pre-images of z_1 . It is not hard to see that this correspondence gives rise to a bijection $g: f^{-1}\{z_1\} \rightarrow f^{-1}\{z_2\}$ such that $d(a, f(a)) \leq L^2 R$ for any

$a \in f^{-1}\{z_1\}.$ ”

“Good, good. And then?” asked Urist.

“And then we fix a point $z_0 \in V$, denote

$$k := \#(B(x_0, R) \cap f^{-1}\{z_0\})$$

and note that for any other point $w \in V$ we have

$$\#(B(x_0, L^2 R) \cap f^{-1}\{w\}) \geq k,”$$

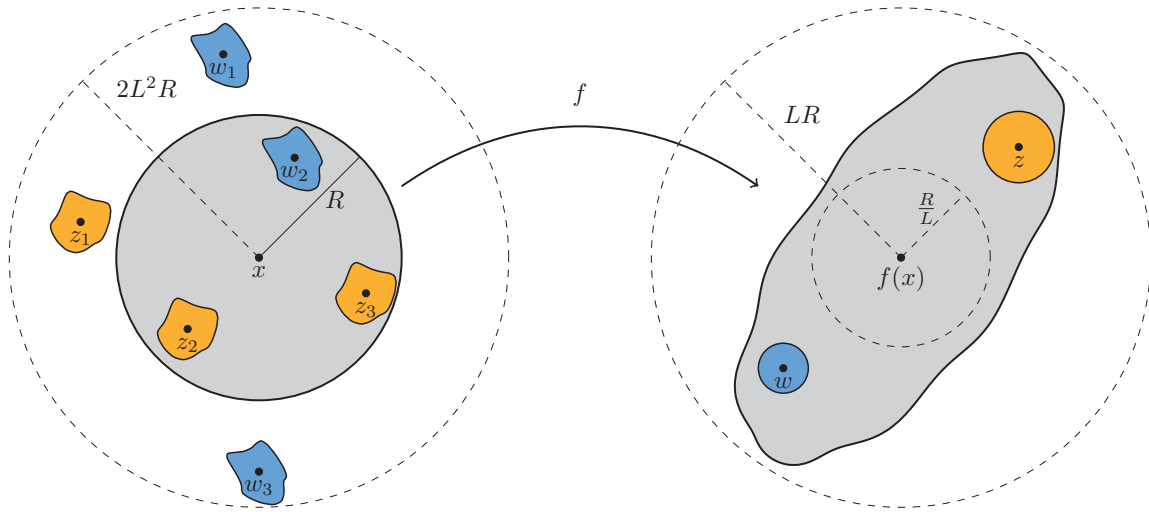
Erlin said. “The situation is kinda symmetric, so the amount of pre-images w has in $B(x_0, R)$ is similarly bounded by the number of pre-images z_0 has in the larger ball $B(x_0, L^2 R)$.”

“You’re not concerned about the pre-image being infinite?” Urist prodded.

“Well, we didn’t consider that possibility, but if the pre-image were infinite in any precompact set it would have an accumulation point whence the mapping would not be locally injective at that point,” Alnis replied after a moment’s hesitation.

“Very good, just checking,” Urist said with a grin. “So you now know that all points in V have roughly equal amount of pre-images, if you are allowed to expand the ball in the domain a bit.”

“Exactly!” Alnis pulled out some technical drawings from her bag before continuing. “Now we just take for each $y \in V$ a



Technical drawings of Alnis

radius $r_y > 0$ in such a way that the restriction of f to any of the pre-image components V_1, \dots, V_N of $f^{-1}B(y, r_y)$ that intersect $B(x_0, L^2R)$ is L -bilipschitz. In particular they will all have volume comparable to r_y^q .”

“And then you take a mutually disjoint subcover, I presume,” probed Urist.

“Yep!” Erlin replied with a grin. “We know by the regularity assumptions that the volume of V is comparable to r^q , so we find a mutually disjoint subcover of $\{B(y, r_y)\}$ with the same measure. Now after calculating some volumes, we see that the measure of $B(x_0, R)$ is comparable to Nr^q , which gives a quantitative upper bound to N .”

“Any estimates on the upper bound?” asked Urist.

“It’s bounded by something like $2L^2C^2$ where C is the multi-

plicative constant in the Ahlfors-regularity, we didn't calculate too hard since the whole aim is to show that the multiplicity is one," Alnis replied.

"Well done, both of you," Urist said with a gentle smile. He was feeling very content. The students had been properly helpful during the project, not yet autonomous but rising up to challenges in the craft. It was hard to say at this point if they had the perseverance to keep up the focus for the decades needed for mastership, but this project was a promising sign.



Besides cursing and blessing in the name of Armok, there is very little organized religion in Geshud Kab. There are several gods, all under Armok of course, together with lesser divine beings. Historical champions are also revered, as are sometimes the great beasts they triumphed – even when terrible, the deep beasts are magnificent in their might and some feel that offerings might keep them placid.

Thus it should not have surprised Urist that his creation has risen to the status of a minor deity, but surprised he is. He first thought it as a friendly prank by some fellow carvers, but several long distance haulers seem to have started to regard *Y* as a guardian spirit of sorts. He has even heard a few curses in its name in the mouths of lost dwarves: "In the name of _{yyy} where in the bright woods are we?!" It is not unheard of that

legendary constructions might be worshipped, but this tends to happen centuries after the origin of these objects, not in the span of a few years. He amuses himself on inner musings he does not wish to utter out loud in fear of blasphemy: “If gods create things and dwarves, what am I for creating a god?”



Chapter 7

The Lectio of Urist

UERY reluctantly did the mountain surrender new pieces of space to be filled with air, making all new areas very precious. Narrow were many of the tunnels, more like veins of a gargantuan being rather than passages for a small one. And the same air pumped through it all. Some disappeared through cracks to the surface, and some new cold drifts could be felt from strange depths, but most of it continuously flowed in a neverending cycle. Strange hybrids of plant and fungus purified it as it was stuffened by the breathing of the inhabitants. If ever someone not born here would be allowed entrance to the fortress proper, the breathing might have been a disgusting experience with the feel and taste of the air that had travelled through countless dwarves for centuries. For ev-

everyone currently present at Geshud Kab, however, breathing was an act of meditation towards a feeling of connectedness and unity. In the first breath you took you inhaled the air passed by your ancestors and kings, and your final exhalation returned your last airs back to your family. All would share the same air and breathe the same flow like different organs of a single body.



The throne room of Geshud Kab is a hall of greatness and great weight, both literal and figurative, tunneled and gated away from most of the fort, respectfully deep within the mountain.

Legendary gates of shining metal bar entry to the room. Of gold is the floor, covered in rich carpets woven from precious weaves. Of gold are the walls as well, filled with a thousand immaculate carvings and covered with long tapestries of delicate ringmail. Useless are the nobles gathered here, around the grand throne of indescribable lavishness and size, facing the speaker's pedestal on which Urist McMapmaker stands.

"I greet you, my Low King, and you, honoured nobles of depth," begins Urist, "I am here today to finish the story of the workings of my great construction."

"Master Urist is recognized by the court. You may tell us your

story,” says someone of no importance with a voice radiating self-importance.

“On the previous telling of my story, we were able to conclude without a chance of error that any path planned on Y , finite length or not, can be realized on our home as well,” continues Urist. Most of the listeners nod without understanding, fantasizing about telling others about being present in this grand event.

“It was a long story and not easy to follow, but we may now pick the fruits it gave us. Indeed, suppose again that there in fact was two different points, call them x_0 and y_0 , in our home that were represented by the same point z_0 in the map Y . Since anywhere can be traversed in our home, we can then take a tether $\alpha: [0, 1] \rightarrow X$ connecting these two points x_0 and y_0 . This tether can then be mapped to our map Y via the local correspondence $U: X \rightarrow Y$, as previously discussed.”

Urist had practiced his story several times, but he could still feel the tension in his body. He would often see their Low King in their fort, but rarely in his ceremonial regalia. By a very good approximation the Low King could currently be thought as a large golden sphere with a dwarven center, his eyes barely visible through some narrow gaps. He radiated authority and golden light.

“Now the mapped tether $\beta: [0, 1] \rightarrow Y$ is necessarily a *loop*

starting from the point z_0 . As per our checking that the map Y is tether safe, this implies that the tether β can be pulled in to the point z without any snatches or breaks.” Urist was accompanying his explanation with some hand gestures, clarifying the exposition to those following the argument and providing something to look at for those who didn’t.

“But by the lifting property of last time, we can actually realize the pulling back of this tether in our fort as well. Indeed for a tether-yank operation $H: [0, 1]^2 \rightarrow Y$ pulling the loop α into a point, each of the paths $H(t, \cdot): [0, 1] \rightarrow Y$, $t \in [0, 1]$, has a lift in X starting from x_0 . Since our local correspondence U is locally one-to-one, this means that the endpoints of the lifts cannot change. But this is a contradiction, since they begin at different points x_0 and y_0 for the path $H(0, \cdot)$, whereas the path $H(1, \cdot)$ is just a singular point x_0 .” Some of Urist’s pompousness was disappearing, as he was getting again excited about the ideas and forgetting that he should be important in this occasion. But a few noticed and fewer minded.

“Thus we see that under our assumptions, the mapping needs to be globally injective, and thus a bijection to its image. To see that this bijective locally bilipschitz correspondence is actually a globally bilipschitz, it suffices to remember that the mapping is an L -Lipschitz Quotient map, and thus automatically L -Lipschitz. Furthermore for any two points x and y in

the fort we see that on the one hand the images of the disjoint balls $B(x, d(x, y)/2)$ and $B(y, d(x, y)/2)$ need to be disjoint due to the injectivity of the map, and on the other hand these disjoint images both contain balls of radii $d(x, y)/(2L)$ centered at x and y , respectively. Thus we may conclude that the mapping is indeed an L -bilipschitz mapping.”



Alnis and Erlin were watching Urist describe the result of their long work, already knowing all the details but still paying their full attention to what was being said. Since they were not nobles they were naturally not allowed to attend the royal presentation, but by old traditions they were there anyway, occupying two boxes near the speaker’s stand with eyeholes cut out to the front. The speaker was not allowed to bring students with them, but having a few extra boxes with you wasn’t forbidden while everyone was pretending that there was nothing inside.

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