Steklov Eigenvalue Problems on Nearly Spherical and Annular Domains

Nathan Philip Schroeder
Claremont Graduate University

Follow this and additional works at: https://scholarship.claremont.edu/cgu_etd

Part of the Mathematics Commons

Recommended Citation
Steklov Eigenvalue Problems on Nearly Spherical and Annular Domains

BY

Nathan Schroeder

Claremont Graduate University

2024
Approval of the Dissertation Committee

This dissertation has been duly read, reviewed, and critiqued by the Committee listed below, which hereby approves the manuscript of Nathan Schroeder as fulfilling the scope and quality requirements for meriting the degree of Doctor of Philosophy in Mathematics.

Chiu-Yen Kao, Chair
Claremont McKenna College
Professor of Mathematical Sciences

Marina Chugunova
Claremont Graduate University
Professor of Mathematics

Ali Nadim
Claremont Graduate University
Professor of Mathematics
ABSTRACT

Steklov Eigenvalue Problems on Nearly Spherical and Annular Domains

BY

Nathan Schroeder

Claremont Graduate University: 2024

We consider Steklov eigenvalues on nearly spherical and nearly annular domains in \( d \) dimensions where \( d \) is any given positive integer. By using the Green-Beltrami identity for spherical harmonic functions, the derivatives of Steklov eigenvalues with respect to the domain perturbation parameter can be determined by the eigenvalues of a matrix involving the integral of the product of three spherical harmonic functions. By using the addition theorem for spherical harmonic functions, we determine conditions when the trace of this matrix becomes zero. These conditions can then be used to determine when spherical and annular regions are critical points while we optimize Steklov eigenvalues subject to a volume constraint. In addition, we develop numerical approaches based on particular solutions and show that numerical results in two and three dimensions are in agreement with our analytic results.
ACKNOWLEDGEMENTS

I would like to thank my advisor Chiu-Yen Kao for introducing me to the problems discussed in this dissertation; and for her patience, encouragement, and collaboration during my research into these problems and during the preparation of this monograph.

I would like to thank my teacher Marina Chugunova for helping me begin the process of learning how to think in an applied way.

I would like to acknowledge the contributions of Weaam Alhejaili to the study of the two-dimensional version of the problems discussed in this monograph; and in particular her contributions to initial formulations of Corollaries 6.1 and 6.2.

This material is based upon work supported by the National Science Foundation under Grant No. 2208373. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
CONTENTS

1. Introduction 1
   1.1. Introduction to the Steklov Eigenvalue Problem 1
   1.2. Historical Overview of Steklov Optimization Problems 15
   1.3. Organization and Contributions of This Work 17
2. Spherical Harmonics on Spherical and Annular Domains 20
   2.1. Steklov Eigenvalues and Eigenfunctions on Spherical and Annular Domains 20
   2.2. Overview of Spherical Harmonic Functions 27
3. Shape Derivatives of Steklov Eigenvalues 30
   3.1. Local Optimization and Shape Derivatives 30
   3.2. The EMP Matrix of a Multiple Eigenvalue 33
   3.3. Criticality and the Sub-Differential of a Multiple Eigenvalue 35
4. Nearly Spherical and Nearly Annular Domains in \( \mathbb{R}^d \) 36
   4.1. A Triple Product Integral Identity for Spherical Harmonics 36
   4.2. EMP Matrix of a Spherical Domain 37
   4.3. EMP Matrix of an Annular Domain 39
5. The Local Steklov Eigenvalue Optimization Problem 41
   5.1. Local Optimization for Spherical Domains 41
   5.2. Local Optimization for Annular Domains 43
6. Numerical Implementation and Numerical Results 45
   6.1. Method of Particular Solutions 45
   6.2. Numerical Results in \( \mathbb{R}^2 \) 48
   6.3. Numerical Results in \( \mathbb{R}^3 \) 52
7. Conclusion and Future Work 54
Appendix A. Eigenvalue/Eigenfunction Formulas for Annular Domains 57
Appendix B. Standard Orthonormal Basis for Spherical Harmonics 59
Appendix C. Calculus on Surfaces 61
   C.1. Definition of Smooth Surface 61
C.2. Definition of the Normal Vector Field to a $C^1$-surface 62
C.3. Definition of the boundary integral on a $C^1$-surface 63
C.4. Back Transport 64
Data Availability Statement 68
References 68
1. Introduction

1.1. Introduction to the Steklov Eigenvalue Problem. We consider the Steklov eigenvalue problem on a domain $\Omega \subset \mathbb{R}^d$:

\begin{equation}
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\partial_n u = \sigma u & \text{on } \partial \Omega,
\end{cases}
\end{equation}

which is a second order elliptic partial differential equation with a spectral parameter in the boundary condition. The term $\Delta u$ denotes the Laplacian of $u$ and $\partial_n u$ denotes the derivative in the direction of the outward facing normal to $\Omega$. Also, by a domain we mean a bounded, connected open set. In this thesis we study the behavior of the spectral parameter $\sigma$ when the domain $\Omega$ is subject to small perturbations by a deforming vector field $V : \mathbb{R}^d \to \mathbb{R}^d$.

We limit attention to $\Omega$ either a spherical or annular domain, in which cases we refer to the nearby perturbed domains as nearly spherical and nearly annular respectively. Specifically, in these two cases, we seek conditions on the deformation field $V$ that guarantee a particular Steklov eigenvalue is maximized on the initial unperturbed spherical or annular domain $\Omega$, compared to the corresponding eigenvalue on the nearby perturbed domains. The boundary constraint appearing in (1.1), known as the Steklov boundary condition, was introduced by Vladimir Andreevich Steklov in 1902, see [32] for a discussion of his life and work. In order to motivate its appearance, we digress and consider the sloshing problem.

Example 1.1 (The Mixed Steklov Sloshing Problem). The discussion in this example is primarily drawn from Kopachevsky and Krein [31], specifically material presented in §3.3. We also borrow from Tan et. al. [42], specifically Appendix A. We recommend the introduction of Tan et. al. [42] for a discussion of the relevance of the sloshing problem to both liquefied natural gas transport vehicles and to spacecraft design. We neglect surface tension and instead consider the irrotational flow of an incompressible, inviscid fluid contained in an impermeable container with a single boundary component free to oscillate or ”slosh”. Without surface tension, the fluid in steady state is initially motionless and the free surface
is planar. We place our coordinate system as pictured in Figure 1, with the \( x \)-axis and \( y \)-axis in the plane of the free surface and the \( z \)-axis providing an outward unit normal vector to the free surface.

![Figure 1. Fluid with a free boundary in steady state equilibrium with zero velocity field.](image)

Next imagine that the free surface is set into motion by an initial impulse and undergoes natural oscillations, as pictured in Figure 2. Then at a given time \( t \), the boundary of the oscillating fluid \( \Omega_t \) consists of two components; the free boundary component \( \Gamma_t \) and the wetted boundary component \( S_t = \partial \Omega_t \setminus \Gamma_t \).

![Figure 2. Natural oscillation of a fluid with a free boundary in a container.](image)

We let \( \rho \) and \( g \) denote the density of the fluid and the gravitational acceleration, and we assume both quantities are constant. With this assumption the conservation of mass
equation and conservation of momentum equation on $\Omega_t$ for an incompressible, inviscid fluid are given by

\begin{align*}
(1.2) \quad \text{Conservation of Mass} : \quad \nabla \cdot u &= 0 \quad \text{on } \Omega_t. \\
(1.3) \quad \text{Conservation of Momentum} : \quad \frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\frac{1}{\rho} \nabla p - \nabla (g z) \quad \text{on } \Omega_t,
\end{align*}

where $u = u(x, y, z, t)$ is a smooth fluid velocity field, $-\nabla (g z)$ is the gravitational potential, and $p$ is a smooth pressure function. Furthermore, the natural oscillation of the fluid is a small motion about steady state equilibrium of the fluid at $t = 0$, when the velocity field $u(x, y, z, 0) \equiv 0$ is identically zero.

The assumption that the fluid is irrotational implies the velocity field $u(x, y, z, t)$ is conservative and so there exists a smooth potential function $\phi = \phi(x, y, z, t)$ such that

\begin{equation}
(1.4) \quad \text{Irrotational and Incompressible Fluid Assumption} : \quad u = \nabla \phi \quad \text{on } \Omega_t.
\end{equation}

Substituting (1.4) into the conservation of energy equation (1.2) we find that the potential function satisfies Laplace’s equation on $\Omega_t$

\begin{equation}
(1.5) \quad \Delta \phi = 0 \quad \text{on } \Omega_t.
\end{equation}

Next notice that under our smoothness assumptions we have

\begin{equation}
(1.6) \quad \partial_t \nabla \phi = \nabla \phi_t.
\end{equation}

Also for an irrotational fluid we have

\[
(u \cdot \nabla) u = \frac{1}{2} \nabla |u|^2,
\]

So substituting (1.4) into the conservation of momentum equation (1.3) we obtain

\begin{equation}
(1.7) \quad \nabla \phi_t + \frac{1}{2} \nabla |\nabla \phi|^2 = -\frac{1}{\rho} \nabla p - \nabla (g z) \quad \text{on } \Omega_t.
\end{equation}
This equation may be rewritten
\[
\nabla \left( \phi_t + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{\rho} p + gz = 0 \right)
\]
which implies
\[
(1.8) \quad \phi_t + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{\rho} p + gz = h(t).
\]
Observe that on the free boundary \( \Gamma_t \) the pressure \( p(x, y, z, t) \) equals the atmospheric pressure \( p_{\text{atm}} \) and \( \zeta(x, y, t) = z \). So we make the choice \( h(t) = p_{\text{atm}}/\rho \) and simplifying equation (1.8), we obtain
\[
(1.9) \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g\zeta(x, y, t) = 0 \quad \text{on} \quad \Gamma_t.
\]
We have obtained versions of equation (1.2) on \( \Omega_t \) and of equation (1.3) on the free boundary \( \Gamma_t \) which apply to the potential \( \phi \):
\[
(1.10) \quad \text{Laplace’s Equation :} \quad \Delta \phi = 0 \quad \text{on} \quad \Omega_t.
\]
\[
(1.11) \quad \text{Bernoulli’s Law on the Free Boundary :} \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g\zeta(x, y, t) = 0 \quad \text{on} \quad \Gamma_t.
\]
Further, we observe that the assumption of impermeability is expressed by requiring the velocity has no component in the direction of the outward unit normal vector field \( \hat{n} \) on the wetted boundary
\[
\text{Impermeable Container Condition :} \quad u \cdot \hat{n} = 0 \quad \text{on} \quad S_t.
\]
Substituting (1.4) into the impermeable boundary condition equation and recalling that for smooth potential functions \( \nabla \phi \cdot \hat{n} = \partial \phi/\partial \hat{n} \) we obtain the following Neumann condition for the potential function on the wetted boundary
\[
(1.12) \quad \text{Neumann Boundary Condition for the Potential :} \quad \frac{\partial \phi}{\partial \hat{n}} = 0 \quad \text{on} \quad S_t.
\]
Next, we derive a second boundary condition on the free boundary $\Gamma_t$. We assume fluid particles on the free surface must remain on the free surface, i.e., the kinematic boundary condition. Denote the surface implicitly by $\zeta(x, y, t) - z = 0$ and use the condition (1.4). We have

\begin{equation}
(1.13) \quad \text{Kinematic Boundary Condition : } \quad \zeta_t(x, y, t) + \nabla \phi \cdot \nabla (\zeta(x, y, t) - z) = 0 \text{ on } \Gamma_t.
\end{equation}

The system of equations (1.10)-(1.13) describe the motion of liquid sloshing for unknown $\phi$ and $\zeta$.

We assuming that magnitude of the liquid motion has small amplitude and we consider the asymptotic expansions: $\zeta = \zeta_0 + \epsilon \zeta_1 + \cdots$ and $\phi = \phi_0 + \epsilon \phi_1 + \cdots$ where $\zeta_0 = 0$ is the undisturbed free surface and $\phi_0$ is a constant velocity potential. We substitute these asymptotic expansions into (1.3), collect the $O(\epsilon)$ terms, and neglect the $O(\epsilon^2)$ and higher order terms. With a slight abuse of notation, we will use $\zeta$ and $\phi$ for $\zeta_1$ and $\phi_1$ in the following. We have obtained the following system of partial differential equations expressed in terms of the fluid potential $\phi$ and the free surface displacement $\zeta$:

\begin{align*}
(1.14) & \quad \text{Laplace’s Equation for the Potential : } \quad \Delta \phi = 0 \text{ on } \Omega_t, \\
(1.15) & \quad \text{Conservation of Momentum for the Free Boundary : } \quad \frac{\partial \phi}{\partial t} = g\zeta \text{ on } \Gamma_t, \\
(1.16) & \quad \text{Neumann Boundary Condition for the Potential : } \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } S_t, \\
(1.17) & \quad \text{Kinematic Constraint for the Free Boundary : } \quad \frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z} \text{ on } \Gamma_t.
\end{align*}

As discussed in [31, §3.3.4], we model the notion of a "natural oscillation" with the following proper oscillation solution ansatz for equations (1.14) - (1.17):

\begin{align*}
(1.18) & \quad \text{Velocity Potential Ansatz : } \quad \phi(x, y, z, t) = \Phi(x, y, z)e^{i\omega t} \text{ on } \Omega_t, \\
(1.19) & \quad \text{Free Boundary Displacement Ansatz : } \quad \zeta(x, y, t) = \eta(x, y)e^{i\omega t} \text{ on } \Gamma_t.
\end{align*}
Here $\omega$ is the frequency of the proper oscillation and the spatially dependent factors $\Phi(x, y, z)$ and $\eta(x, y)$ are called proper modes. Because the time dependent factor in the fluid potential proper oscillation is never zero, it passes through the spatial differential operators in equations (1.14) - (1.16) and may be divided out, giving the following two equations for the proper potential mode.

(1.20) Laplace’s Equation for the Potential Proper Mode: \[ \Delta \Phi = 0 \text{ on } \Omega_t, \]

(1.21) Neumann Condition for the Potential Proper Mode: \[ \frac{\partial \Phi}{\partial n} = 0 \text{ on } S_t, \]

Substituting the proper oscillation ansatz equations (1.18) and (1.19) into (1.15), performing the time derivative, cancelling out the common exponential factor, and isolating the free surface displacement proper mode, we obtain

(1.22) Conservation of Momentum for the Proper Modes: \[ \frac{i\omega}{g} \Phi(x, y, z) = \eta(x, y) \text{ on } \Gamma_t. \]

Substituting the proper oscillation ansatz equations (1.18) and (1.19) into (1.17), performing the time derivative, cancelling out the common exponential factor, and isolating the velocity potential proper mode, we obtain

(1.23) Kinematic Constraint for the Proper Modes: \[ \frac{\partial \Phi}{\partial z} = i\omega \eta(x, y) \text{ on } \Gamma_t. \]

Eliminating the free surface displacement proper mode by combining (1.22) and (1.23), using the condition that $\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial n}$ in the first order expansion when the free surface displacement is small, absorbing the resulting negative sign $i^2 = -1$ into the velocity potential proper mode, and writing $\sigma = \omega^2 / g$, we obtain our final equation.

(1.24) Steklov Condition for the Potential Proper Mode: \[ \frac{\partial \Phi}{\partial n} = \sigma \Phi \text{ on } \Gamma_t. \]
Collecting together (1.20), (1.21), and (1.24) we obtain the sought after formulation of the sloshing problem which involves a Steklov boundary condition

\begin{equation}
\begin{aligned}
\text{Mixed Steklov Sloshing Equations For} & \quad \Delta \Phi = 0 \quad \text{on } \Omega_t, \\
\text{the Velocity Potential Proper Mode } \Phi & \quad \frac{\partial \Phi}{\partial n} = 0 \quad \text{on } S_t, \\
& \quad \frac{\partial \Phi}{\partial n} = \sigma \Phi \quad \text{on } \Gamma_t.
\end{aligned}
\end{equation}

A discussion of the sloshing problem would not be complete without mention of its use to analyze coffee spilling phenomenon. See, for instance, the 2012 Ig Nobel Prize winning work of Mayer and Krechetnikov [35] where the question is asked:

**Important Question:** Why, when walking with a coffee cup, does one so often spill?

Their analysis is based on the mixed Steklov sloshing equations (1.25) and their answer is two-fold:

1. Walking with a coffee cup tends to drive the fundamental proper sloshing mode.
2. The fundamental proper sloshing mode in a cylindrical container, like a coffee cup, has its ”high spot” on the boundary of the cup.

Therefore, unless preventative action is taken, 1 and 2 render a spill nearly inevitable when walking with a standard cylindrical coffee cup.

Having demonstrated that the Steklov boundary condition is not entirely theoretical, we return to the study of the Steklov eigenvalue problem (1.1). A number \( \sigma \) for which the Steklov problem has a non-trivial solution \( u \) is called a *Steklov eigenvalue of* \( \Omega \) and the collection of all Steklov eigenvalues is called the *Steklov spectrum of* \( \Omega \). Fairly general statements can be made regarding the Steklov spectrum of a bounded, connected open set subject to regularity conditions on the boundary. First, we must examine the weak Steklov problem and its relationship to the Dirichlet-to-Neumann map \( D_0 \).
Example 1.2 (Nonlocalization of the Steklov Problem via the Dirichlet-to-Neumann Map). Our discussion closely follows the account given in [33, Section 2.1 and 7.1]. Here and throughout this monograph, we make the assumption that our domain $\Omega$ has at least a $C^1$-boundary, see Appendix C.1 for a definition. Note that it is standard to only require $\Omega$ have Lipschitz boundary, but the exposition of calculus on surfaces provided in Appendix C is considerably simplified by the smoothness assumption. In addition, we reference [33] which in turn references [21, Chapter 5] where the $C^1$-boundary assumption is also used to simplify the analysis. Our goal in this section is to replace the equation (1.1) on $\Omega$ with an equivalent, nonlocal formulation on the boundary $\partial \Omega$. Having formulated the nonlocal equation, we cite a standard theorem concerning operators on Hilbert spaces, which provides detailed information about the equation’s eigenvalues as well as a regularity result for the corresponding eigenfunctions. This last regularity result shows that the spectrum of the nonlocal problem coincides with the spectrum of the classical problem, and that the eigenfunctions of either problem can be computed from the eigenfunctions of the other.

In order to formulate a weak version of (1.1), we take as test function space the Sobolev space $H^1(\Omega)$, see [33, Definition 2.1.3]. We integrate the Laplacian of a classical solution $u \in C^2(\Omega)$ against a test function $v \in H^1(\Omega)$ and then apply Green's formula [33, Lemma 2.1.13] to obtain

$$0 = \int_{\Omega} (\Delta u)v \, dV = -\int_{\Omega} \nabla u \cdot \nabla v \, dV + \int_{\partial \Omega} (\partial_n u)v \, dS \tag{1.26}$$

Next, we apply the Steklov boundary condition to the argument of the surface integral in (1.26) and obtain the following integral formulation of the problem

$$\int_{\Omega} \nabla u \cdot \nabla v \, dV - \sigma \int_{\partial \Omega} uv \, dS = 0. \tag{1.27}$$

We pause and note that because the eigenfunction $u$ is not identically zero, and both $u$ and $\nabla u$ are continuous on $\overline{\Omega}$, we have upon setting $v = u$, that both

$$\int_{\Omega} \|\nabla u\|^2 \, dV > 0 \quad \text{and} \quad \int_{\partial \Omega} u^2 \, dS > 0. \tag{1.28}$$
We conclude that the Steklov spectrum consists of positive real numbers. Indeed

\begin{equation}
\sigma = \frac{\int_{\Omega} \|\nabla u\|^2 \, dV}{\int_{\partial \Omega} u^2 \, dS} > 0.
\end{equation}

Because \( u \in C^2(\Omega) \), these computations are well defined. In particular, \( u \) restricted to the boundary, \( u|_{\partial \Omega} \), is well defined, as is the Laplacian \( \Delta u \), and the normal derivative at the boundary, \( \partial_n u \). However, we would like to weaken the regularity assumption and only require \( u \in H^1(\Omega) \); and it is here that we encounter an issue. Functions in Lebesgue spaces like \( H^1(\Omega) \) are only defined up to sets of measure zero; and the boundary \( \partial \Omega \) has Lebesgue measure zero. Therefore, for an arbitrary function \( u \in H^1(\Omega) \) the restriction to the boundary and the normal derivative at the boundary have no meaningful definition in the classical sense.

First, we deal with the problem of defining boundary values for a function \( u \in H^1(\Omega) \). What is needed is an operator \( \gamma_0 : H^1(\Omega) \to L^2(\partial \Omega) \) which assigns to each function \( u \in H^1(\Omega) \) a canonical boundary value \( \gamma_0(u) \in L^2(\partial \Omega) \). Furthermore, we require that this assignment agrees with the classical restriction \( u|_{\partial \Omega} \) when \( u \in C(\Omega) \). The following result, stated in [33, Theorem 2.1.11], shows that there is a bounded linear operator \( \gamma_0 \) on \( H^1(\Omega) \) which achieves these goals.

**Theorem 1.3** (The Trace Theorem). Let \( \Omega \) be a bounded, connected, open set with Lipschitz boundary. There exists a bounded linear operator \( \gamma_0 : H^1(\Omega) \to L^2(\partial \Omega) \), called the trace operator, such that \( \gamma_0(u) = u|_{\partial \Omega} \) when \( u \in H^1(\Omega) \cap C(\Omega) \).

Regarding functions that lie in the range of the trace operator, we note that in general they form a proper subset of \( L^2(\partial \Omega) \), and so we define

**Definition 1.4** (Boundary Trace Functions). Given \( u \in H^1(\Omega) \), we call \( \gamma_0(u) \) the boundary trace of \( u \). We denote the space of all such functions as follows

\[
\text{Boundary Trace Functions : } H^{\frac{1}{2}}(\partial \Omega) = \{ u \in L^2(\partial \Omega) : \exists \tilde{u} \in H^1(\Omega) : \gamma_0(\tilde{u}) = u \}
\]

See [8, Proposition 5.6.3] for a proof that the above definition of \( H^{\frac{1}{2}}(\partial \Omega) \) coincides with the usual definition, which makes use of the Fourier transform.
Armed with the Trace Operator, we may now formulate a weak version of the Steklov problem.

**Definition 1.5 (Weak Steklov Problem).**

We say \( u \in H^1(\Omega) \setminus \{0\} \) is a weak solution to the Steklov problem with eigenvalue \( \sigma \), if for all \( v \in H^1(\Omega) \) we have

\[
\int_\Omega \nabla u \cdot \nabla v \, dV - \sigma \int_{\partial \Omega} \gamma_0(u) \gamma_0(v) \, dS = 0.
\]

(1.30) **Weak Steklov Problem**

Notice that if \((\sigma, u)\) are an (eigenvalue, eigenfunction) pair for the classical Steklov problem (1.1), then (1.26) implies that \((\sigma, u)\) is also a solution to weak Steklov problem (1.30).

Having noted that the trace operator resolves the boundary value question for \( u \in H^1(\Omega) \) and having formulated a weak Steklov problem for \( u \in H^1(\Omega) \), we return to our project of constructing a nonlocal version of the classical Steklov problem (1.1). For this we must make sense of the Laplacian \( \Delta u \) on \( \Omega \), and of the normal derivative \( \partial_n u \) on \( \partial \Omega \) for an arbitrary \( u \in H^1(\Omega) \). With a caveat, we find the appropriate definitions in [46].

**Definition 1.6 (Weak Laplacian).** If \( u \in H^1(\Omega) \), we say that \( \Delta u \in L^2(\Omega) \) if there exists an \( f \in L^2(\Omega) \) such that

\[
\int_\Omega fv \, dV = \int_\Omega \nabla u \cdot \nabla v \, dV \quad \text{for all} \quad v \in H^1_0(\Omega)
\]

We then write \( \Delta u \) for \( f \). Here \( H^1_0(\Omega) \) is the space of \( u \in H^1(\Omega) \) which are compactly supported in \( \Omega \).

**Definition 1.7 (Weak Normal Derivative).** If \( u \in H^1(\Omega) \) and \( \Delta u \in L^2(\Omega) \), we say that \( \partial_n u \in L^2(\partial \Omega) \) if there exists a \( b \in L^2(\partial \Omega) \) such that

\[
\int_{\partial \Omega} bv \, ds = \int_\Omega (\nabla u \cdot \nabla v - (\Delta u)v) \, dV \quad \text{for all} \quad v \in H^1(\Omega)
\]

We then write \( \partial_n u \) for \( b \).

The following weak version of Green’s formula follows from these definitions.
Proposition 1.1. Given \( u \in H^1(\Omega) \) such that \( \Delta u \in L^2(\Omega) \) and \( \partial_n u \in L^2(\partial \Omega) \) we have

\[
\int_{\Omega} (\nabla u \cdot \nabla v - \Delta u \, v) \, dV = \int_{\partial \Omega} (\partial_n u \, v) \, dS \quad \text{for all } v \in H^1(\Omega).
\]

Definitions 1.6 and 1.7 make sense for arbitrary \( u \in H^1(\Omega) \) and even provide a generalization of Green’s formula. Here is the caveat. Whatever nonlocal equation we produce, it must be solved by the boundary trace of any solution to the weak Steklov problem (1.30). Therefore, if some notion of a normal derivative is used to formulate the nonlocal equation, it must be well defined for any \( u \in H^1(\Omega) \). So to employ Definition 1.7 would require \( \partial_n u \in L^2(\partial \Omega) \) for every \( u \in H^1(\Omega) \); but the following result, see [16, Theorem A.5], demonstrates that this is not true, because in general \( H^1(\partial \Omega) \subsetneq H^{\frac{1}{2}}(\partial \Omega) \).

Proposition 1.2. Suppose \( \Omega \subset \mathbb{R}^d \) is a domain with Lipschitz boundary and \( u \in H^1(\Omega) \) is such that \( \Delta u \in L^2(\Omega) \), then we have that

\[
\partial_n u \in L^2(\partial \Omega) \iff \gamma_0(u) \in H^1(\partial \Omega).
\]

Despite Proposition 1.2, we would like our nonlocal equation to express the Steklov boundary condition in an appropriate weak sense. So we will require a notion normal derivative defined for an arbitrary \( u \in H^1(\Omega) \). We overcome the obstacle presented in Proposition 1.2 by only requiring that \( \partial_n u \) be defined in sense of distribution, i.e. as a linear functional on the nonlocal equation solution space \( H^{\frac{1}{2}}(\partial \Omega) \). The exact formulation is given in definition 1.9 below. First we need to introduce some preliminary results.

Observe that if \( \tilde{u} \in H^1_0(\Omega) \) then the trace \( \gamma_0(\tilde{u}) = 0 \). This follows from the fact that \( C^1_0(\Omega) \) is dense in \( H^1_0(\Omega) \) together with the fact that the trace operator agrees with classical boundary values when they are defined. In fact \( \ker \gamma_0 = H^1_0(\Omega) \), and therefore, given \( u \in H^{\frac{1}{2}}(\partial \Omega) \) there are in general many \( \tilde{u} \in H^1(\Omega) \) with \( \gamma_0(\tilde{u}) = u \). Indeed, if \( \tilde{u} \) is such a function then \( \gamma_0(\tilde{u} + v) = u \) for any \( v \in H^1_0(\Omega) \). Despite this, making use of the weak differential operators defined above, we are able to pick out a canonical \( \tilde{u} \) which satisfies Laplace’s equation and whose boundary trace is \( u \).
**Definition 1.8** (Harmonic Extension Operator). Given \( u \in H^{\frac{1}{2}}(\partial\Omega) \) we formulate the

\[
\begin{aligned}
\Delta \tilde{u} &= 0 \quad \text{on } \Omega \\
\gamma_0(\tilde{u}) &= u \quad \text{on } \partial\Omega
\end{aligned}
\]

where \( \tilde{u} \in H^1(\Omega) \) and the Laplacian \( \Delta \tilde{u} \) is understood in the weak sense of definition (1.6). It can be shown that the weak Laplace equation has a unique solution, for instance see [23, Theorem 8.3]. We let \( \mathcal{H}_0 = \{ U \in H^1(\Omega) : \Delta U = 0 \} \) be the space of harmonic functions in \( H^1(\Omega) \), and define the *harmonic extension operator* \( \mathcal{E}_0 : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}_0 \) by \( \mathcal{E}_0(u) = \tilde{u} \) the unique solution to (1.31).

Having defined the harmonic extension operator and taking our cue from the weak Green’s formula Proposition 1.1, we are now able to define in the sense of distribution, an appropriate notion of normal derivative.

**Definition 1.9** (Normal Derivative in the Sense of Distribution). Given any \( u \in H^{\frac{1}{2}}(\partial\Omega) \) we define the linear functional \( \partial_n \mathcal{E}_0(u) \in H^{\frac{1}{2}}(\partial\Omega) \) by the pairing

\[
\langle \partial_n \mathcal{E}_0(u), v \rangle = \int_\Omega \nabla \mathcal{E}_0(u) \cdot \nabla \mathcal{E}_0(v) \, dV \quad \text{for all } v \in H^{\frac{1}{2}}(\partial\Omega).
\]

With this final definition, we have assembled the ingredients needed to define the Dirichlet-to-Neumann operator.

**Definition 1.10** (The Dirichlet-to-Neumann Operator). Given any \( u \in H^{\frac{1}{2}}(\partial\Omega) \) we define the *Dirichlet-to-Neumann operator* \( \mathcal{D}_0 : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega) \) as follows

\[
\mathcal{D}_0(u) = \partial_n \mathcal{E}_0(u).
\]

The Dirichlet boundary data \( u \) of the harmonic function \( \mathcal{E}_0(u) \) is mapped to it’s Neumann boundary data \( \partial_n \mathcal{E}_0(u) \) defined in the sense of distribution.

Making use of definition 1.10, we formulate a nonlocal version of the Steklov boundary condition.
Theorem 1.11 (The Dirichlet-to-Neumann Eigenvalue Problem). The nonlocalization of the classical Steklov boundary condition is provided by the following eigenvalue problem

\[(1.32) \quad \mathcal{D}_0u = \sigma u \quad \text{with } u \in H^{\frac{1}{2}}(\partial \Omega)\]

We have observed that an eigenvalue, eigenfunction pair \((\sigma, u)\) for the classical Steklov problem (1.1) is also a solution to the weak Steklov problem (1.30). Also, it follows from the definitions that if \((\sigma, u)\) is a solution to the weak Steklov problem then \((\sigma, \gamma_0(u))\) provides a solution to the Dirichlet-to-Neumann eigenvalue problem. It can be shown, see [14] and the references therein, that if \((\sigma, u)\) is a solution to the Dirichlet-to-Neumann eigenvalue problem, then \(u \in C^\infty(\partial \Omega)\); and, therefore, \((\sigma, \mathcal{E}_0(u))\) provides a solution to the classical Steklov problem. We conclude that all three versions, classical, weak, and nonlocal, of the Steklov problem are equivalent; and that the Dirichlet-to-Neumann eigenvalue problem is the correct nonlocalization of the classical Steklov problem.

Theorem 1.12 (The Nonlocal Steklov Problem). The nonlocalization of the classical Steklov problem (1.1) is provided by

\[(1.33) \quad \mathcal{D}_0u = \sigma u \quad \text{with } u \in H^{\frac{1}{2}}(\partial \Omega)\]

Furthermore, the spectrum of (1.1) and (1.32) coincide, while the respective eigenfunctions are \(C^\infty\) and are related through the trace operator \(\gamma_0\) and the harmonic extension operator \(\mathcal{E}_0\).

The relationship between the Steklov eigenvalue problem and the Dirichlet-to-Neumann eigenvalue problem is important on the one hand because of the numerous applications of the later to various fields such as electrical engineering, where it is known as the current to impedance map. It also plays an important role in the study of inverse scattering problems, see [41]. On the other hand, it is important for our purposes because of the coincidence of the spectral parameters described in Theorem (1.12) allows the use of any form of the Steklov equation when determining general properties of the spectrum. The relevant techniques
and facts are nicely summarized in [10, Chapter 3] where the analysis is based on the weak Steklov problem. The results are stated in the following theorem.

**Theorem 1.13** (The Classical Steklov Spectrum). Given a bounded, open $\Omega \subset \mathbb{R}^d$ with $C^1$-boundary we have that the Steklov spectrum of $\Omega$ forms a discrete set starting with the eigenvalue 0 and proceeding through an increasing sequence of finite multiplicity eigenvalues that diverge to infinity, i.e., $\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty$; and, furthermore, the corresponding eigenfunctions are $C^\infty$ and when restricted to the boundary provide a orthogonal basis for $L^2(\partial \Omega)$.

The eigenvalues also have a variational characterization,

$$
\sigma_k(\Omega) = \min_{v \in H^1(\Omega)} \left\{ \frac{\int_{\Omega} |\nabla v|^2 \, dx}{\int_{\partial \Omega} v^2 \, ds} : \int_{\partial \Omega} vu_j = 0, \ j = 0, \ldots, k-1 \right\}
$$

where $u_j$ is the corresponding $j$-th eigenfunction.

We also have the following homothety property for Steklov eigenvalues.

**Proposition 1.3** (Homothety Property for Steklov Eigenvalues). Given a bounded, open $\Omega \subset \mathbb{R}^d$ with $C^1$-boundary and a real number $h > 0$, we write $h\Omega = \{hx : x \in \Omega\}$ we have

$$
\sigma_k(h\Omega) = \frac{1}{h} \sigma_k(\Omega)
$$

**Proof.** Notice that the transformation that takes $\Omega$ to $h\Omega$ is given by

$$
T_h(x) = hI_d x \quad \text{where} \ I \text{ is the } d \times d \text{ identity matrix.}
$$

It follows that the transformation Jacobian matrix is given by

$$
DT_h = hI_d
$$

which implies that given any vector $v \in \mathbb{R}^d$

$$
(DT_h^*)^{-1}v = \frac{1}{h} v.
$$
Suppose first that $\sigma_h$ is an Steklov eigenvalue of $h\Omega$ with eigenfunction $u_h$, then applying the back transport formulas in Appendix C.4 we have

$$\nabla u_h \cdot \hat{n}_h = \sigma_h \nabla u_h \implies (DT^*_h)^{-1}u^b_h \cdot \frac{(DT^*_h)^{-1}\hat{n}}{||(DT^*_h)^{-1}\hat{n}||} = \sigma_h u^b_h$$

$$\implies \frac{1}{h} \nabla u^b_h \cdot \frac{1}{h} \hat{n} = \sigma_h u^b_h$$

$$\implies \nabla u^b_h \cdot \hat{n} = h\sigma_h u^b_h$$

We have shown

$$\sigma_h \in \text{spectrum}(\Omega_h) \implies h\sigma_h \in \text{spectrum}(\Omega).$$

An identical argument applying back transport to $T^{-1}_h$ starting with an eigenvalue $\sigma$ of $\Omega$ shows

$$\sigma \in \text{spectrum}(\Omega) \implies \frac{1}{h} \sigma \in \text{spectrum}(\Omega_h).$$

We conclude that

$$\text{spectrum}(\Omega_h) = \frac{1}{h}\text{spectrum}(\Omega)$$

which proves the result since $1/h > 0$ does not change the order of the eigenvalues.

\[\square\]

1.2. **Historical Overview of Steklov Optimization Problems.** The problem of finding a global optimizing shape for a Steklov eigenvalue $\sigma_k$ among a constrained family of admissible shapes is classic and has been studied extensively. For instance, Weinstock [45] showed that, among planar domains with fixed perimeter, the first Steklov eigenvalue $\sigma_1$ is maximized by a disk provided that $\Omega$ is simply-connected. Furthermore, for simply-connected planar domains with fixed perimeter, the $k$-th Steklov eigenvalue with $k \geq 1$ is maximized in the limit by a disjoint union of $k$ identical disks [25]. In higher dimension $d \geq 3$, Brock [15] showed that the ball maximizes $\sigma_1$ among open sets of a given volume. Ftouhi [22] considers a family of doubly connected domains of the form $\mathbb{B}_1 \setminus \mathbb{B}_2$ with $\mathbb{B}_2 \subset \mathbb{B}_1$, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are not necessarily concentric $d$-dimensional balls. It is shown that among all such domains
σ₁ is maximized uniquely when the balls are concentric. A comprehensive review of these types of Steklov eigenvalue problems can be found in [26].

An alternative type of Steklov eigenvalue problem asks when a given initial shape Ω locally optimizes a Steklov eigenvalue σₖ among all nearby shapes that are small perturbations of Ω by members of some fixed class of deformation fields. Local optimization problems, where the initial shape is either a spherical or annular domain, have lately received attention. For instance, Dambrine et al. [18] show that any ball in dimension d = 2 or d = 3 locally maximizes the first Steklov eigenvalue σ₁ under smooth, volume preserving perturbation. Viator and Osting [44, Theorem 1.1] show in dimension d = 3 that for any k = 1, 2, 3, · · · , a ball B in R³ is stationary for σₖ². In dimension d = 2, Quinones shows that an annulus is, in an appropriate sense, locally critical for σ₁ when perturbed by smooth, perimeter length preserving deformation fields, see [37, Proposition 5]. Viator and Osting rule out any disk in dimension d = 2 as a local maximizer of the Steklov eigenvalues σ₂ₖ, for k = 1, 2, 3, · · · , see the discussion following [43, Theorem 4,3].

Turning to numerical techniques, we note that to solve extremal Steklov eigenvalue problems most approaches start with an initial guess of the domain and deform it iteratively based on a gradient-ascent approach until it converges to an optimal domain. Numerical methods based on finite element approaches [13, 38] can handle complex geometries and allow adaptive meshes to improve accuracy and efficiency but they require a mesh on the whole domain Ω. To reduce the computational cost and achieve high accuracy, methods which only require discretization of the boundary ∂Ω are preferred, e.g., conformal mapping approaches [3, 4, 29, 36], boundary integral methods [2, 5], method of particular solutions (MPS) [29, 36], and method of fundamental solutions (MFS) [6, 11, 12]. Advances in these numerical techniques enabled the discovery of local maximizers of σₖ subject to a fixed volume constraint in dimension two [2, 4, 11] and higher [6]. In two dimensions, the optimal domains of σₖ looks like a ruffled edge pie dish. Comparable results are observed in both three- and four-dimensional calculations by MFS [6]. Recently, MPS methods have been
developed to solve maximal Steklov eigenvalues problem among two-dimensional surfaces with zero genus and several boundary components [29, 36].

1.3. Organization and Contributions of This Work. In this thesis, we study the local optimization problem in $d \geq 2$ dimensions where the initial domain is either spherical or annular; and the nearby perturbed nearly spherical or nearly annular domains are induced by a smooth volume preserving at first order deformation field. Analytically, we base our approach on fundamental results from Dambrine et al. [18] applied to the Steklov case. In Theorem 5.1 we generalize the result of Viator and Osting [44, Theorem 1.1], which is limited to dimension $d = 3$, and the result of Dambrine et al. [18, Theorem 1.6], which is limited to the first nonzero Steklov eigenvalue. We show that in an appropriate sense, for any dimension $d \geq 2$ any spherical domain is critical for any Steklov eigenvalue. In the same theorem, we also generalize a result discussed just after Theorem 1.6 in Dambrine et al. [18], which again is limited to the first nonzero Steklov eigenvalue. We show that when a certain matrix is nonzero, we have in any dimension $d \geq 2$ an infinite family of Steklov eigenvalues locally minimized by any spherical domain. Unfortunately, the needed assumptions about a nonzero matrix is unnatural, and so the result falls short of the goal of producing satisfactory sufficient conditions for a spherical or annular domain to locally optimize an arbitrary Steklov eigenvalue in dimension $d \geq 2$. Note that Dambrine et al. [18, Corollary 1.8] do produce a satisfactory sufficient condition, but their result is limited to dimensions $d = 2$ or $d = 3$ and only applies to the first nonzero Steklov eigenvalue. To generalize this result to higher eigenvalues in higher dimensions, a second order shape derivative would need to be computed. Although not undertaken in this thesis, we believe the conceptual and computational simplifications introduced below would render such a computation significantly more tractable, compared to the approach taken in [18].

The organization and contributions of this thesis are as follows. In Section 2, we review Steklov eigenvalues on a ball and an annular domain, referring to Appendix A, where we present general formulas for the eigenvalues and eigenvectors of annular domains in
$d$-dimensions. The remainder of section 2 reviews properties of spherical harmonic functions. In Section 3, we discuss perturbations of Steklov eigenvalues. We note that, nonzero Steklov eigenvalues for spherical and annular domains are multiple, and so split in a non-differentiable way under perturbation. We restate a result of Dambrine et al. [18, Theorem E.1] which, despite the nondifferentiability, shows the existence, after reordering, of a family of smooth eigenvalue branches emerging from a given multiple Steklov eigenvalue under perturbation. Furthermore, they show the existence of a matrix whose eigenvalues provide the slope of the tangent lines through zero of the aforementioned eigenvalue branches. For brevity and ease of reference, we refer to this matrix as the EMP matrix of a given eigenvalue.

We end our survey of [18] by recalling the notion of a subdifferential of an eigenvalue and discuss how the trace of an eigenvalue’s EMP matrix may be used to infer local optimization results. Section 4 begins with a derivation of a triple product integral identity, Proposition 4.1, for $d$-dimensional spherical harmonic functions. Based on this identity together with results from Section 2, we show for spherical domains, Theorem 4.1, and for annular domains, Theorem 4.2, that for dimension $d \geq 2$ and for any Steklov eigenvalue the entries of the EMP matrix may be computed in terms of finite sums of triple products of spherical harmonics. These representations are decisive for what follows. Numerically, because integrals of triple products may be expressed and computed in terms of the Wigner-3j symbols, see Section 6 Formula 6.5, we are able to easily produce highly accurate approximations to the eigenvalues of the EMP matrices for spherical and annular domains. Analytically, based on these representations, we are able to compute the trace of an EMP matrix only making use of basic results about spherical harmonics described in section 2; and so computations previously involving daunting technicalities, become immediate, almost trivial, when viewed from this perspective. Indeed, in Section 5 for any dimension $d \geq 2$ and for any eigenvalue, we explicitly compute the trace of the EMP matrix for a spherical domain, Proposition 5.1, and for an annular domain Proposition 5.2; and in both cases the derivation is simple and clear. We close Section 5 by proving the local criticality and local optimality for spherical domains in Theorem 5.1. We also prove the corresponding result for annular domains in
Theorem 5.2. Both theorems are straightforward consequences of the trace computations. In Section 6 numerical approaches based on MPS are described and in dimensions 2 and 3, are used to generate eigenvalue branches under various perturbations for both spherical and annular domains. The numerical results are consistent with theoretical predictions of the slope of the tangent line of the eigenvalues given by Theorems 4.1 and 4.2. Section 7 concludes the findings and discusses future work.
2. Spherical Harmonics on Spherical and Annular Domains

2.1. Steklov Eigenvalues and Eigenfunctions on Spherical and Annular Domains.

In this section we consider examples of Steklov eigenvalues and eigenfunctions for spherical and annular domains. We also offer a brief overview of spherical harmonics, highlighting some properties required for the proof of our main results. We let $B_{r_o}$ denote the $d$-dimensional open ball of radius $r_o$ and let $S_{r_o}^{d-1}$ denote its boundary the $(d-1)$-dimensional sphere of radius $r_o$. We also let $A_{r_i,r_o}$ denote the $d$-dimensional annular domain with inner radius $r_i$ and outer radius $r_o$.

$$B_{r_o}^d = \{ x \in \mathbb{R}^d : \|x\| < r_o \},$$

$$S_{r_o}^{d-1} = \{ x \in \mathbb{R}^d : \|x\| = r_o \},$$

$$A_{r_i,r_o}^d = \{ x \in \mathbb{R}^d : r_i < \|x\| < r_o \}.$$

We work in spherical coordinates $(r, \vec{\theta}) = (r, \theta_1, \cdots, \theta_{d-1})$ on $\mathbb{R}^d$ which are related to the standard Cartesian coordinates as follows

\[
\begin{align*}
x_1 &= r \cos(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{d-1}), \\
x_2 &= r \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{d-1}), \\
x_3 &= r \cos(\theta_2) \sin(\theta_3) \cdots \sin(\theta_{d-1}), \\
\vdots \\
x_{d-1} &= r \cos(\theta_{d-2}) \sin(\theta_{d-1}), \\
x_d &= r \cos(\theta_{d-1})
\end{align*}
\]

where $r \geq 0, 0 \leq \theta_1 < 2\pi, 0 \leq \theta_j \leq \pi$ for $j = 2, \cdots, d-1$.

In general, the Laplacian in Cartesian coordinates

\[
\Delta_{\mathbb{R}^d} = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}
\]
may be expressed in spherical coordinates as follows [7, formula 3.2]

\[(2.2) \quad \Delta_{\mathbb{R}^d} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_{S^{d-1}}\]

where the Laplace-Beltrami operator \( \Delta_{S^{d-1}} \) on the unit sphere \( S^{d-1} \) can be defined recursively in spherical coordinates as follows

\[(2.3) \quad \Delta_{S^1} = \frac{\partial^2}{\partial \theta_1^2} \quad d = 2 \]

\[(2.4) \quad \Delta_{S^{d-1}} = \frac{1}{\sin^{d-2}(\theta_{d-1})} \frac{\partial}{\partial \theta_{d-1}} \left( \sin^{d-2}(\theta_{d-1}) \frac{\partial}{\partial \theta_{d-1}} \right) + \frac{1}{\sin^2(\theta_{d-1})} \Delta_{S^{d-2}} \quad d \geq 3.\]

For an arbitrary domain \( \Omega \) we attempt to find a solution to Laplace’s equation in spherical coordinates making use of the ansatz

\[(2.5) \quad \text{Laplace’s Equation Separability Ansatz : } u(r, \vec{\theta}) = R(r)Y(\vec{\theta}).\]

Applying the method of separation of variables as usual, we obtain a pair of differential equations

\[(2.6) \quad \text{Laplace-Beltrami Eigenvalue Problem : } -\Delta_{S^{d-1}} Y(\vec{\theta}) = \lambda Y(\vec{\theta}), \]

\[(2.7) \quad \text{Radial Cauchy-Euler Equation : } r^2 \frac{\partial^2 R(r)}{\partial r^2} + (d - 1)r \frac{\partial R(r)}{\partial r} - \lambda R(r) = 0.\]

It is well known that the eigenfunctions of the spherical Laplace-Beltrami operator are given by the spherical harmonics. Indeed, we offer the following proposition taken directly from Atkinson-Han [7, Proposition 3.5].

**Proposition 2.1.** Non-zero functions in the vector space \( \mathcal{Y}_l^d \) of degree \( l \) spherical harmonics on \( \mathbb{R}^d \), are eigenfunctions of the Laplace-Beltrami operator \(-\Delta_{S^{d-1}}\) on \( S^{d-1} \) corresponding to the eigenvalue \( l(l + d - 2) \), \( l = 0, 1, \cdots \). The dimension \( N_{l,d} = \dim \mathcal{Y}_l^d \) is the multiplicity of the eigenvalue \( l(l + d - 2) \).
A more detailed discussion of the spherical harmonic vector spaces $\mathbb{Y}_l^d$ is given in Section 2.2. Here we note that according to [7, formula 2.10], we have

\begin{equation}
N_{l,d} = \left( \begin{array}{c}
d + l - 1 \\
-1 \\
\end{array} \right) - \left( \begin{array}{c}
d + l - 3 \\
-1 \\
\end{array} \right) = \frac{(d + 2l - 2)(d + l - 3)!}{l!(d - 2)!}.
\end{equation}

So if we let $\{Y_l^m : m = 1, \cdots, N_{l,d}\}$ denote an arbitrary orthonormal basis for $\mathbb{Y}_l^d$, the $d$-dimensional spherical harmonics of degree $l$, see [7, section 2.1]; then we have that equation 2.6 constrains the separation parameter to the values

\begin{equation}
\lambda = l(l + d - 2) \quad \text{for } l = 0, 1, 2, \cdots,
\end{equation}

and so for a particular value of $l = 0, 1, 2, \cdots$, according to the solution ansatz (2.5), the solutions in spherical coordinates to the Laplacian problem on a domain $\Omega$ take the form

\begin{equation}
R(r)Y_l^m(\vec{\theta}) \quad \text{for } m = 1, \cdots, N_{l,d}
\end{equation}

where the radial factor $R(r)$ solves the Cauchy-Euler equation

\begin{equation}
\text{Radial Cauchy-Euler Equation : } r^2 \frac{\partial^2 R(r)}{\partial r^2} + (d - 1)r \frac{\partial R(r)}{\partial r} - l(l + d - 2)R(r) = 0.
\end{equation}

For equation (2.11), the indicial equation is given by

\begin{equation}
\text{Cauchy-Euler Indicial Equation : } s^2 + (d - 2)s - l(l + d - 2) = 0.
\end{equation}

which has roots

\begin{equation}
\text{Roots of the Indicial Equation : } s = l \quad \text{and} \quad s = -(l + d - 2).
\end{equation}

Notice that when $d = 2$ and $l = 0$ the indicial equation has a multiple root $s = 0$, but otherwise the roots are real and distinct. It follows, see [1, Section 5.3], that depending on
When \( l = 0 \), \( R(r) = \begin{cases} A + B \ln(r) & \text{for } d = 2, \\ A + B \ r^{-d+2} & \text{for } d \geq 3. \end{cases} \)

When \( l \geq 1 \), \( R(r) = A \ r^l + B \ r^{-l+d-2} \) for \( d \geq 3 \).

Next, we restrict attention to the cases when \( \Omega \) is either a spherical or annular domain, and use the Steklov boundary condition to determine the constants \( A \) and \( B \) in the formula for the radial factor \( R(r) \), as well possible values of the Steklov eigenvalues \( \sigma \).

**Example 2.1 (Steklov Eigenvalues and Eigenfunctions of a Spherical Domain).**

When \( \Omega = B^d_{r_\alpha} \) is the \( d \)-dimensional ball with radius \( r_\alpha \), we have that for all values of \( d \) and \( l \), the radial factor \( R(r) \) must be defined at 0. As a consequence, for all \( d \) and \( l \) we have that the constant \( B \) in equations (2.14) and (2.15) must be zero. Therefore, for \( d \geq 2 \) and for each \( l = 0, 1, 2, \ldots \), our solution ansatz equation (2.5) has produced the following potential solutions to the Steklov eigenvalue problem (1.1)

\[
A \ r^l Y_m^l(\vec{\theta}) \quad \text{for } m = 1, \ldots, N_{l,d}.
\]

It remains to substitute the functions (2.16) into the Steklov boundary condition in order to determine the possible Steklov eigenvalues \( \sigma \) and any additional constraints on \( A \neq 0 \) required for their existence. Indeed, on the surface of the spherical domain \( B^d_{r_\alpha} \) we have that \( \partial/\partial \hat{n} = \partial/\partial r \), and so

\[
\frac{\partial A \ r^l Y_m^l(\vec{\theta})}{\partial r} = A \ l r^{l-1} Y_m^l(\vec{\theta}).
\]

It follows that satisfaction of the Steklov boundary condition requires

\[
A \ l r^{l-1} Y_m^l(\vec{\theta}) = \sigma A \ r^l Y_m^l(\vec{\theta}).
\]

Based on equation (2.17), we first observe that with \( \sigma = \frac{l}{r_\alpha} \) each of the functions in (2.16) is in fact a Steklov eigenfunction of \( \sigma \). Second, we note that beyond being nonzero, no additional
constraint is placed on $A$; and so we choose $A$ in order to normalize each eigenfunction on the boundary $S^{d-1}_{r_0}$.

We have shown that for each $l = 0, 1, 2, \cdots$, the Steklov problem on $B^d_{r_0}$, the $d$-dimensional ball with radius $r_0$, has an eigenvalue $\sigma = \frac{l}{r_0}$ which repeats with multiplicity $N_{l,d}$

$$0, \ldots, \frac{l}{r_0}, \ldots, \frac{l}{r_0}, \ldots.$$

Furthermore the eigenspace of $\sigma = \frac{l}{r_0}$ has a basis of eigenfunctions

$$u^m_l(r, \theta_1, \cdots, \theta_{d-1}) = r_0^{-\frac{d-1}{2}} \left( \frac{r}{r_0} \right)^l Y^m_l(\theta_1, \cdots, \theta_{d-1}) \quad m = 1, \cdots, N_{l,d},$$

where $\{Y^m_l : m = 1, \cdots, N_{l,d}\}$ is an arbitrary orthonormal basis for the $d$-dimensional spherical harmonics of degree $l$, see [7, section 2.1]; and we write $N(r, l, d)$ for the radial dependence $r$ of the dimension $d$ eigenfunctions of $\sigma = \frac{l}{r_0}$ orthonormalized on the boundary $S^{d-1}_{r_0}$.

**Example 2.2 (Steklov Eigenvalues of Unit Disk).** In the case of the unit disk $B^2$ in $\mathbb{R}^2$, the multiplicity $N_{l,2}$ evaluates to 1 when $l = 0$ and to 2 for all choices of $l \geq 1$. In addition, a basis for the spherical harmonics is provided by the standard exponential Fourier series functions orthonormalized on $S^1$. It follows that the Steklov eigenvalues and eigenfunctions are listed according to multiplicity as follows

$$0 < 1 \leq 1 < 2 \leq 2 < \cdots < l \leq l < \cdots$$

and the corresponding eigenfunctions are given by

$$\frac{1}{\sqrt{2\pi}}, \frac{r}{\sqrt{2\pi}} e^{-i\theta}, \frac{r}{\sqrt{2\pi}} e^{i\theta}, \frac{r^2}{\sqrt{2\pi}} e^{-i2\theta}, \frac{r^2}{\sqrt{2\pi}} e^{i2\theta}, \cdots \frac{r^l}{\sqrt{2\pi}} e^{-il\theta}, \frac{r^l}{\sqrt{2\pi}} e^{il\theta}, \cdots$$

In Figure 3 we show a view of eigenfunctions corresponding to the first four non-zero distinct Steklov eigenvalues of the unit ball $B^2$ in $\mathbb{R}^2$. 24
Example 2.3 (Steklov Eigenvalues of Unit Ball). In the case of the unit ball $B^3$ in $\mathbb{R}^3$, the multiplicity $N_{l,2}$ evaluates to $2l + 1$ for all $l \geq 0$. In addition, a basis for the spherical harmonics is provided by the standard orthonormal spherical harmonic functions described in Appendix B. It follows that the Steklov eigenvalues and eigenfunctions are listed according to multiplicity as follows

$$0 < 1 \leq 1 \leq \frac{2}{3} < 2 \leq 2 \leq 2 \leq 2 \leq \cdots < l \leq \cdots \leq \frac{2l+1}{l} \leq \cdots$$

and, using the notation of (B.1) the corresponding eigenfunctions for a given eigenvalue $l$ are given by

$$r^l Y_l^{-1}(\theta, \phi), \cdots r^l Y_l^{-1}(\theta, \phi), r^l Y_l^0(\theta, \phi), r^l Y_l^{-1}(\theta, \phi), \cdots r^l Y_l^l(\theta, \phi)$$

In Figure 4 we show an internal view of eigenfunctions corresponding to the first four non-zero distinct Steklov eigenvalues of the unit ball $B^3$ in $\mathbb{R}^3$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Steklov eigenfunctions of the unit disk $B^3$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Steklov eigenfunctions of the unit ball $B^3$.}
\end{figure}
Example 2.4 (Steklov Eigenvalues and Eigenfunctions of an Annular Domain). As with spherical domains, each eigenfunction of an annular domain can be written as a spherical harmonic multiplied by a radially dependent factor. However, for annular domains the Steklov eigenvalues are no longer conveniently ordered according to multiplicity. In general, because of the presence of a pair of independent boundary conditions, we have for each space of spherical harmonics $Y^d_l$ that there is a pair of eigenvalues $\mu_{l,1}$ and $\mu_{l,2}$. Where, for $k = 1, 2$, the eigenspace of $\mu_{l,k}$ has a basis $\{N(r_i, \mu_{l,k}, d)Y^m_l, m = 1, \cdots, N_{l,d}\}$. Here $Y^m_l, m = 1, \cdots, N_{l,d}$ is an arbitrary basis for the spherical harmonics of degree $l$, and each eigenfunction is boundary normalized by a radially dependent factor $N(r_i, \mu_{l,k}, d)$, which depends explicitly on the associated eigenvalue $\mu_{l,k}$. In Appendix A, we provide general formulas for the eigenvalues and eigenfunctions of a $d$-dimensional annular domain $A^d_{r_i,r_o}$ with inner radius $r_i$ and outer radius $r_o$.

Example 2.5 (Steklov Eigenvalues of the Annular Domain $A^2_{0.4,1}$). In Figure 5 we show a view of a selection of eigenvalues and associated eigenfunctions of the three-dimensional annular domain $A^2_{0.4,1}$ with inner radius $r_i = 0.4$ and outer radius $r_o = 1$.

![Figure 5. Steklov eigenfunctions of the annular domain $A^2_{0.4,1}$.](image-url)
Example 2.6 (Steklov Eigenvalues of the Annular Domain $A_{0.4,1}^3$). In Figure 6, we show an internal view of a selection of eigenvalues and associated eigenfunctions of the three-dimensional annular domain $A_{0.4,1}^3$ with inner radius $r_i = 0.4$ and outer radius $r_o = 1$.

![Steklov eigenfunctions of the annular domain $A_{0.4,1}^3$.](image)

**Figure 6.** Steklov eigenfunctions of the annular domain $A_{0.4,1}^3$.

2.2. Overview of Spherical Harmonic Functions. For both spherical and annular domains, spherical harmonics appear as a factor in the expressions for the Steklov eigenfunctions; and so, play an important role in the development that follows. For this reason, we briefly recount some useful properties of spherical harmonics and refer readers to [7, chapters 2 and 3] for a detailed exposition. The vector space of *degree* $l$ spherical harmonics $\mathbb{Y}_l^d$ is defined by restricting to the unit sphere $S^{d-1}$, the polynomials on $\mathbb{R}^d$ that are both harmonic and homogeneous of degree $l$ [7, definition 2.7]. Where a polynomial $P : \mathbb{R}^d \to \mathbb{C}$ is harmonic if it satisfies Laplace’s equation $\Delta P(\vec{x}) = 0$ for all $\vec{x} \in \mathbb{R}^d$; and homogeneous of degree $l$ if it satisfies $P(t\vec{x}) = t^lP(\vec{x})$ for all $t \in \mathbb{R}, \vec{x} \in \mathbb{R}^d$. Here and below, we let $\{Y_l^m : l = 0, 1, \cdots; m = 1, \cdots, N_{l,d}\}$ denote an arbitrary orthonormal basis for the vector space of all $d$-dimensional spherical harmonics $\bigoplus_{l=0}^{\infty} \mathbb{Y}_l^d$. Orthonormality is defined with
respect to the Hilbert space $L^2(\mathbb{S}^{d-1})$, i.e.,

$$
(2.18) \quad \int_{\mathbb{S}^{d-1}} Y_{l_1}^{m_1} Y_{l_2}^{m_2} dS = \delta_{l_1,l_2} \delta_{m_1,m_2}.
$$

Note that the spherical harmonics $\bigoplus_{l=0}^{\infty} \mathbb{Y}_l^d$ are complete in $L^2(\mathbb{S}^{d-1})$, [7, Theorem 2.38]; and so any $f \in L^2(\mathbb{S}^{d-1})$ has a Fourier-Laplace expansion in spherical harmonics, i.e., for some choice of coefficients $\alpha_{l,m} \in \mathbb{C}$ we have

$$
(2.19) \quad f = \sum_{l=0}^\infty \sum_{m=1}^{N_{l,d}} \alpha_{l,m} Y_l^m.
$$

It follows from the definitions that the degree 0 spherical harmonics $\mathbb{Y}_0^d$ consist of complex-valued constant polynomials; and the requirement that the basis be normalized in $L^2(\mathbb{S}^{d-1})$ implies that for any basis we have $Y_0^1 = \frac{1}{\sqrt{\omega_{d-1}}}$, where $\omega_{d-1}$ is the surface area of the unit sphere $\mathbb{S}^{d-1}$. This observation allows us to evaluate the integral of any spherical harmonic basis element as follows.

**Proposition 2.2.** For any $l = 0, 1, 2, \cdots$ and $1 \leq m \leq N_{l,d}$ we have that

$$
(2.20) \quad \int_{\mathbb{S}^{d-1}} Y_l^m \, dS = \begin{cases} 
\sqrt{\omega_{d-1}} & \text{when } l = 0, \\
0 & \text{otherwise}.
\end{cases}
$$

*Proof.* If $l = 0$ then

$$
\int_{\mathbb{S}^{d-1}} Y_0^1 \, dS = \int_{\mathbb{S}^{d-1}} \frac{1}{\sqrt{\omega_{d-1}}} \, dS = \frac{\omega_{d-1}}{\sqrt{\omega_{d-1}}} = \sqrt{\omega_{d-1}},
$$

while if $l > 0$ we have by orthonormality, see (2.18), that for any $m = 1, \cdots, N_{l,d}$

$$
\int_{\mathbb{S}^{d-1}} Y_l^m \, dS = \sqrt{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} Y_0^1 Y_l^m \, dS = 0.
$$

$\square$
It also follows from the definition that the product of any two spherical harmonics of degree \( l \) is a homogeneous polynomial of degree \( 2l \); and so, in particular, we have the following expansion result.

**Proposition 2.3.** The product of any two spherical harmonics in \( \mathbb{Y}_l^d \) admits an expansion in spherical harmonics of even degree bounded above by \( 2l \), i.e., for all \( 1 \leq m_1, m_2 \leq N_{l,d} \) we have

\[
Y_{l_1}^{m_1} Y_{l_2}^{m_2} = \sum_{l' = 0}^{2l} \sum_{m' = 1}^{N_{l',d}} c_{l',m'} Y_{l'}^{m'}
\]

for some choice of coefficients \( c_{l',m'} \in \mathbb{C} \).

**Proof.** See [7, Theorem 2.18 and Corollary 2.19].

We also make use of a special case of the Addition theorem, which in three dimensions is known as Unsöld’s Theorem, and in the general case takes the following form.

**Proposition 2.4.** Given any \( l = 0, 1, 2, \cdots \) and any point \( \bar{\theta} \in S^{d-1} \) we have that

\[
\sum_{m=1}^{N_{l,d}} Y_{l}^{m}(\bar{\theta}) Y_{l}^{m}(\bar{\theta}) = \frac{N_{l,d}}{\omega_{d-1}}.
\]

**Proof.** See [7, formula 2.35].

We close this review of spherical harmonics by the following version of Greens Theorem on the unit sphere \( S^{d-1} \)

**Proposition 2.5.** (Green-Beltrami Identity) Given smooth functions \( f : S^{d-1} \to \mathbb{C} \) and \( g : S^{d-1} \to \mathbb{C} \) we have

\[
\int_{S^{d-1}} g \Delta f \, dS = - \int_{S^{d-1}} \nabla f \cdot \nabla f \, dS.
\]

**Proof.** See [7, Proposition 3.3].
3. SHAPe DERIVATIVES OF STEKLOV EIGENVALUES

3.1. Local Optimization and Shape Derivatives. We consider the problem of optimizing Steklov eigenvalue shape functionals among domains which are small perturbations of either a ball or an annular domain in $\mathbb{R}^d$. In general, given a shape $\Omega$ and a deformation field $V : \mathbb{R}^d \to \mathbb{R}^d$, we define for small $t \in \mathbb{R}$ the perturbed shape $\Omega_t = \{x + tV(x) : x \in \Omega\}$. For a given shape functional $J$ and shape $\Omega$, the local shape optimization problem seeks conditions on a deformation field $V$ that guarantees $\Omega$ optimizes $J$ when restricted to the perturbation $\{\Omega_t\}$; i.e.,

$$\Omega = \arg\max_{t \in (-\delta, \delta)} J(\Omega_t) \quad \text{or} \quad \Omega = \arg\min_{t \in (-\delta, \delta)} J(\Omega_t).$$

In this case we say that the pair $(\Omega, V)$ locally optimizes $J$. In the event that for all $t \in (-\delta, \delta)$ we have $J(\Omega) > J(\Omega_t)$ or alternatively $J(\Omega) < J(\Omega_t)$, then we say $(\Omega, V)$ locally strictly maximizes $J$ or $(\Omega, V)$ locally strictly minimizes $J$ respectively.

The function $J(\Omega, V)$ defined by the correspondence $t \mapsto J(\Omega_t)$ may be differentiable; and we write $dJ(\Omega, V)$ for its derivative at zero when it exists; i.e.,

$$dJ(\Omega; V) = \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

and say is $J$ locally differentiable at $(\Omega, V)$. Clearly, by Fermat’s theorem, if $dJ(\Omega, V) \neq 0$ then $(\Omega, V)$ cannot locally optimize $J$. We say $(\Omega, V)$ is critical for $J$ if either $dJ(\Omega; V) = 0$ or $dJ(\Omega, V)$ is not defined. In general, the determination of pairs $(\Omega, V)$ critical for $J$ is a natural way to start investigating a local shape optimization problem.

We present some examples of shape functionals and their Eulerian derivatives. But first we need some additional definitions which, with the exception of the first, require that we assume the domain $\Omega \subset \mathbb{R}^2$ has a $C^2$-boundary. With this assumption any function $u \in C^1(\partial \Omega)$ or vector field $v \in C^1(\partial \Omega)^d$ has a canonical $C^1$ extension to a tubular neighborhood of $\partial \Omega$. Such an extension may be constructed for a $C^2$-boundary by making use of the projection operator defined by the signed distance function to $\partial \Omega$ restricted to a suitable tubular neighborhood of $\partial \Omega$, see [19, Section 9.5]. Tangential differential operators on functions $u$ or vector fields $v$
on the surface $\partial \Omega$, may then be defined in terms of the standard differential operators applied to canonical extensions defined on a tubular neighborhood of $\partial \Omega$. We will also require that any deformation field $V : \mathbb{R}^d \to \mathbb{R}^2$ is at a minimum $C^2$, so that the perturbed domains are also $C^2$. See Appendix C.1.

Given a deformation field $V : \mathbb{R}^d \to \mathbb{R}^d$, we define the normal vector field $V_n$ on $\partial \Omega$ as

\begin{equation}
\text{(3.1) Normal Vector Field on } \partial \Omega \quad V_n = V \cdot \hat{n}
\end{equation}

where $\hat{n}$ is the unit normal vector field on $\partial \Omega$. Define the mean curvature $H$ of $\partial \Omega$ to be the sum of the principal curvatures $\kappa_i$; $i = 1, \cdots, d - 1$ of $\partial \Omega$

\begin{equation}
\text{(3.2) Mean Curvature of } \partial \Omega \quad H = \sum_{i=1}^{d-1} \kappa_i.
\end{equation}

Notice that, as defined here, the "mean curvature" is not the average of the principal curvature. This form turns out to be more useful for shape derivative applications. See [19, Section 3.3]. Given $u \in C^1(\partial \Omega)$ define the tangential gradient $\nabla_\tau$ on $\partial \Omega$ to be

\begin{equation}
\text{(3.3) Tangential Gradient on } \partial \Omega \quad \nabla_\tau u = \nabla U - (\partial_\hat{n} U) \hat{n}
\end{equation}

where $U$ is the canonical extension of $u$ to a tubular neighborhood of $\partial \Omega$. See [19, Definition 9.5.1]. Given $v \in C^1(\partial \Omega)^2$ define the tangential divergence $\text{div}_\tau$ on $\partial \Omega$ to be

\begin{equation}
\text{(3.4) Tangential Divergence on } \partial \Omega \quad \text{div}_\tau v = \text{div} V |_{\partial \Omega} - (DV \hat{n}) \cdot \hat{n}
\end{equation}

where $V$ is the canonical extension of $v$ to a tubular neighborhood of $\partial \Omega$ and $DV$ is the Jacobian matrix of $V$. See [19, Section 9.5.2]. Given $u \in C^1(\partial \Omega)$ define the Laplace-Beltrami operator $\Delta_\tau$ on $\partial \Omega$ to be

\begin{equation}
\text{(3.5) Laplace-Beltrami Operator on } \partial \Omega \quad \Delta_\tau u = \text{div} (\nabla_\tau u)
\end{equation}

See [19, Section 9.5.3].
Example 3.1 (Surface Area Shape Functional). Let $\omega$ be a bounded, open set with $C^2$-boundary and $V : \mathbb{R}^d \to \mathbb{R}^d$ a $C^1$ deformation field, then the surface area shape functional

$$J(\Omega) = \int_{\partial \Omega} dS,$$

has Eulerian derivative

$$J(\Omega; V) = \int_{\partial \Omega} HV_n dS.$$

We generally write $|\partial \Omega|$ for the surface area of a shape $\Omega$. When $d|\partial \Omega|(\Omega; V) = 0$ we say $V$ is surface area preserving at first order on $\Omega$. See [19, Section 9.4.3].

Example 3.2 (Volume Shape Functional). Let $\omega$ be a bounded, open set with $C^2$-boundary and $V : \mathbb{R}^d \to \mathbb{R}^d$ a $C^2$ deformation field, then the volume shape functional defined by

$$J(\Omega) = \int_{\Omega} dV,$$

has Eulerian derivative

$$dJ(\Omega; V) = \int_{\partial \Omega} V_n dS.$$

We generally write $|\Omega|$ for the volume of a shape $\Omega$. When $d|\Omega|(\Omega; V) = 0$ we say $V$ is volume preserving at first order on $\Omega$. See [19, Section 9.4.3].

Example 3.3 (Simple Steklov Eigenvalue Shape Functional). Let $\Omega$ be a bounded, open set with $C^2$-boundary and $V : \mathbb{R}^d \to \mathbb{R}^d$ a $C^2$ deformation field. Define $\sigma_k(\Omega)$ to be the $k^{th}$ Steklov eigenvalue of the domain $\Omega$ when the eigenvalues are ordered according to multiplicity, and let $u$ be an eigenfunction for $\lambda_k(\Omega)$, then the shape functional defined by

$$J(\Omega) = \lambda_k(\Omega),$$

which we generally denote by $\lambda_k$, has at a domain $\Omega$ where $\lambda_k(\Omega)$ is simple, Eulerian derivative

$$d\lambda_k(\Omega; V) = \int_{\partial \Omega} (|\nabla_{\tau} u|^2 - |\nabla_n u|^2 - \lambda_k Hu^2) V_n dS$$

See [18, Theorem E.1].
3.2. The EMP Matrix of a Multiple Eigenvalue. Returning to the specific case of a ball or an annular domain in $\mathbb{R}^d$, we note that unfortunately, when $\Omega$ is either a ball or an annular domain, the multiplicity of the Steklov eigenvalue $\sigma_n(\Omega)$ is greater than one and eigenvalue multiplicity is not preserved under perturbation. As a consequence, depending on the deformation field, the shape functionals $\sigma_n$ are not differentiable at 0, see [28, section 2.5]. Therefore, most pairs $(\Omega, V)$ are critical for $\sigma_n$ simply because $d\sigma_n(\Omega, V)$ does not exist. Despite this fact, the notion of local differentiability can still be used to investigate local optimization problems. To explain how, we need some additional definitions. Given a perturbation $\{\Omega_t\}$ generated by $(\Omega, V)$, we introduce the eigenvalue branch functions $\lambda_0, \lambda_1, \lambda_2, \ldots$ defined by the rule that for each perturbation parameter $t$, we have that $\lambda_0(t) \leq \lambda_1(t) \leq \lambda_2(t) \leq \cdots$ enumerate the Steklov eigenvalues of $\Omega_t$ repeated according to multiplicity, i.e. $\lambda_k(t) = \sigma_k(\Omega_t)$. Next, we define the index of an arbitrary Steklov eigenvalue $\sigma(\Omega)$ to be the smallest integer $n$ such that $\sigma(\Omega) = \sigma_n(\Omega)$. It follows from the definitions that if $\sigma(\Omega)$ has index $n$ and multiplicity $p$ then $\lambda_n(0) = \lambda_{n+1}(0) = \cdots = \lambda_{n+p-1}(0) = \sigma$, in this case we call the functions $\lambda_n(t), \ldots, \lambda_{n+p-1}(t)$ the eigenvalue branches of $\sigma(\Omega)$. Note that some or all of these eigenvalue branches may coincide.

Example 3.4. The situation is illustrated in Figure 7 where the unit ball $B^3$ is radially perturbed by the real spherical harmonic $Y_{2,1}$ according to the correspondence

$$V : \begin{bmatrix} r \\ \theta \end{bmatrix} \mapsto \begin{bmatrix} r \\ \theta \end{bmatrix} + \begin{bmatrix} rY_{2,1}(\theta) \\ 0 \end{bmatrix}.$$

In Figure 7, $\sigma = 1$ has index 1 and multiplicity 3 with eigenvalue branches $\lambda_1(t), \lambda_2(t), \lambda_3(t)$. Also observe that for each perturbation parameter $t$ we have that $\sigma_1(\Omega_t) = \lambda_1(t)$; and so, since $\lambda_1(t)$ is not differentiable at 0, we have an example where $d\sigma_1(B^3, V)$ fails to exist. Finally observe that there is a reordering of the branches on the left-hand side of the eigenvalue, $\sigma = 1$ that will result in new branch functions $\tilde{\lambda}_k$ that are differentiable at 0. For
instance, if we let $s$ denote the permutation $1 \mapsto 3, 2 \mapsto 2, 3 \mapsto 1$ and define for $k = 1, 2, 3$.

$$\tilde{\lambda}_k(t) = \begin{cases} 
\lambda_k(t) & t \geq 0, \\
\lambda_{s(k)}(t) & t < 0.
\end{cases}$$

Then each of the functions $\tilde{\lambda}_k(t)$ is differentiable at 0. Notice that $\tilde{\lambda}'_1(0) < 0$ and $\tilde{\lambda}'_3(0) > 0$, and so, from the fact that $\sigma_1(\Omega_t) = \lambda_1(t)$ and $\sigma_3(\Omega_t) = \lambda_3(t)$, we can infer that $(\mathbb{B}^3, V)$ locally strictly maximizes $\sigma_1$ and locally strictly minimizes $\sigma_3$.

The next theorem shows that these observations are neither mistaken nor specific to this particular perturbation. This result is a reformulation of parts of Theorem E.1 of Dambrine-Kateb-Lamboley [18].

**Theorem 3.5.** Let $\Omega$ be bounded, open, and have Lipschitz boundary, and $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$. Suppose $\sigma$ is a Steklov eigenvalue of $\Omega$ with multiplicity $p$ and index $n$. Let the functions $\{u_i : \overline{\Omega} \to \mathbb{C} : i = 1, \ldots, p\}$, be a basis for the Steklov eigenspace of $\sigma$, orthonormalized with respect to the $L^2(\partial \Omega)$ inner product. Then we have
• If \( \lambda_n(t), \cdots, \lambda_{n+p-1}(t) \) are the eigenvalue branches of \( \sigma \), then there exists a permutation \( s \) of \( \{n, \cdots, n+p-1\} \) such that for \( k = n, \cdots, n+p-1 \) the shape functional

\[
\tilde{\lambda}_k(t) = \begin{cases} 
\lambda_k(t) & t \geq 0, \\
\lambda_{s(k)}(t) & t < 0,
\end{cases}
\]

is differentiable at 0; i.e. \( d\tilde{\lambda}_k(\Omega; V) \) exists.

• The derivatives \( d\tilde{\lambda}_k(\Omega; V) \), \( k = n, \cdots, n+p-1 \) are equal to the eigenvalues of the \( p \times p \) matrix \( M = M(\Omega, V, \sigma) \) with entries defined by

\[
M_{ij} = \int_{\partial\Omega} (\nabla_\tau u_i \cdot \nabla_\tau \bar{w}_j - (\sigma^2 + \sigma H) u_i \bar{w}_j) V_n \, dS.
\]

Here \( \nabla_\tau \) is the tangential gradient, \( H \) is the additive curvature, i.e., the sum of the principal curvatures; and \( V_n \) is the normal component of the deformation field \( V \) on the boundary. We refer to \( M \) as an eigenvalue multiplicity perturbation matrix or EMP matrix for short.

3.3. Criticality and the Sub-Differential of a Multiple Eigenvalue. Based on Theorem 3.5, Dambrine et. al [18] introduce the sub-differential \( \partial\sigma[\Omega, V] \) of the eigenvalue \( \sigma(\Omega) \) as follows

\[
\partial\sigma[\Omega, V] = \left[ \min_{k \in \{n, \cdots, n+p-1\}} \tilde{\lambda}_k'(\Omega, V)(0), \max_{k \in \{n, \cdots, n+p-1\}} \tilde{\lambda}_k'(\Omega, V)(0) \right].
\]

Notice that if \( \sigma(\Omega) \) has index \( n \) and multiplicity \( p \) and \( 0 \notin \partial\sigma[\Omega, V] \) then \( (\Omega, V) \) cannot locally optimize any of the shape functionals \( \sigma_n, \cdots, \sigma_{n+p-1} \). On the other hand if \( 0 \in \partial\sigma[\Omega, V] \) and \( \min_k \tilde{\lambda}_k'[\Omega, V](0) < 0 \) while \( \max_k \tilde{\lambda}_k'[\Omega, V](0) > 0 \) then as in Example 3.4, we may conclude that \( (\Omega, V) \) locally strictly maximizes \( \sigma_n \) and locally strictly minimizes \( \sigma_{n+p-1} \). Based on these observations we define \( (\Omega, V) \) to be critical for \( \sigma(\Omega) \) provided \( 0 \in \partial\sigma[\Omega, V] \). Finally, because the trace of a matrix is the sum of its eigenvalues, we have that if the trace of the EMP matrix \( M(\Omega, V, \sigma) \) is zero, then we must have that \( (\Omega, V) \) is critical for \( \sigma(\Omega) \). In what follows we seek a condition on \( V \) which implies that the trace \( [M(\Omega, V, \sigma)] = 0 \), which in turn implies that \( (\Omega, V) \) is critical for \( \sigma(\Omega) \).
4. Nearly Spherical and Nearly Annular Domains in $\mathbb{R}^d$

4.1. A Triple Product Integral Identity for Spherical Harmonics. We first establish a triple product integral identity for spherical harmonic functions. The result is based on the Green-Beltrami Identity Proposition 2.5, together with the fact that on the unit sphere $S^{d-1}$, the spherical harmonics are simultaneously eigenfunctions for the surface Laplacian as well as the the Steklov boundary condition, i.e. for any spherical harmonic function $Y^m_l$ we have

\begin{equation}
\Delta_\tau Y^m_l = -l(l - d - 2)Y^m_l,
\end{equation}
\begin{equation}
\partial_n Y^m_l = lY^m_l.
\end{equation}

This fact seems to be far from coincidental, see [24] for a discussion of this point.

**Proposition 4.1. (Triple Product Integral Identity)** Given any spherical harmonic basis element of degree $l$ and any pair of spherical harmonic basis elements of degree $n$ we have

\begin{equation}
\int_{S^{d-1}} Y^m_l \nabla_\tau Y^i_n \cdot \nabla_\tau \overline{Y^j_n} \, dS = \left( n(n + d - 2) - \frac{l(l + d - 2)}{2} \right) \int_{S^{d-1}} Y^m_l Y^i_n \overline{Y^j_n} \, dS.
\end{equation}

**Proof.** Applying Formula (4.1), Proposition 2.5, and the product rule for surface gradients, we have

\[
\int_{S^{d-1}} Y^m_l \nabla_\tau Y^i_n \cdot \nabla_\tau \overline{Y^j_n} \, dS = \int_{S^{d-1}} \nabla_\tau Y^i_n \cdot \nabla_\tau (\overline{Y^j_l} Y^m_l) \, dS - \int_{S^{d-1}} \overline{Y^j_l} \nabla_\tau Y^i_n \cdot \nabla_\tau Y^m_l \, dS
\]
\[
= -\int_{S^{d-1}} \overline{Y^j_l} Y^m_l \Delta_\tau Y^i_n \, dS - \int_{S^{d-1}} \overline{Y^j_l} \nabla_\tau Y^i_n \cdot \nabla_\tau Y^m_l \, dS
\]
\[
= n(n + d - 2) \int_{S^{d-1}} Y^m_l Y^i_n \overline{Y^j_n} \, dS - \int_{S^{d-1}} \overline{Y^j_l} \nabla_\tau Y^i_n \cdot \nabla_\tau Y^m_l \, dS
\]
\[
= n(n + d - 2) \int_{S^{d-1}} Y^m_l Y^i_n \overline{Y^j_n} \, dS - \int_{S^{d-1}} Y^i_n \nabla_\tau \overline{Y^j_l} \cdot \nabla_\tau Y^m_l \, dS.
\]

where the last equality follows from the fact that the previous three equalities are symmetric in $Y^i_n$ and $\overline{Y^j_l}$. From this last equality we deduce

\[
\int_{S^{d-1}} Y^i_n \nabla_\tau \overline{Y^j_l} \cdot \nabla_\tau Y^m_l \, dS = \int_{S^{d-1}} \overline{Y^j_l} \nabla_\tau Y^i_n \cdot \nabla_\tau Y^m_l \, dS.
\]
A result which we apply to obtain the last equality in the derivation of the next formula.

\[
\int_{S^{d-1}} Y_i^i \nabla_\tau \overline{Y}_j^j \cdot \nabla_\tau Y_l^m dS = \int_{S^{d-1}} \nabla_\tau \left( Y_i^i \overline{Y}_j^j \right) \cdot \nabla_\tau Y_l^m dS - \int_{S^{d-1}} Y_i^i \nabla_\tau \nabla_\tau Y_l^m dS
\]

\[
= - \int_{S^{d-1}} Y_i^i \nabla_\tau Y_l^m dS - \int_{S^{d-1}} \overline{Y}_j^j \nabla_\tau Y_l^m dS
\]

\[
= l(l + d - 2) \int_{S^{d-1}} Y_l^m Y_i^i dS - \int_{S^{d-1}} Y_l^m \nabla_\tau Y_i^i dS
\]

It follows that

\[
\int_{S^{d-1}} Y_i^i \nabla_\tau \overline{Y}_j^j \cdot \nabla_\tau Y_l^m dS = \frac{l(l + d - 2)}{2} \int_{S^{d-1}} Y_l^m Y_i^i dS.
\]

Combining this result with

\[
\int_{S^{d-1}} Y_l^m \nabla_\tau Y_i^i \cdot \nabla_\tau Y_l^m dS = n(n + d - 2) \int_{S^{d-1}} Y_l^m Y_i^i dS - \int_{S^{d-1}} Y_l^m \nabla_\tau Y_i^i dS,
\]

we have proved the result. \(\square\)

### 4.2. EMP Matrix of a Spherical Domain.

Based on Proposition 4.1, we show that the entries of the EMP matrix of a Steklov eigenvalue of a spherical domain may be expressed as a finite sum of integrals of triple products of spherical harmonics.

**Theorem 4.1.** Given a deformation field \(V \in W^{3, \infty}(B_{r_o}, \mathbb{R}^d)\), the entries of the EMP matrix \(M = M(B_{r_o}, V, \sigma)\) for the Steklov eigenvalue \(\sigma = \frac{n}{r_o}\) of \(B_{r_o}\) are given by

\[
M_{i,j} = \sum_{l=0}^{2n} \sum_{m=1}^{N_{l,d}} \alpha_{l,m,r_o} B(l, r_o) \int_{S^{d-1}} Y_l^i Y_j^j Y_l^m dS
\]

where

\[
B(l, r) = -\frac{1}{r^2} \left( \frac{l(l + d - 2)}{2} + n \right)
\]

The \(\alpha_{l,m,r_o}\) are the coefficients of the Laplace-Fourier expansion of \(V_{n,r_o}\) in the orthonormal basis \(\{Y_l^m : l \geq 0; m = 1, \ldots, N_{l,d}\}\) of \(d\)-dimensional spherical harmonics on the unit sphere \(S^{d-1}\).
Proof. According to Example 2.1 and Theorem 3.5, an entry of the EMP matrix of a spherical domain is given by

\[
M(B^d_{r_o}, V, \sigma)_{ij} = N(r_o, n, d)^2 \int_{S^{d-1}_{r_o}} \left( \nabla_\tau Y^i_n \cdot \nabla_\tau Y^j_n - (\sigma^2 + H_{r_o} \sigma) Y^i_n Y^j_n \right) V_{n,r_o} dS,
\]

where we write \(V_{n,r_o}\) for the normal component of \(V\) on \(S^{d-1}_{r_o}\); and we write \(H_{r_o}\) for the additive curvature on \(S^{d-1}_{r_o}\). Our goal is to simplify this integral expression. We start by expanding \(V_{n,r_o}\) into a Fourier-Laplace series as in Formula (2.19). Next, we substitute and reduce the integration to the unit sphere by making use of the fact that \(\int_{S^{d-1}_{r_o}} dS = r_o^{d-1} \int_{S^{d-1}} dS\) together with the fact that the surface gradient \(\nabla_\tau\) evaluated on \(S^{d-1}_{r_o}\) equals \(1/r_o \nabla_\tau\) evaluated on \(S^{d-1}\).

\[
\int_{S^{d-1}_{r_o}} \left( \nabla_\tau Y^i_n \cdot \nabla_\tau Y^j_n - (\sigma^2 + H_{r_o} \sigma) Y^i_n Y^j_n \right) V_{n,r_o} dS = \sum_{l=0}^{\infty} N_{l,d} \sum_{m=1}^{N_{l,d}} \alpha_{l,m,r_o} r_o^{d-1} \left( \frac{1}{r_o^2} \int_{S^{d-1}} \nabla_\tau Y^i_n \cdot \nabla_\tau Y^j_n Y^m_l dS - (\sigma^2 + H_{r_o} \sigma) \int_{S^{d-1}_{r_o}} Y^i_n Y^j_n Y^m_l dS \right).
\]

Applying Proposition 4.1 to the first integral we have

\[
\int_{S^{d-1}} \left( \nabla_\tau Y^i_n \cdot \nabla_\tau Y^j_n - (\sigma^2 + H \sigma) Y^i_n Y^j_n \right) V_{n,r_o} dS = \sum_{l=0}^{\infty} N_{l,d} \sum_{m=1}^{N_{l,d}} \alpha_{l,m,r_o} r_o^{d-1} \left( \frac{n(n + d - 2) - \frac{l(l + d - 2)}{2}}{r_o^2} \right) - (\sigma^2 + H_{r_o} \sigma) \int_{S^{d-1}} Y^i_n Y^j_n Y^m_l dS.
\]

The sphere \(S^{d-1}_{r_o}\) has additive curvature \(H_{r_o} = \frac{d-1}{r_o}\) and the Steklov eigenvalue \(\sigma = \frac{n}{r_o}\). So, substituting and simplifying, we find that

\[
\frac{1}{r_o^2} \left( n(n + d - 2) - \frac{l(l + d - 2)}{2} \right) - (\sigma^2 + H \sigma) = \frac{1}{r_o^2} \left( -\frac{l(l + d - 2)}{2} - n \right).
\]

Also, from Example 2.1, we have \(N(r_o, n, d)^2 = r_o^{-(d-1)}\), and so plugging into 4.4 and simplifying we obtain

\[
M_{i,j} = \sum_{l=0}^{\infty} \sum_{m=1}^{N_{l,d}} \alpha_{l,m,r_o} \left[ \frac{1}{r_o^2} \left( -\frac{l(l + d - 2)}{2} - n \right) \right] \int_{S^{d-1}} Y^i_n Y^j_n Y^m_l dS.
\]
To show the sum is finite, we apply Proposition 2.3 to the product $Y_n^i Y_n^j$ to obtain the following expression for the triple product integral

$$
\int_{S^{d-1}} Y_n^i Y_n^j Y_l^m dS = \sum_{l' = 0}^{2n} \sum_{m' = 1}^{N_{l',d}} c_{l',m'} \int_{S^{d-1}} Y_{l'}^m Y_l^m dS.
$$

It follows by orthonormality, see Formula (2.18), that the integral is guaranteed to be 0 when $l$ is odd and also for all $l > 2n$. The result follows.

4.3. EMP Matrix of an Annular Domain. Again, based on Proposition 4.1, we have a similar result for annular domains.

**Theorem 4.2.** Given a deformation field $V \in W^{3,\infty}(A_{r_i, r_o}, \mathbb{R}^d)$, the entries of the EMP matrix $M = M(A_{r_i, r_o}, V, \mu_{n,k})$ for the Steklov eigenvalue $\mu_{n,k}$ of $A_{r_i, r_o}$ are given by

$$
M_{i,j} = \sum_{l = 0}^{2n} \sum_{m = 1}^{N_{l,d}} \alpha_{l,m,r_o} A(r_o, l, \mu_{n,k}, d) \int_{S^{d-1}} Y_n^i Y_n^j Y_l^m dS
$$

$$
- \sum_{l = 0}^{2n} \sum_{m = 1}^{N_{l,d}} \alpha_{l,m,r_i} A(r_i, l, \mu_{n,k}, d) \int_{S^{d-1}} Y_n^i Y_n^j Y_l^m dS
$$

where

$$
A(r_o, l, \mu_{n,k}, d) = N(r_o, \mu_{n,k}, d)^2 r_o^{d-3} \left( n(n + d - 2) - \frac{l(l + d - 2)}{2} - r_o^2 \mu_{n,k}^2 - (d - 1)r_o \mu_{n,k} \right),
$$

and

$$
A(r_i, l, \mu_{n,k}, d) = N(r_i, \mu_{n,k}, d)^2 r_i^{d-3} \left( n(n + d - 2) - \frac{l(l + d - 2)}{2} - r_i^2 \mu_{n,k}^2 + (d - 1)r_i \mu_{n,k} \right).
$$

The $\alpha_{l,m,r_o}$ (respectively the $\alpha_{l,m,r_i}$) are the coefficients of the Laplace-Fourier expansion of $V_{n,r_o}$ (respectively $V_{n,r_i}$) in the orthonormal basis $\{Y_l^m : l \geq 0; m = 1, \cdots, N_{l,d}\}$ of $d$-dimensional spherical harmonics on the unit sphere $S^{d-1}$. The radial normalization coefficient $N(r, \mu_{n,k}, d)$ is defined in Appendix A.
Proof. According to Theorem 3.5, an entry of the EMP matrix of an annular domain is given by

\[
M_{ij} = N(r_o, n, k, d)^2 \int_{S_{r_o}} \left( \nabla_\tau Y_n^i \cdot \nabla_\tau Y_n^j - (\mu_{n,k}^2 + H_{r_o}\mu_{n,k}) Y_n^i Y_n^j \right) V_{n,r_o} dS
- N(r_i, n, k, d)^2 \int_{S_{r_i}} \left( \nabla_\tau Y_n^i \cdot \nabla_\tau Y_n^j - (\mu_{n,k}^2 + H_{r_i}\mu_{n,k}) Y_n^i Y_n^j \right) V_{n,r_i} dS.
\]

We have that the additive curvatures on the outer and inner boundary are given by \(H_{r_o} = \frac{d-1}{r_o}\) and \(H_{r_i} = -\frac{d-1}{r_i}\). So, the result follows immediately by applying the derivation in the proof of Theorem 4.1 to the individual terms of the EMP matrix \(M\). \(\Box\)
5. The Local Steklov Eigenvalue Optimization Problem

5.1. Local Optimization for Spherical Domains. In this section we return to the local shape optimization problem for Steklov eigenvalue functionals and provide partial solutions for spherical and annular domains. We first consider spherical domains; and start by calculating the trace of an EMP matrix.

**Proposition 5.1.** Let \( V \in W^{3,\infty}(\mathbb{B}^d_{r_o}, \mathbb{R}^d) \) and \( \sigma = \frac{n}{r_o} \) be a Steklov eigenvalue of \( \mathbb{B}^d_{r_o} \), then we have for the EMP matrix \( M = M(\mathbb{B}^d_{r_o}, V, \sigma) \)

\[
tr(M) = -\alpha_{0, r_o} \frac{n N_{n,d}}{r_o \sqrt{\omega_{d-1}}}. 
\]

**Proof.** If first we apply the result of Theorem 4.1; and second, we exchange the order of summation and integration; and third, we apply Proposition 2.4; and finally, we apply Proposition 2.2, then we obtain

\[
tr(M) = \sum_{j=1}^{N_{n,d}} \sum_{l=0}^{2n} \sum_{m=1}^{N_{l,d}} \alpha_{l,m, r_o} B(l, r_o) \int_{\mathbb{S}^2} Y^j_n Y^m_l Y^m_l dS
\]

\[
= \sum_{l=0}^{2n} \sum_{m=1}^{N_{l,d}} \alpha_{l,m, r_o} B(l, r_o) \int_{\mathbb{S}^2} \left( \sum_{j=1}^{N_{n,d}} Y^j_n Y^j_n \right) Y^m_l dS
\]

\[
= \sum_{l=0}^{2n} \sum_{m=1}^{N_{l,d}} \alpha_{l,m, r_o} B(l, r_o) \frac{N_{n,d}}{\omega_{d-1}} \int_{\mathbb{S}^2} Y^m_l dS
\]

\[
= \alpha_{0,1, r_o} B(0, r_o) \frac{N_{n,d}}{\sqrt{\omega_{d-1}}}.
\]

Noting that \( B(0, r_o) = -\frac{n}{r_o} \), the result follows. \(\square\)

We now state our main result for spherical domains.

**Theorem 5.1.** Let \( V \in W^{3,\infty}(\mathbb{B}^d_{r_o}, \mathbb{R}^d) \) be a boundary component volume preserving at first order deformation field on \( \mathbb{B}^d_{r_o} \), i.e., \( \int_{\mathbb{S}^d_{r_o}} V_n dS = 0 \), and let \( \sigma = \frac{n}{r_o} \) be a Steklov eigenvalue of \( \mathbb{B}^d_{r_o} \), then \( (\mathbb{B}^d_{r_o}, V) \) is critical for \( \sigma \). If, in addition, the EMP matrix \( M = M(\mathbb{B}^d_{r_o}, V, \sigma) \) is not
the zero matrix, then \((\mathbb{B}^d_{r_o}, V)\) locally strictly maximizes \(\sigma_{\text{ind}_\sigma}\) and locally strictly minimizes \(\sigma_{\text{ind}_\sigma + N(n,d) - 1}\). Where \(\text{ind}_\sigma\) is the index of \(\sigma\), while \(N(n,d)\) is the multiplicity of \(\sigma\).

**Proof.** Again applying Proposition 2.2, we have that

\[
\int_{S^{d-1}} V_n \, dS = r_o^{d-1} \sum_{l=0}^{N_l,d} \sum_{m=1}^{N_l,d} \alpha_{m,l,r_o} \int_{S^{d-1}} Y_l^m \, dS = r_o^{d-1} \sqrt{\omega_{d-1}} \alpha_{0,1,r_o}
\]

and it follows that \(V\) is volume preserving at first order on \(\mathbb{B}^d_{r_o}\) if and only if \(\alpha_{0,1,r_o} = 0\), which implies that \(\text{trace}(M) = 0\). We conclude that \(0 \in \partial\sigma[\mathbb{B}^d_{r_o}, V]\), and therefore \((\mathbb{B}^d_{r_o}, V)\) is critical for \(\sigma\), (see the definition at the end of Section 3.1).

Turning to the local optimization result, we assume that the EMP matrix \(M\) has been computed using the standard orthonormal basis given by B.3 in Appendix B. For clarity, we assume the basis of \(\mathbb{Y}^d_n\) has been enumerated in some way and retain the generic notation for these spherical harmonics while using the standard basis indices for the spherical harmonics used in the expansion of \(V_{n,r_o}\). Despite the notation it should be understood that all three spherical harmonics appearing in the integral are elements of the standard orthonormal basis.

With this convention we have

\[
\overline{M}_{ij} = \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} \bar{\sigma}_{\mu_1,\ldots,\mu_{d-3},l,m,r_o} B(l,r_o) \int_{S^{d-1}} \overline{Y}_n \overline{Y}_n Y_{\mu_1,\ldots,\mu_{d-3},l,m} dS
\]

\[
= \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} \alpha_{\mu_1,\ldots,\mu_{d-3},l,m,r_o} B(l,r_o) \int_{S^{d-1}} \overline{Y}_n Y_{\mu_1,\ldots,\mu_{d-3},l,m} dS
\]

\[
= \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} \alpha_{\mu_1,\ldots,\mu_{d-3},l,m,r_o} B(l,r_o) \int_{S^{d-1}} Y_n \overline{Y}_n Y_{\mu_1,\ldots,\mu_{d-3},l,m} dS = M_{ij}
\]

where for the second equality we have used Proposition B.1 applied to \(V_{n,r_o}\); and for the final equality we have rearranged the sum with respect to the symmetric index \(m\) which ranges over \(-l \leq m \leq l\). We have shown the EMP matrix is Hermitian; and furthermore, because any pair of orthonormal bases for \(\mathbb{Y}^d_n\) are unitarily equivalent, it is still the case that the EMP matrix is non-zero when computed with respect to the standard basis. It follows that if all the eigenvalues of \(M\) were zero, then \(M\) would have to be the zero matrix, a contradiction.
We conclude that the EMP matrix $M$ must have a non-zero eigenvalue. The conclusion that $(B_{d \rightarrow r_o}, V)$ locally maximizes $\sigma_{ind_\mu}$ and locally minimizes $\sigma_{ind_\mu+N(n,d)-1}$, follows from the discussion at the end of Section 3.1 together with the fact that $\text{trace}(M) = 0$. □

5.2. Local Optimization for Annular Domains. For an annular domain we have the following formula for the trace of an EMP matrix.

**Proposition 5.2.** Let $V \in W^{3,\infty}(A_{d \rightarrow r_o}, \mathbb{R}^d)$ and let $\mu_{l,k}$ be a Steklov eigenvalue of $B_{r_o}$, then we have for the EMP matrix $M = M(A_{d \rightarrow r_o}, V, \sigma)$

$$tr(M) = \left[\alpha_{0,1,r_o} A(r_o, 0, \mu_{n,k}, d) - \alpha_{0,1,r_i} A(r_i, 0, \mu_{n,k}, d)\right] \frac{N_{n,d}}{\sqrt{\omega_{d-1}}}.$$ 

**Proof.** If first we apply the result of Theorem 4.2 and then separately apply the proof of Proposition 5.1 to the inner and outer expressions, we obtain the result.

$$tr(M) = \sum_{j=1}^{N_{n,d}} \sum_{l=0}^{2n} \sum_{m=1}^{N_{l,d}} \alpha_{l,m,r_o} A(r_o, l, \mu_{n,k}, d) \int_{\mathbb{S}^{d-1}} Y_n^l Y_m^l dS$$

$$- \sum_{j=1}^{N_{n,d}} \sum_{l=0}^{2n} \sum_{m=1}^{N_{l,d}} \alpha_{l,m,r_i} A(r_i, l, \mu_{n,k}, d) \int_{\mathbb{S}^{d-1}} Y_n^l Y_m^l dS$$

$$= \alpha_{0,1,r_o} A(r_o, 0, \mu_{n,k}, d) \frac{N_{n,d}}{\sqrt{\omega_{d-1}}} - \alpha_{0,1,r_i} A(r_i, 0, \mu_{n,k}, d) \frac{N_{n,d}}{\sqrt{\omega_{d-1}}}.$$ □

We also have the corresponding local optimization result for annular domains.

**Theorem 5.2.** Let $V \in W^{3,\infty}(A_{d \rightarrow r_o}, \mathbb{R}^d)$ be a boundary component volume preserving at first order deformation field on $A_{d \rightarrow r_o}$, i.e., $\int_{\mathbb{S}^{d-1}} V_n dS = \int_{\mathbb{S}^{d-1}} V_n dS = 0$, and let $\mu_{n,k}$ be a Steklov eigenvalue of $A_{r_o}$, then $(A_{r_o}, V)$ is critical for $\mu_{n,k}$. If, in addition, the EMP matrix $M = M(A_{d \rightarrow r_o}, V, \mu_{n,k})$ is not the zero matrix, then $(A_{d \rightarrow r_o}, V)$ locally maximizes $\sigma_{ind_\mu}$ and locally minimizes $\sigma_{ind_\mu+N(n,d)-1}$ where $ind_\mu$ is the index of $\mu$, while $N(n,d)$ is the multiplicity of $\mu$. 43
Proof. Arguing as in the proof of Theorem 5.1, we have that

\[ \int_{S_{r_0}} V_{n,r_0} \, dS = r_o^{d-1} \sqrt{\omega_{d-1}} \alpha_{0,1,r_o} \quad \text{and} \quad \int_{S_{r_i}} V_{n,r_i} \, dS = r_i^{d-1} \sqrt{\omega_{d-1}} \alpha_{0,1,r_i} \]

and it follows that \( V \) is boundary component volume preserving at first order on \( A_{r_i,r_o} \) if and only if \( \alpha_{0,1,r_o} = 0 \), which implies that \( \text{trace}(M) = 0 \). We conclude that \( 0 \in \partial\sigma[A_{r_i,r_o}, V] \), and therefore \( (A_{r_i,r_o}, V) \) is critical for \( \sigma \). Therefore \( (A_{r_i,r_o}, V) \) is critical for \( \sigma \).

Because the sum of two Hermitian matrices is Hermitian, we show the EMP matrix is Hermitian by arguing as in the proof of Theorem 5.1, that the inner and outer components of the EMP matrix are Hermitian. The local optimization result follows by arguing as in the proof of Theorem 5.1, that a non-zero and Hermitian EMP matrix must have a non-zero eigenvalue. \( \square \)
6. Numerical Implementation and Numerical Results

6.1. Method of Particular Solutions. Making use of the method of particular solutions, we numerically investigate and illustrate Theorems 5.1 and 5.2. We consider perturbations of spherical domains \( \Omega = \mathbb{B}^d_{r_o} \) and annular domains \( \Omega = \mathbb{A}^d_{r_i, r_o} \), where for each small perturbation parameter \( t \in \mathbb{R} \), we have

\[
\Omega_t = \left\{ \begin{bmatrix} r \\ \theta \end{bmatrix} + t \begin{bmatrix} r V(\bar{\theta}) \\ 0 \end{bmatrix} : \begin{bmatrix} r \\ \theta \end{bmatrix} \in \Omega \right\}.
\]

We limit attention to deformation fields \( V \) such that the normal components \( V_{r_o} \) and \( V_{r_i} \) can be written as a finite linear combination of spherical harmonics taken from an arbitrary orthonormal basis \( \{ Y_{l,m}^r : l = 0, 1, \cdots ; m = 1, \cdots, N_{l,d} \} \). In this case, a solution \( u \) of the Steklov eigenvalue problem (1.1) on the perturbed domain \( \Omega_t \) may be expanded in a Fourier-Laplace series of regular and singular solid harmonics. Indeed, if \( \Omega_t \) is nearly spherical, then we have an expansion in regular solid harmonics

\[
u(r, \bar{\theta}) = \sum_{l=0}^{\infty} \sum_{m=1}^{N_{l,d}} a_{l,m} r^l Y_{l,m}^r(\bar{\theta}),
\]

while if \( \Omega_t \) is nearly annular and \( d \geq 3 \), then the expansion also includes singular solid harmonics

\[
u(r, \bar{\theta}) = \sum_{l=0}^{\infty} \sum_{m=1}^{N_{l,d}} a_{l,m} r^l Y_{l,m}^r(\bar{\theta}) + \sum_{l=0}^{\infty} \sum_{m=1}^{N_{l,d}} b_{l,m} r^{-(d+l-2)} Y_{l,m}^r(\bar{\theta}).
\]

When \( \Omega_t \) is nearly annular and \( d = 2 \), care must be taken because the leading singular solid harmonic includes a logarithmic dependency on \( r \). So, noting that \( N_{l,2} = 2 \) for all \( l \geq 1 \), we have

\[
u(r, \bar{\theta}) = a_{0,1} Y_{0}^1 + b_{0,1} \log(r) Y_{0}^1 + \sum_{l=1}^{\infty} \sum_{m=1}^{2} a_{l,m} r^l Y_{l,m}^r(\bar{\theta}) + \sum_{l=1}^{\infty} \sum_{m=1}^{2} b_{l,m} r^{-(d+l-2)} Y_{l,m}^r(\bar{\theta}).
\]
The natural logarithm occurs because when the Steklov equation is solved on $A_{r_1,r_o}$ by separation of variables, we find that the radially dependent factor must satisfy a Cauchy-Euler equation; and in two dimensions, the solution of this equation includes a logarithmic term when $l = 0$.

Based on these Fourier-Laplace expansions, if we denote the regular solid harmonics by

$$u_{l,m}^r(r,\vec{\theta}) = r^l Y_{l}^m(\vec{\theta}),$$

and the singular solid harmonics by

$$u_{0,1}^s(r,\vec{\theta}) = \begin{cases} 
\log(r) Y_0^1 & d = 2, \\
-r^{-(d-2)} Y_0^1 & d \geq 3,
\end{cases}$$

$$u_{l,m}^s(r,\vec{\theta}) = r^{-(d+l+2)} Y_{l}^m(\vec{\theta}) \text{ for } l \geq 1,$$

then, for a fixed choice of maximum spherical harmonic degree $L$, we have the following approximate solution ansatz for the Steklov problem. When $\Omega_t$ is nearly spherical

$$u^L(r,\vec{\theta}) = \sum_{l=0}^{L} \sum_{m=1}^{N_l,d} a_{l,m} u_{l,m}^r(r,\vec{\theta}),$$

and when $\Omega_t$ is nearly annular

$$u^L(r,\vec{\theta}) = \sum_{l=0}^{L} \sum_{m=1}^{N_l,d} a_{l,m} u_{l,m}^r(r,\vec{\theta}) + \sum_{l=0}^{L} \sum_{m=1}^{N_l,d} b_{l,m} u_{l,m}^s(r,\vec{\theta}).$$

In both cases $u^L$ clearly satisfies Laplace’s equation on $\Omega_t$, i.e., $\Delta u^L(r,\vec{\theta}) = 0$ for $(r,\vec{\theta}) \in \Omega_t$.

Suppose for a given $\Omega_t$, we can find approximate eigenvalues $\sigma$ and corresponding coefficients $a_{l,m}$ and $b_{l,m}$ such that $u^L$ also satisfies the Steklov boundary condition $\partial_n u^L(r,\vec{\theta}) = \sigma u^L(r,\vec{\theta})$ on some collection of points $(r,\vec{\theta})$ distributed across the boundary $\partial \Omega_t$. Then letting the perturbation parameter $t$ vary over a discrete set of values around and including zero and
ordering the resulting approximate eigenvalues according to multiplicity, we produce a numerical approximation to the eigenvalue branches for \( A_{r_i,r_o}^d \) as described in Theorem 3.5. To obtain the unknown eigenvalues \( \sigma \) and corresponding coefficients \( a_{l,m} \) and \( b_{l,m} \), we solve a generalized eigenvalue problem

\[
B^T A \tilde{\alpha} = \sigma B^T B \tilde{\alpha}.
\]

For clarity we describe the derivation of Equation (6.1) for the unperturbed case when \( t = 0 \), and later indicate the changes required in the general perturbed case of \( \Omega_t \), with \( t \neq 0 \).

In the case when \( \Omega = B_{r_o}^d \) we have \( \tilde{\alpha} = [a_{0,1}, \ldots, a_{L,N_{L,d}}]^T \) is the column vector of unknown coefficients of \( u^L_r \); and the matrices \( A \) and \( B \) are defined as follows. Let \( (r_o, \vec{\theta}_{o,k})_{k=1}^{K_o} \) be a collection of points distributed on the sphere \( S_{r_o}^{d-1} \) as described in [20]. If we define \( A \) and \( B \) to be the \( K_o \times (\sum_{l=0}^L N_{l,d}) \) matrices whose \( k^{th} \) rows are respectively given by

\[
A(k,:) = \left[ \partial_n u_{0,1}^r(r_o, \vec{\theta}_{o,k}), \ldots, \partial_n u_{L,N_{L,d}}^r(r_o, \vec{\theta}_{o,n}) \right],
\]

\[
B(k,:) = \left[ u_{0,1}^r(r_o, \vec{\theta}_{o,k}), \ldots, u_{L,N_{L,d}}^r(r_o, \vec{\theta}_{o,k}) \right],
\]

then the equation \( A \tilde{\alpha} = \sigma B \tilde{\alpha} \) expresses the requirement that the Steklov eigenvalue equation \( \partial u = \sigma u \) is true on the boundary \( S_{r_o}^{d-1} \) at each of the points \( (r_o, \vec{\theta}_{o,k})_{k=1}^{K_o} \). To prevent an ill-conditioned problem, we multiply both sides of this equation by \( B^T \) and obtain our final generalized eigenvalue problem (6.1).

When \( \Omega = A_{r_i,r_o}^d \) we have \( \tilde{\alpha} = [a_{0,1}, \ldots, a_{L,N_{L,d}}, b_{0,1}, \ldots, b_{L,N_{L,d}}]^T \) is the column vector of unknown coefficients of \( u^L_r \); and the matrices \( A \) and \( B \) are defined as follows. Let \( (r_o, \vec{\theta}_{o,k})_{k=1}^{K_o} \) and \( (r_i, \vec{\theta}_{i,k})_{k=1}^{K_i} \) be collections of points distributed on the spheres \( S_{r_o}^{d-1} \) and \( S_{r_i}^{d-1} \) respectively. Define \( A_o, B_o \) to be the \( K_o \times (\sum_{l=0}^L N_{l,d}) \) matrices and \( A_i, B_i \) to be the \( K_i \times (\sum_{l=0}^L N_{l,d}) \)
matrices whose $k^{th}$ rows are respectively given by

\[
A_o(k,:) = \begin{bmatrix}
\partial_n u_{0,1}^r(r_o, \vec{\theta}_{o,k}), \ldots, \partial_n u_{L,N,L,d}^r(r_o, \vec{\theta}_{o,k}), \\
\partial_n u_{0,1}^s(r_o, \vec{\theta}_{o,k}), \ldots, \partial_n u_{L,N,L,d}^s(r_o, \vec{\theta}_{o,k})
\end{bmatrix},
\]

\[
B_o(k,:) = \begin{bmatrix}
u_{0,1}^r(r_o, \vec{\theta}_{o,k}), \ldots, u_{L,N,L,d}^r(r_o, \vec{\theta}_{o,k}), \\
u_{0,1}^s(r_o, \vec{\theta}_{o,k}), \ldots, u_{L,N,L,d}^s(r_o, \vec{\theta}_{o,k})
\end{bmatrix},
\]

\[
A_i(k,:) = \begin{bmatrix}
\partial_n u_{0,1}^r(r_i, \vec{\theta}_{i,k}), \ldots, \partial_n u_{L,N,L,d}^r(r_i, \vec{\theta}_{i,k}), \\
\partial_n u_{0,1}^s(r_i, \vec{\theta}_{i,k}), \ldots, \partial_n u_{L,N,L,d}^s(r_i, \vec{\theta}_{i,k})
\end{bmatrix},
\]

\[
B_i(k,:) = \begin{bmatrix}
u_{0,1}^r(r_i, \vec{\theta}_{i,k}), \ldots, u_{L,N,L,d}^r(r_i, \vec{\theta}_{i,k}), \\
u_{0,1}^s(r_i, \vec{\theta}_{i,k}), \ldots, u_{L,N,L,d}^s(r_i, \vec{\theta}_{i,k})
\end{bmatrix},
\]

If we define

\[
A = \begin{bmatrix} A_o \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_o \\ B_i \end{bmatrix},
\]

then the equation $A \vec{\alpha} = \sigma B \vec{\alpha}$ expresses the requirement that the Steklov eigenvalue equation $\partial u^L = \sigma u^L$ is true on both the outer and inner boundary at each of the points $(r_o, \vec{\theta}_{o,k})_{k=1}^{K_o}$ and $(r_i, \vec{\theta}_{i,k})_{k=1}^{K_i}$. To prevent an ill-conditioned problem, we multiply both sides of this equation by $B^T$ and obtain our final generalized eigenvalue problem (6.1).

For a perturbed domain $\Omega_t$ with $t \neq 0$, the derivation of Equation (6.1) requires only that all occurrences of the points $(r_o, \vec{\theta}_{o,k})_{k=1}^{K_o}$ in the nearly spherical case, and $(r_o, \vec{\theta}_{o,k})_{k=1}^{K_o}$ together with $(r_i, \vec{\theta}_{i,k})_{k=1}^{K_i}$ in the nearly annular case, be replaced with their images under the deformation field, i.e., $(r_{t,o,k}, \vec{\theta}_{o,k})_{k=1}^{K_o}$ and $(r_{t,i,k}, \vec{\theta}_{i,k})_{k=1}^{K_i}$, where $r_{t,o,k} = r_o + tr_o V(\vec{\theta}_{o,k})$ and $r_{t,i,k} = r_i + tr_i V(\vec{\theta}_{i,k})$. With this substitution the solutions of Equation (6.1) provide approximations to the Steklov eigenvalues of $\Omega_t$.

6.2. **Numerical Results in $\mathbb{R}^2$.** In two dimensions we have that the multiplicity of any non-zero Steklov eigenvalue for either a disk or an annulus is given by $N_{n,2} = 2$. Therefore, the EMP matrices described in Theorems 4.1 and 4.2 are $2 \times 2$ and so may be explicitly calculated. Indeed, with the following basis for the 2D spherical harmonics

\[
Y_l^0 = \frac{1}{\sqrt{2\pi}} \quad \text{for } l = 0,
\]

\[
Y_l^m = \frac{1}{\sqrt{2\pi}} e^{i(-1)^m \theta} \quad \text{for } l \geq 0, \quad m = 1, 2,
\]

48
In this basis the triple product integral appearing in the expressions for either $M_{B_{r_o}, \sigma_n}(V)$ or $M(A_{r_o, r_o}^2, V, \mu_{n,k})$ may be evaluated as follows.

$$\int_{S^{d-1}} Y_l^m Y_n^\mu Y_n^\nu dS = \frac{1}{2\pi \sqrt{2\pi}} \int_0^{2\pi} e^{i(-1)^m l \theta} e^{i(-1)^{n} n \theta} e^{i(-1)^{\nu+1} n \theta} d\theta$$

$$= \frac{1}{2\pi \sqrt{2\pi}} \int_0^{2\pi} e^{i[-(-1)^ml+((-1)^{\mu}+(-1)^{\nu+1})n] \theta} d\theta$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{if } [(-1)^m l + ((-1)^{\mu} + (-1)^{\nu+1}) n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

Notice

$$[(-1)^m l + ((-1)^{\mu} + (-1)^{\nu+1}) n] = 0 \iff (-1)^m l = ((-1)^{\mu+1} + (-1)^{\nu}) n.$$ 

So the specific values of $l$ and $m$ that lead to a nonzero triple product integral depends on the values of $\mu$ and $\nu$. Indeed, for diagonal entries of $M$ we have

$$(6.2) \quad \mu = \nu \implies (-1)^{\mu+1} + (-1)^{\nu} = 0 \implies l = 0, m = 0.$$ 

While for off diagonal entries of $M$ we have

$$(6.3) \quad \mu = 1, \nu = 2 \text{ or } \mu = 2, \nu = 1 \implies (-1)^{\mu+1} + (-1)^{\nu} = (-1)^{\nu} 2 \implies l = 2n, m = \nu.$$ 

Applying we obtain the following two corollaries to Theorems 4.1 and 4.2 respectively.

**Corollary 6.1.** Given a deformation field $V \in W^{3,\infty}(B^2_{r_o}, \mathbb{R}^2)$, we have that the EMP matrix $M(B^2_{r_o}, V, \sigma)$ for the Steklov eigenvalue $\sigma = \frac{n}{r_o}$ of $B^2_{r_o}$ is given by

$$-\frac{n}{r_o^2 \sqrt{2\pi}} \begin{bmatrix} \alpha_{0,1,r_o} & (2n+1)\alpha_{2n,2,r_o} \\ (2n+1)\alpha_{2n,1,r_o} & \alpha_{0,1,r_o} \end{bmatrix}.$$ 


Given a deformation field $V \in W^{3,\infty}(A^2_{r_o,r_i}, \mathbb{R}^2)$, we have that the EMP matrix $M(A^2_{r_o,r_i}, V, \mu_{n,k})$ for the Steklov eigenvalue $\mu_{n,k}$ of $A^2_{r_o,r_i}$ is given by

$$
\frac{1}{\sqrt{2\pi}} \begin{bmatrix}
A(r_o, 0, \mu_{n,k}, 2) \alpha_{0,1,r_o} & A(r_o, 2n, \mu_{n,k}, 2) \alpha_{2n,2,r_o} \\
A(r_o, 2n, \mu_{n,k}, 2) \alpha_{2n,1,r_o} & A(r_o, 0, \mu_{n,k}, 2) \alpha_{0,1,r_o}
\end{bmatrix}
$$

$$
- \frac{1}{\sqrt{2\pi}} \begin{bmatrix}
A(r_i, 0, \mu_{n,k}, 2) \alpha_{0,1,r_i} & A(r_i, 2n, \mu_{n,k}, 2) \alpha_{2n,2,r_i} \\
A(r_i, 2n, \mu_{n,k}, 2) \alpha_{2n,1,r_i} & A(r_i, 0, \mu_{n,k}, 2) \alpha_{0,1,r_i}
\end{bmatrix},
$$

Note that when $V$ is boundary component volume preserving at first order, as in Theorems 5.1 and 5.2, we have that $\alpha_{0,1,r_o} = \alpha_{0,1,r_i} = 0$. It follows that given a Steklov eigenvalue $\sigma = \frac{1}{r_o}$ of a disk or $\mu_{n,k}$ of an annulus, a necessary condition for the EMP matrix to be non-zero is that at least one of $\alpha_{2n,1,r_o}$, $\alpha_{2n,2,r_o}$ respectively $\alpha_{2n,1,r_i}$, $\alpha_{2n,2,r_i}$ be non-zero. For a disk this condition is also sufficient. Interestingly, for a 2D annulus it is possible for the EMP matrix to be zero even when all of $\alpha_{2n,1,r_o}$, $\alpha_{2n,1,r_i}$, $\alpha_{2n,2,r_o}$, $\alpha_{2n,2,r_i}$ are non-zero. Indeed, this will be the case provided the Fourier coefficients are selected non-zero and satisfying

$$
\frac{\alpha_{2n,1,r_o}}{\alpha_{2n,1,r_i}} = \frac{\alpha_{2n,2,r_o}}{\alpha_{2n,2,r_i}} = \frac{A(r_i, 2n, \mu_{n,k}, 2)}{A(r_o, 2n, \mu_{n,k}, 2)}.
$$

In Figure 8(a) we visualize some eigenvalue branches and their tangent lines at zero when $A^2_{0,4,1}$ is perturbed by a deformation field $V$ with $V_{n,r_o} = V_{n,r_i} = 2 \cos(6\theta)$. In this case, condition (6.4) is not satisfied for $n = 3$, and the EMP matrices of $\mu_{3,1} = 2.944$ and $\mu_{3,2} = 7.642$ are non-zero. We see that the branches of these two eigenvalues demonstrate the characteristic “bow tie” response to the perturbation. The EMP matrices of all other eigenvalues are zero, and their branches all have slope zero at the initial shape. In Figure 8(b) the same domain has been perturbed by a deformation field $V$ with $V_{n,r_o} = V_{n,r_i} = 2 \cos(5\theta)$. Since there are no non-zero even indexed Fourier coefficients, it follows that the EMP matrix of every eigenvalue is identically zero; and every eigenvalue branch has slope zero at the initial shape. In both cases, observe the robust agreement between the slope at zero of the
numerically generated eigenvalue branches and the corresponding tangent lines determined analytically from Corollary 6.2.

![Figure 8](image)

**Figure 8.** Eigenvalue Branches of the annulus $A_{2.0,4.1}^2$ where $L = 7$, $K_o = 28$, and $K_i = 20$. In (a) we use $V_{n,r_i} = V_{n,r_o} = 2 \cos(6\theta)$ and in (b) we use $V_{n,r_i} = V_{n,r_o} = 2 \cos(5\theta)$.

The local maximization results obtained in Theorems 5.1 and 5.2 required that the EMP matrix under consideration be non-zero, raising the question of the possibility of weakening or eliminating this assumption. In Figure 9, we see that for unit disk $B^2$ perturbed by $V$ with normal component $V_{n,1} = \cos(7\theta)$ we have that $(B^2, V)$ appears to locally strictly minimizes both the first and last eigenvalue branches of $\sigma = 5$, while when $V$ has normal component $V_{n,1} = \sin(5\theta)$ we have that the opposite appears to be true, and $(B^2, V)$ appears to locally strictly maximizes the first and last eigenvalue branches of $\sigma = 5$. In the same figure we also see that for the annulus $A_{0.4,1}^2$ perturbed by $V$ with normal components $V_{n,0.4} = \cos(7\theta), V_{n,1} = \cos(7\theta)$ it appears that $(A_{0.4,1}^2, V)$ locally strictly minimizes the first and last eigenvalue branches of $\mu_{5,2}$, while when $V$ has normal component $V_{n,0.4} = 0, V_{n,1} = \sin(5\theta)$ it appears that $(A_{0.4,1}^2, V)$ locally strictly maximizes the first and last eigenvalue branches of $\mu_{5,2}$. Notice that in all four examples the associated EMP matrix is identically zero; and so, based on the numerics, we conclude that our assumption is required to conclude local strict maximization of the bottom eigenvalue branch and local strict minimization of the top eigenvalue branch.
6.3. Numerical Results in $\mathbb{R}^3$. In three dimensions we have that the multiplicity of any Steklov eigenvalue $\sigma = \frac{n}{r_o}$ for a disk or $\mu_{n,k}$ for an annulus is given by $N_{n,2} = 2n + 1$, and so the size of the corresponding EMP matrix grows with $n$. We determine the eigenvalues of these matrices numerically. We compute the entries of the EMP matrix using the standard orthonormal basis given by Appendix B.1. With this choice of basis, we have that the triple product integrals appearing in Theorems 4.1 and 4.2 may be expressed in terms of Wigner-3j symbols as follows

\[
\int_{\mathbb{S}^2} Y_{l_1}^{m_1} Y_{l_2}^{m_2} Y_{l_3}^{m_3} dS = (-1)^{m_2} \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & -m_2 & m_3 \end{pmatrix}.
\]

A convenient expression for the numerical determination of the eigenvalues of the EMP matrix in the three-dimensional case [30].

An examination of the formulas in Theorems 4.1 and 4.2 yields a necessary condition for the non-vanishing of the EMP matrix of a Steklov eigenvalue $\sigma = \frac{n}{r_o}$ or $\mu_{n,k}$ of either a spherical or annular domain, respectively. Under the assumption that $V$ is boundary component volume
preserving at first order, for either a spherical or annular domain a necessary condition that the EMP matrix of a Steklov eigenvalue be non-zero is provided by the non-vanishing of at least one Fourier coefficients $\alpha_{l,m,r_o}$ for spherical domains, and $\alpha_{l,m,r_i}, \alpha_{l,m,r_o}$ for annular domains, where $2 \leq l \leq 2n, l$ even and $-l \leq m \leq l$, (see [44, Corollary 2.3] where this condition is also observed). As in dimension 2, this condition is also sufficient for spherical domains, while not being sufficient in the case of annular domains because of the possibility of cancellation between the inner and outer components of the EMP matrix.

In Figure 10 we visualize the eigenvalue branches and their tangent lines at zero for a selection of eigenvalues $\mu_{n,1}$ of $A_{0,4,1}$. Where the annular domain is perturbed by a deformation field $V$ with $V_{n,r_o} = V_{n,r_i} = Y_{8,1}$. Note that $Y_{8,1}$ denotes the real spherical harmonic of degree 8 and order 1. In this case the EMP matrix of $\mu_{n,1}$ is identically zero for $n < 3$, and the eigenvalue branches all have slope zero at the initial shape. On the other hand, for $n \geq 4$ the EMP matrix is non-zero, and the eigenvalue branches demonstrate the "bowtie" response to perturbation of the initial shape, characteristic of a non-zero EMP matrix. Again we observe robust agreement between the analytically determined dotted black tangent lines and the numerically generated eigenvalue branches.

**Figure 10.** $A_{0,4,1}^3$ Eigenvalue Branches with $V_{n,r_i} = Y_{8,1}, V_{n,r_o} = Y_{8,1}$. Here we have used $L = 7, K_o = 28, K_i = 20$. 

53
Finally, in Figure 11, we see that when its EMP matrix under perturbation is identically zero, the eigenvalue \( \mu_{2,2} \) of \( A_{0.4,1}^3 \) can both locally maximize and locally minimize both the first and last branch of its eigenvalue branches. Again, the numerics indicate that for local strict optimization the assumption that the EMP matrix be non-zero is required in 5.2. Interestingly an example of similar behavior for the first branch for an eigenvalue of a spherical domain in dimension 3 was not forthcoming, and so the numerics does not immediately support the requirement that the EMP matrix be non-zero. For instance, we see that perturbing the unit ball by \( Y_{7}^{1} \) or by \( Y_{5}^{5} \) results in the unit ball \( B^3 \) locally maximizing the first branch of its eigenvalue branches.

\[ V_{n,1} = Y_{7,1} \quad V_{n,1} = Y_{5,5} \quad V_{n,0.4} = 0, V_{n,1} = Y_{7,1} \quad V_{n,0.4} = Y_{5,5}, V_{n,1} = Y_{5,5} \]

\[ \sigma = 2 \quad \sigma = 2 \quad \mu_{2,2} = 7.744 \quad \mu_{2,2} = 7.744 \]

**Figure 11.** Eigenvalue Branches for the eigenvalue \( \sigma = 2 \) of \( B_{1}^{3} \) and for the eigenvalue \( \mu_{2,2} \) of \( A_{0.4,1}^3 \). Here we have used \( L = 7, K_{o} = 28, \) and \( K_{i} = 20. \)

### 7. Conclusion and Future Work

In this thesis, we studied how Steklov eigenvalues vary when a spherical domain or an annular domain in dimensions \( d \geq 2 \) is perturbed by a sufficiently smooth deformation field. By using a Green-Beltrami identity and that spherical harmonic functions are eigenfunctions of the surface Laplacian, we demonstrated that the derivatives of multiple Steklov eigenvalue branches are eigenvalues of a matrix whose entries are determined by finite sums of terms that involve the integral of the product of three spherical harmonic functions. It would be of interest to determine if a similar analytic result could be obtained for other symmetric
star shaped domains and their corresponding concentric "annular" versions, for instance ellipsoidal domains and ellipsoidal annular domains. It would also be of interest to consider even more general domains, and to determine the EMP matrix eigenvalues numerically based directly on Theorem 3.5.

Also, by determining sufficient conditions that imply the trace of its EMP matrix is zero, we show that for a Steklov eigenvalue \( \sigma \) of a spherical or annular domain, the pair \((B^{d}_{r_{o}}, V)\), respectively \((A^{d}_{r_{i},r_{o}}, V)\), is critical for the eigenvalue provided the deformation field \( V \) is sufficiently smooth. In addition, we show that if the EMP matrix is not identically zero, then \((B^{d}_{r_{o}}, V)\), respectively \((A^{d}_{r_{i},r_{o}}, V)\), locally maximizes the first branch and locally minimizes the last branch of the eigenvalues branches of \( \sigma \). For spherical domains our sufficient condition, \( V \) is sufficiently smooth and volume preserving at first order, is equivalent to the trace of the EMP equaling zero. In the case of annular domains, the corresponding condition, \( V \) is sufficiently smooth and boundary component volume preserving at first order, implies but is not equivalent to the EMP matrix trace equaling zero; because of the possibility of cancellation between the perturbations on the inner and outer domains. It would be of interest, in the case of annular domains, to develop a natural geometric condition on \( V \) which is equivalent to the vanishing of the EMP matrix trace.

Finally, numerically, we observe robust agreement between the tangent lines obtained from EMP matrix eigenvalues and the simulated eigenvalue branches obtained using the method of particular solutions. We also used numerics to investigate the assumption that the EMP matrix not be identically zero, required in our proof that the initial shape locally maximizes the first branch and locally minimizes the last branch of the eigenvalues branches of a given Steklov eigenvalue. Numerically it appears that in two dimensions when the EMP matrix is zero, depending on the deformation field \( V \), both a disk and annulus can either maximize or minimize the first branch among a particular Steklov eigenvalues branches. So numerically it appears the non-zero EMP matrix assumption is required. In three dimensions, we observe similar behavior for annular domains; but we could not find an example where a given eigenvalues first branch is both maximized and minimized by a spherical domain. Instead,
the numerics supports the implication that the first branch is always maximized by a spherical domain. It would be of interest to either establish or refute this result analytically.

More generally it would be of interest to develop analytic methods which determine the solution to the local optimization problem, even when the EMP matrix is zero; and so a second order analysis like that carried out in Dambrine et al. [18] would be of interest. The results in this thesis were achieved making use of elementary properties of the spherical harmonics, in particular no theoretical use of the cumbersome Wigner-3j formulas was needed. It is our expectation that the techniques developed in this thesis will render the second order analysis far more tractable compared to the difficulties encountered when employing the Wigner-3j symbol.
APPENDIX A. EIGENVALUE/EIGENFUNCTION FORMULAS FOR ANNULAR DOMAINS

We discuss the eigenvalues and eigenfunctions for the Steklov problem on $\mathbb{A}^d_{r_i,r_o}$, the $d$-dimensional annulus with outer radius $r_o$ and inner radius $r_i$. For annular domains the Steklov eigenvalues are no longer conveniently ordered according to multiplicity. In general, to each space of spherical harmonics $\mathbb{Y}^d_l$ is associated a pair of eigenvalues $\mu_{l,1}$ and $\mu_{l,2}$ whose eigenspaces are distinguished by a radially dependent multiplicative factor. For details of the derivation see [34] and for formulas similar to those given below see [22, Section 4]. We breakup our description of the Steklov eigenvalues and corresponding eigenspaces into two cases, because in dimension $d = 2$ we have an eigenfunction with a logarithmic radial term, which does not occur in dimension $d \geq 3$. In what follows $\{Y^m_l : l = 0, 1, \cdots ; m = 1, \cdots , N_{l,d}\}$ denotes an arbitrary orthonormal basis for the vector space of all $d$-dimensional spherical harmonics $\bigoplus_{l=0}^{\infty} \mathbb{Y}^d_l$.

When $l = 0$ we have a pair of eigenvalues given by

$$\mu_{0,1} = 0 \quad \text{and} \quad \mu_{0,2} = \begin{cases} \frac{-r_i + r_o}{r_i r_o \ln \left( \frac{r_o}{r_i} \right)}, & d = 2, \\ \frac{(d-2)(r_o^{d-1} + r_i^{d-1})}{r_i r_o (r_o^{d-2} - r_i^{d-2})}, & d \geq 3. \end{cases}$$
Each eigenvalue $\mu_{0,k}, k = 1, 2$ is simple with corresponding eigenfunction

$$u_{0,1}(r, \theta_1, \cdots, \theta_{d-1}) = \frac{1}{\sqrt{r_i^{2(d-1)} + r_o^{2(d-1)}}} Y_0^1(\theta_1, \cdots, \theta_{d-1})$$

$$u_{0,2}(r, \theta_1, \cdots, \theta_{d-1}) = \begin{cases} \frac{\ln(r)}{\sqrt{\ln(r_i)^2 r_i + \ln(r_o)^2 r_o}} Y_0^1(\theta_1) & d = 2 \\ \left(\frac{r_i^{-(d-2)} + 1}{r_i^{d-1} + r_i} + \frac{r_o^{-(d-1)} + r_o}{r_o^{d-1} + r_o}\right)^{-1} \frac{Y_0^1(\theta_1, \cdots, \theta_{d-1})}{N(r_i, \mu_{0,2,d})} & d \geq 3 \end{cases}$$

Here and below we write $N(r_i, \mu_{l,k}, d)$ for the radial dependence $r$ of the dimension $d$ eigenfunctions of $\mu_{n,k}$ orthonormalized on $S_{r_o}^1 \cup S_{r_i}^1$.

For $l \geq 1$ we have a pair of eigenvalues $\mu_{l,1}$ and $\mu_{l,2}$ given by the zeros of the quadratic equation:

$$\mu^2 - B\mu + \frac{l(l + d - 2)}{r_i r_o} = 0$$

where

$$B = \frac{(l + d - 2) \left(r_o^{2l+d-1} + r_i^{2l+d-1}\right) + lr_i r_o \left(r_o^{2l+d-3} + r_i^{2l+d-3}\right)}{r_i r_o \left(r_o^{2l+d-2} - r_i^{2l+d-2}\right)}$$

Each eigenvalue $\mu_{l,k}, k = 1, 2$ has multiplicity $N_{l,d}$ with corresponding basis of Eigenfunctions

$$u_l^m(r, \theta_1, \cdots, \theta_{d-1}) = \left(\frac{a(l, k)}{c(l, k)} r^l + \frac{b(l, k)}{c(l, k)} r^{-(d+l-2)}\right) Y_l^m(\theta_1, \cdots, \theta_{d-1}) \quad m = 1, \cdots, N_{l,d}$$
where

\[
(d + l - 2) \left( r_o^{-(d+l-1)} - r_i^{-(d+l-1)} \right) + \mu_{l,k} \left( r_o^{-(d+l-2)} + r_i^{-(d+l-2)} \right) \\
b(l, k) = l \left( r_o^{l-1} - r_i^{l-1} \right) - \mu_{l,k} \left( r_o^l + r_i^l \right)
\]

\[
c(l, k) = \sqrt{r_i^{l-1} \left( a(l, k) r_i^l + b(l, k) r_o^{-(d+l-2)} \right)^2 + r_o^l}
\]

Take note that the particular choice of coefficients \(a(l, k)\) and \(b(l, k)\) are not unique. In fact they are obtained by solving a singular system of equations whose determinant set equal to zero gives rise to the quadratic equation for \(\mu_{l,k}\).

**APPENDIX B. STANDARD ORTHONORMAL BASIS FOR SPHERICAL HARMONICS**

In this example, for \(d \geq 3\), we define a standard orthonormal basis for \(\bigoplus_{l=0}^\infty \mathbb{Y}_l^d\) which has particularly nice behavior under conjugation. In dimension \(d = 3\), define the standard orthonormal basis for the spherical harmonics

\[
Y^m_l(\theta, \phi) = \frac{(2l + 1)(l - m)!}{4\pi(l + m)!} P_l^m(\cos \theta) e^{im\phi} \quad \text{for } l \geq 0, \ m = -l, \ldots, l.
\]

where \(P_l^m(x)\) denotes an associated Legendre polynomial (see [9, Section 3.4]). Here \(l\) is the degree of the spherical harmonics and \(m\) enumerates the particular basis elements for the space \(\mathbb{Y}_l^d\). Notice that the index \(m\) ranges from \(-l\) to \(l\). This choice allows easy representation of the fact that the standard basis has symmetry under conjugation. Indeed, it is well known that the associated Legendre polynomials satisfy the condition \(P_l^m(x) = (-1)^m \frac{(l+m)!}{(l-m)!} P_l^{-m}\) from which it follows that the standard basis elements satisfy the following conjugation relation

\[
\bar{Y}_l^m = (-1)^m Y_l^{-m}.
\]
For dimension $d \geq 4$, we may build up a standard basis recursively starting with the basis elements in B.1 (see [9, Sections 3.5-3.6]) and we obtain the following

$$Y_{\mu_1, \cdots, \mu_{d-3}, l, m}(\theta_1, \cdots, \theta_2, \phi) = N(\vec{\mu}, l, m) \prod_{j=1}^{d-3} C_{\mu_j - \mu_{j+1}}^{\alpha_j + \mu_{j+1}} (\cos \theta_j) (\sin \theta_j)^{\mu_{j+1}} Y_{l}^{m}(\theta_{d-2}, \phi)$$

where $\mu_1$ is the degree of the spherical harmonic basis element, the indices $\vec{\mu}, l, m = \mu_1, \cdots, \mu_{d-3}, l, m$ satisfy $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{d-3} \geq l \geq |m|$, and $2\alpha_j = d - j - 1$. The coefficient $N(\vec{\mu}, l, m)$ normalizes $Y_{\vec{\mu}, l, m}$ in $L^2(S^{d-1})$ and $C_{\mu_j - \mu_{j+1}}^{\alpha_j + \mu_{j+1}}$ are Gegenbauer polynomials. The normalizing coefficient and the product of the Gegenbauer polynomials are real quantities; and so, it follows from (B.2) that the higher dimensional standard basis elements also satisfy a conjugation relation

$$\overline{Y_{\vec{\mu}, l, m}} = (-1)^m Y_{\vec{\mu}, l, -m}.$$  

Making use of (B.4) we prove the following conjugation property for real-valued function on the sphere.

**Proposition B.1.** If $f : S^{d-1} \to \mathbb{R}$ has Fourier-Laplace expansion in the standard orthonormal basis

$$f = \sum_{\mu_1=0}^{\infty} \sum_{\mu_2, \cdots, \mu_{d-3}, l, m} \alpha_{\vec{\mu}, l, m} Y_{\vec{\mu}, l, m}$$

then we have the following conjugation formula

$$\overline{\alpha_{\vec{\mu}, l, m} Y_{\vec{\mu}, l, m}} = \alpha_{\vec{\mu}, l, -m} Y_{\vec{\mu}, l, -m}.$$
Proof. Below we first use that \( f \) is real-valued, second, we use B.4, and third we reindex the second sum and combine like terms.

\[
0 = \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} \alpha_{\vec{\mu},l,m} Y_{\vec{\mu},l,m} - \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} \overline{\alpha}_{\vec{\mu},l,m} \overline{Y}_{\vec{\mu},l,m}
\]

\[
= \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} \alpha_{\vec{\mu},l,m} Y_{\vec{\mu},l,m} - \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} (-1)^m \overline{\alpha}_{\vec{\mu},l,m} Y_{\vec{\mu},l,m}
\]

\[
= \sum_{\mu_1=0}^{\infty} \sum_{\mu_1,\ldots,\mu_{d-3},l,m} [\alpha_{\vec{\mu},l,m} - (-1)^m \overline{\alpha}_{\vec{\mu},l,m}] Y_{\vec{\mu},l,m}
\]

Because 0 has a unique Fourier-Laplace expansion we conclude

\[
\alpha_{\vec{\mu},l,m} - (-1)^m \overline{\alpha}_{\vec{\mu},l,-m} = 0
\]

So we have \( \overline{\alpha}_{\vec{\mu},l,m} = (-1)^m \alpha_{\vec{\mu},l,-m} \) which together with \( \overline{\overline{\alpha}}_{\vec{\mu},l,m} = (-1)^m Y_{\vec{\mu},l,-m} \) gives the result.

\[\square\]

Appendix C. Calculus on Surfaces

In this appendix, for the reader’s convenience, we collect together various facts regarding calculus on surfaces.

C.1. Definition of Smooth Surface. The discussion in this subsection, including figure 12, is taken from Chapter 2 Section 3.1 of [19].

![Diffeomorphism](image)}
Let \( \{e_1, \cdots, e_N\} \) denote the standard basis of \( \mathbb{R}^N \). Given a point of \( \zeta = (\zeta_1, \cdots, \zeta_N) \) of \( \mathbb{R}^N \) we write \( \eta = (\zeta', \zeta_N) \), where \( \zeta' = (\zeta_1, \cdots, \zeta_{N-1}) \). Let \( B = \mathbb{B}_1^N \) denote the unit ball in \( \mathbb{R}^N \) and define the sets:

\[
B_0 = \{ \zeta \in \mathbb{R}^N : \zeta_N = 0 \}, \\
B_+ = \{ \zeta \in \mathbb{R}^N : \zeta_N > 0 \}, \\
B_- = \{ \zeta \in \mathbb{R}^N : \zeta_N < 0 \},
\]

We say a bounded open subset \( \Omega \subset \mathbb{R}^N \) has \( C^k \)-boundary if given any point \( x \in \partial \Omega \) there exists an open neighborhood \( U(x) \subset \mathbb{R}^N \) of \( x \) and a bijective map \( g_x : U(x) \to B \) such that:

\[
g_x \in C^k(U(x); B) \quad \text{and} \quad h_x := g_x^{-1} \in C^k(B, U(x)),
\]

\[
\text{interior}(\Omega) \cap U(x) = h_x(B_+), \\
\Gamma_x := \partial \Omega \cap U(x) = h_x(B_0) \quad \text{and} \quad B_0 = g_x(\Gamma_x).
\]

Recall that given two open subsets \( \Omega_1 \) and \( \Omega_2 \) of \( \mathbb{R}^N \) we say a vector field \( g : \Omega_1 \to \Omega_2 \) is \( C^k \) provided each coordinate function has partial derivatives up to and including order \( k \) and, in addition, the \( k^{th} \)-order derivatives are continuous.

**C.2. Definition of the Normal Vector Field to a \( C^1 \)-surface.** The discussion in this subsection is taken from Chapter 2 Remark 3.2 of [19]. For a domain \( \Omega \) with \( C^1 \)-boundary we may define an outward unit normal vector field on \( \partial \Omega \) by employing the Jacobian matrix \( Dh_x \) of the map \( h_x \) from the definition of a smooth surface in Appendix C.1. Noting that \( \{e_1, \cdots, e_{N-1}\} \subset B_0 \), we have that \( \{Dh_x(0,0)e_1, \cdots, Dh_x(0,0)e_{N-1}\} \) spans the tangent space \( T_x\Gamma_x \) at \( x \in \partial \Omega \). It follows that \( (Dh_x(0,0)^{-1})^*e_N \) is orthogonal to \( T_x\Gamma_x \). Indeed,

\[
\langle Dh_x(0,0)e_i, (Dh_x(0,0)^{-1})^*e_N \rangle = \langle Dh_x(0,0)^{-1}Dh_x(0,0)e_i, e_N \rangle = \langle e_i, e_N \rangle = 0.
\]

It follows that

\[
n_x = -\frac{(Dh_x(0,0)^{-1})^*e_N}{\| (Dh_x(0,0)^{-1})^*e_N \|}
\]
defines an outward unit normal vector field on $\partial \Omega$. The minus sign is a consequence of the fact that $B_+$ is required to map into the interior of $\Omega$ in Appendix C.1.

C.3. **Definition of the boundary integral on a $C^1$-surface.** The discussion in this subsection is taken from Chapter 2 Section 3.2.1 of [19]. The sets $U(x); x \in \partial \Omega$ defined in Appendix C.1 provide an open cover of $\partial \Omega$. So, because $\partial \Omega$ is compact, there is a finite subcover which we denote by $U_1, \cdots, U_n$. We define a *partition of unity subordinate to the cover $U_1, \cdots, U_n$* as follows

\[
\begin{cases}
  r_j \in C^\infty_o(U_j), \\
  0 \leq r_j(x) \leq 1, \\
  \sum_{j=1}^n r_j(x) = 1 \text{ in a neighborhood } U \text{ of } \partial \Omega
\end{cases}
\]

where the open neighborhood $U$, is such that $\overline{U} \subset \cup_{j=1}^n U_j$. For the existence of such a partition of unity, see [27, Theorem 2.2].

Using the notation of appendix C.1, we have that if $f \in C(\partial \Omega)$ is a continuous function on $\partial \Omega$, then

\[(fr_j) \circ h_j \in C(B_0) \quad \text{for } j = 1, \cdots, n,\]

and we define the boundary integral of $fr_j$ on $\partial \Omega_j = U_j \cap \partial \Omega$ by

\[
\int_{\partial \Omega_j} fr_j \, dS \defeq \int_{B_0} (fr_j) \circ h_j(\zeta', 0) \omega_j(\zeta') \, d\zeta.
\]

Here $(\zeta', 0)$ ranges over $B_0$ and $\omega_j = \omega_{x_j}$ is an instance of the surface density term $\omega_x$, where

\[
\omega_x(\zeta') = |(Dh_x^*)^{-1}(\zeta', 0)|| \det Dh_x(\zeta', 0)|.
\]

Finally we define the integral $f \in C(\partial \Omega)$ by

\[
\int_{\partial \Omega} f \, dS \defeq \sum_{j=1}^n \int_{\partial \Omega_j} fr_j \, dS.
\]
C.4. **Back Transport.** Given a bounded open set $\Omega \subset \mathbb{R}^d$, with $C^1$-boundary and a smooth $C^k$ vector field $V : \mathbb{R}^d \to \mathbb{R}^d$, we define for $h \in \mathbb{R}$ the transformation

$$T_h : \Omega \to \Omega_h \quad \text{by} \quad x \mapsto (I + hV)(x) = x + hV(x)$$

where $I$ is the $d \times d$ identity matrix. We have the following result from [40, Section 1.1]

If $\|hV\|_k < \frac{1}{2}$, where we use the supremum norm on $C^k$, then $I + hV$ is invertible and furthermore,

$$(I + hV)^{-1} \equiv I + \tilde{V} \quad \text{for some } \tilde{V} \in C^k.$$ 

It follows that if $h \ll 1$, then the transformation $T_h$ is invertible and the inverse is also smooth, as pictured in Figure 13. Although, in this monograph, we are primarily interested in perturbations of the identity $I + hV$, the results in this appendix apply to more general smooth transformations $T$ provided a smooth inverse $T^{-1}$ exists.

Given a function, vector field, differential operator, integral operator, etc... defined on $\Omega_h$, "back transport" refers to the process of discovering corresponding objects defined on $\Omega$, in terms of which the original objects may be computed without reference to the perturbed domain. In this appendix we catalog the back transport formulas for a collection of objects defined on a perturbed domain.

**Figure 13.** A smooth transformation of the domain $\Omega$ with a smooth inverse transformation
**Example C.1** (Back Transport of a Function). Given a function \( f : \Omega_h \to \mathbb{R} \), we define \( f^b : \Omega \to \mathbb{R} \), the *back transport* of \( f \) to \( \Omega \), as follows

\[
\text{Back Transport of a Function} \quad f^b(x) = (f \circ T_h)(x).
\]

![Diagram](image)

**Figure 14.** defining a smooth structure on a perturbed domain

A smooth structure, as defined in Appendix C.1, is defined by the composing the smooth structure on \( \Omega \) with the transformation \( T_h \) and \( T_h^{-1} \), see Figure 14.

\[
\text{Smooth Structure on } \partial \Omega \quad h_y \equiv T_h \circ h_x \quad \text{and} \quad g_y \equiv g_x \circ T_h^{-1}.
\]

Making use of this smooth structure, together with the definition of the unit normal vector field in appendix C.2 and surface integral in appendix C.3, we apply the chain rule to obtain the following back transport results.

**Example C.2** (Back Transport of a Unit Normal Vector Field). Given the unit normal vector field \( \hat{n}_h(y) \) on \( \Omega_h \), with \( y = T_h(x) \), we compute its back transport to \( \Omega \), as follows

\[
\text{Back Transport of a Unit Normal Vector Field} \quad \hat{n}_h(y) = \frac{(DT_h(x)^*)^{-1}\hat{n}(x)}{||(DT_h(x)^*)^{-1}\hat{n}(x)||}.
\]

**Example C.3** (Back Transport of a Surface Integral). Given a function \( f \) defined on \( \partial \Omega_h \), we compute its back transport to \( \partial \Omega \), as follows

\[
\text{Back Transport of a Surface Integral} \quad \int_{\partial \Omega_h} f(y) \, dS = \int_{\partial \Omega} f^b(x) \omega_h(x) \, dS
\]
where \( \omega_h \) is the surface Jacobian given by

\[
\text{Surface Jacobian} \quad \omega_h(x) = |(DT_h(x)^*)^{-1}\hat{n}(x)||\det(T_h(x)|.
\]

Of course, in this case, the back transport is just the change of variables formula for surface integrals.

**Example C.4 (Back Transport of the Gradient).** Given a function \( f : \Omega_h \to \mathbb{R} \), we compute the back transport of its gradient to \( \partial\Omega \), as follows

\[
\text{Back Transport of the Gradient} \quad \nabla f(y) = (DT_h(x)^*)^{-1}\nabla f^b(x).
\]

**Example C.5 (Back Transport of the Dirichlet-to-Neumann Map).** Recall from Definition 1.10, that given a function \( f \in H^{1/2}(\partial\Omega_h) \), the Dirichlet-to-Neumann map sends \( f \) to the weak normal derivative of its harmonic extension

\[
\mathcal{D}_0(f) = \partial_n\mathcal{E}_0(f)
\]

where the harmonic extension \( \mathcal{E}_0(f) \) is defined as equal to the unique solution \( U \) to the weak Dirichlet problem with boundary data \( f \)

\[
\text{Weak Dirichlet Problem} : \begin{cases} 
\Delta U = 0 & \text{on } \Omega_h, \\
T(U) = f & \text{on } \partial\Omega_h.
\end{cases}
\]

In order to back propagate \( \mathcal{D}_0 \) from \( \Omega_h \) to \( \Omega \), we need to find an operator \( \mathcal{D}_h : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \) such that

\[
\mathcal{D}_0(Th\phi) = \mathcal{D}_h(\phi) \quad \text{for all } \phi \in H^{1/2}(\partial\Omega).
\]

We follow the outline discussed just before Lemma 2.5 [17] and start by back transporting the weak Dirichlet problem to \( \Omega \). The back transport of the boundary condition is immediate. The boundary trace of the solution \( U^b \) of the back transported weak Dirichlet problem should equal the back transport of \( f \)

\[
T(U^b) = f^b.
\]
Applying Definition 1.6 of the weak Laplacian and the back transport formulas for the volume integral and gradient, the equation $\Delta u = 0$ on $\Omega_h$ back transports to $\Omega$ as follows

$$0 = \int_{\Omega_h} (-\Delta u) v \, dV = \int_{\Omega_h} \nabla u \cdot \nabla v \, dV$$

$$= \int_{\Omega} (DT_h^a)^{-1} \nabla u^b \cdot (DT_h^a)^{-1} \nabla v^b |\det(DT_h)| \, dV$$

$$= \int_{\Omega} \tilde{A}_h \nabla u^b \cdot \nabla v^b \, dV$$

where $v \in H^1(\Omega_h)$ is arbitrary and we define

$$\tilde{A}_h \overset{\text{def}}{=} (DT_h^{-1})(DT_h^a)^{-1} |\det(DT_h)|.$$
Data Availability Statement

The research datasets/codes associated with this article are available in Zenodo, under the reference

doi.org/10.5281/zenodo.10034741 [39].

References


[41] John Sylvester and Gunther Uhlmann. The Dirichlet to Neumann map and applications. 01 1990.


