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“It’s All for the Best”: Optimization in the History of Science

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Abstract

Many problems, from optics to economics, can be solved mathematically by finding the highest, the quickest, the shortest — the best of something. This has been true from antiquity to the present. Why did we start looking for such explanations, and how and why did we conclude that we could productively do so? In this article we explore these questions and tell a story about the history of optimization. Scientific examples we use to illustrate our story include problems from ancient optics, and more modern questions in optics and classical mechanics, drawing on ideas from Newton’s and Leibniz’s calculus and from the Euler-Lagrange calculus of variations. A surprising role is also played by philosophical and theological ideas, including those of Leibniz, Maupertuis, Maclaurin, and Adam Smith.

Keywords: optimization, history of mathematics, calculus of variations, economics, optics, principle of least action

Optimization is built into the fabric of all the sciences. Evolution requires the maximization of reproductive success, by means of the best adaptations. Physical equilibrium requires the minimization of potential energy. The shape of soap bubbles minimizes the total surface area. And light in space-time travels in the shortest path. Why do we construct scientific concepts that work this way?

Now, if you ask a mathematician “why?” the answer you get will be a proof. If you ask a scientist, or philosopher, or theologian, “why?” the explanation you get will be a deduction from the basic principles of the person’s subject. But if you ask a historian “why?”, the historian will tell you a story.

So, here's a question for me as a historian of mathematics: Why does so much of modern science look for the shortest path, the least time, the greatest volume, when we are trying to figure out how the universe works? Why should laws describing the world maximize, or minimize, something? In this paper, I will tell you the story of how this came to be in the western scientific tradition.¹ You will see the mathematician, scientist, philosopher, and theologian answering the “why” question after their own fashions along the way. And you will see that not only science, but also philosophy and, even more surprisingly, theology, are actors in the history of mathematics.

1. From Antiquity to Fermat

My story begins in the first century of the Common Era, with Heron of Alexandria. Heron was the first to use maximal and minimal principles philosophically to explain a law of physics. What he explained was the equal-angle law of reflection of light. [Heron, *Catoptrics*; see [1, pages 261–265]]. The law was known; Heron asked why it was true. First, he pointed out, everybody thinks that sight travels in straight lines. (The Greeks thought sight went from the eye to the object, not vice versa.) Heron said that all fast-moving objects, not just sight, travel in straight lines — even projectiles at the start of their motion. Why do they go straight? To get where they're going faster. As Heron put it, “By reason of its speed, the object tends to move over the shortest path.” Hence, a straight line.

And light and sight are very fast. Heron said that we see the stars as soon as we look up, even though the distance is infinite, so the rays go infinitely fast. So there is no “interruption, nor curvature, nor changing direction; the rays will move along the shortest path, a straight line.” That is true not only for ordinary unimpeded light, but for reflected light as well. Light is reflected at equal angles precisely because this gives the shortest path. Heron stated and demonstrated that “of all possible incident rays from a given point reflected to a given point, the shortest path is the one that is reflected at equal angles.”

¹ My story tracks the path of science, and the philosophy it grew up with, from the Greek-speaking world through medieval and modern Europe. It would be valuable to know more about the role of optimization in the science, philosophy, mathematics, and religion of other civilizations, especially those of China, India, and the Islamic world, but this is beyond the scope of the present paper.

Here is his diagram and proof:

Figure 1: The diagram for Heron's proof [1, page 264].

Suppose the eye is at point C, the object at D. Let reflecting at A give equal angles. Why is this the shortest path? Extend the line DA to the point C' symmetric to where your eye is at C. The shortest path from D to C' is the straight line DAC'. Since DAC' is a straight line, the vertical angles DAB and MAC' are equal. Thus the equal-angle path DAC equals the straight line DAC', and therefore the path DAC is the shortest path.

If there were reflection around another point B, where the angle of reflection does not equal the angle of incidence, the non-straight-line path DBC', and thus the path symmetric to it, DBC, would be longer. Thus, says Heron, "Reflection at equal angles is in conformity with reason," which means, light takes the shortest path. As we will see, this was an extremely influential argument.

Let us turn now to another ancient mathematician, Pappus of Alexandria (fourth century CE). Pappus was interested in what he called isoperimetric problems in geometry. For instance, of all plane figures with the same perimeter, which has the greatest area? Pappus introduced his discussion of isoperimetric problems by enlisting an unusual mathematical colleague: the honeybee. [Pappus, *The Collection*, Book 5; see [12, pages 185–186]].

Pappus said, “God gave human beings the best and most perfect notion of wisdom in general and of mathematical science in particular, but a partial share in these things he allotted to some of the unreasoning animals as well.” He noted that when bees set up their honeycombs, they divide them into hexagons. He explained why they did so thus: “The bees have contrived this by virtue of a certain geometrical forethought... the figures must be contiguous to one another... their sides common, so that no foreign matter could enter the interstices between them and so defile the purity of their produce.” And the figures have to be regular polygons, he said, “because the bees would have none of figures which are not uniform.” Only three regular polygons, he continued, the square, the triangle, and the hexagon, are capable by themselves of exactly filling up the space about the same point. Finally, he concluded, the bees know that the hexagon has larger area than does the square or the triangle with the same perimeter, so they know the hexagon “will hold more honey for the same expenditure of wax.” Thus the bees have solved what we now call the problem of economically tiling the plane with regular polygons.

Besides the ideas of Heron and Pappus, there is another influential maximal principle that we have inherited from Greek thought. However, this one is not from mathematics or physics, but from philosophy. This principle claims to answer the question, why are there so many different kinds of things in the universe? Plato (427-347 BCE) gave this answer in his *Timaeus* [pages 30, 33]: The universe is made so that it contains the maximal amount of being.² The number of beings in the universe is the greatest possible, wrote Plato, because of the goodness and lack of envy of the creator. That is, the creator exists, and is not envious, so he makes the universe as much like himself as possible, full of things that exist [18, pages 46–55].

² That the universe contains the maximal amount of being is called the Principle of Plenitude, and [18, pages 380–381] lists dozens of important thinkers who spoke favorably of this principle, including (besides Plato), Abelard, Aquinas, Thomas Browne, Victor Hugo, Averroes, Galileo, Henry More, Glanvill, George Herbert, Leibniz, Spinoza, Locke, Addison, Bolingbroke, Petty, Monboddo, Schiller, Schleiermacher. The importance of the principle in the philosophy and literature and theology of the seventeenth and eighteenth centuries is amply documented in [18, pages 99–181]. The important role of the Principle of Plenitude in the Scientific Revolution is discussed in depth in [14, pages 25, 42, 44, 52, 188, 275].

However unusual Plato's principle of maximal existence may seem to modern thinkers, it was picked up by various theologians and philosophers and was remarkably influential. For instance, in the seventeenth century, the principle of maximal existence was used to argue that the universe is infinite, and to argue that the stars have planets around them, even planets with inhabitants.

One could quote hundreds of later statements influenced by Plato, Heron, and Pappus about maximum or minimum principles and economy. I'll settle for four. Olympiodorus, in the sixth century, said, "Nature does nothing superfluous or any unnecessary work." Robert Grosseteste, in the thirteenth century, said, "nature always acts in the mathematically shortest and best possible way." William of Ockham, in the fourteenth century, stated the doctrine known as Occam's razor, which, simplified, has entered the language as telling us that the simplest explanations are the best. In the Renaissance, Leonardo da Vinci said that nature is economical and her economy is quantitative; for instance, living things eat each other so that the maximum amount of life can exist from the minimum amount of material [13, page 580]. So the view that the universe must follow optimal principles has been very pervasive. Nevertheless, this view cannot be said to have been empirically based. So far, we are in the history of philosophy — or, when God is thought to be involved, theology.

But eventually, these philosophical and theological ideas entered the exact sciences. They found their culmination in the seventeenth century in the treatment of the refraction of light by Pierre de Fermat. When Fermat took up the subject, the law of refraction itself had already been discovered, independently by Willebrod Snell and Rene Descartes. That law states that when light goes from one medium into another, like from air into water, the sine of the angle of incidence divided by the sine of the angle of refraction is a constant for this pair of media. That is what we now call Snell's law. (Figure 2).

And now we are ready to look at Fermat and what he did. Calculus and physics textbooks tell us that "Fermat's principle" in optics says that when light is refracted from one medium to another, it takes the path that minimizes the time. But that is not what Fermat said. Fermat was a mathematician, not a physicist, and a truly great mathematician at that.

Snell's (& Descartes') Law:
 $\sin(\theta_1) / \sin(\theta_2)$ is constant

Figure 2: Snell-Descartes Law of Refraction.

He, like Descartes, was an independent inventor of analytic geometry. Fermat invented a method of finding maxima and minima, and methods of finding tangents and areas. He did pathbreaking work in number theory and, with Pascal, helped found probability theory. He also had a career in politics and law. But he got involved in optics only late in his career.

Years before Fermat's work on analytic geometry became known, Descartes published his *Geometry* in 1637. And when Descartes heard about Fermat's similar work, he responded with disrespect. Descartes claimed that Fermat's tangent method was not general (it is in fact better than Descartes' method, which required first finding a circle tangent to the curve, while Fermat's method was not only general but purely algebraic), and Descartes said that Fermat should read Descartes' own *Geometry* to learn what was correct. Fermat was annoyed by this, so when he became acquainted with Descartes' work on optics, Fermat was in no mood to be charitable. He strongly criticized Descartes' derivation of the law of refraction. [22, page 376]

Descartes imagined that light was a mechanical motion of the particles in a medium he called the "ether." When a ray of light crosses a boundary to a place where the ether has different density, Descartes said, it was like when a ball hits a tennis net. The component of velocity parallel to the net is unchanged, but the component perpendicular to it is changed.

One might think it would be slowed down in a denser medium. But no, Descartes said. Coming into the denser medium, it gets a little kick from the net. That is why its path is bent toward the perpendicular. [22, pages 378–380]

Fermat thought Descartes' justification was nonsense, so he attacked the problem himself, but in a quite different way. Fermat's approach was motivated by a man who is hardly a household name: Marin Cureau de la Chambre. In 1657, Cureau wrote about the law of reflection of light exactly the way Heron had, by saying "nature always acts along the shortest paths." Fermat liked this approach, so he wrote to Cureau proposing to link his own mathematics to Cureau's physics; that would show Descartes! [22, page 382]

But the "shortest path" for refraction obviously cannot mean "the shortest distance." Instead, Fermat used an idea of Aristotle's, that velocity in a medium varies inversely as the medium's resistance to motion. Fermat then defined the path length as the product of the distance and the resistance. Now we can understand in detail how Fermat, in 1662, derived the law of refraction. See Figure 3.

For Fermat, the path from C to I to be minimized in refraction is not the sum of the two lines CD and DI, but a sum involving multiples of those lines, the multiples being determined by the ratio of the resistances. Let there exist a line called M, such that the ratio of the resistances is M/DF .

Then Fermat used his method of maxima and minima to minimize the path defined by the expression for the sum of each distance times the corresponding resistance to motion: $CD \times M + DI \times DF$.

Minimizing the quantity that he calls the path in fact minimizes the time travelled, since he follows Aristotle in assuming that resistance varies inversely with velocity, giving "path" the units of time, but Fermat did not say here that he was minimizing the time. This is partly because Cureau thought light travels infinitely fast and partly because Fermat did not want to take a position on the physical question of whether light speeds up or slows down in the denser medium; he chose to stay as mathematical as he could.

Fermat expected that, by applying his method of maxima and minima to find the conditions for the shortest path, he would be able to derive the true

Figure 3: Fermat path resistance derivation: Define M such that the ratio of resistances is M/DF . "Path" is essentially distance times resistance. Minimizing the "path" gives Snell's law: ratio of resistances is DH/DF . Resistance varies inversely with velocity, and we multiply distance by this to get time, so we have also minimized time.

law of refraction, whatever it was. Fermat must have been astounded when he got the result that $M = DH$, which meant that the ratio of resistances was DH/DF — which is equivalent to Snell's Law, since the ratio DH/DF is the ratio of the sines. [22, pages 387–390]. But now, Snell's law is no longer merely a description of how refraction takes place. Now Fermat could say that he knew why that law is followed. It is because refracted light, like reflected light, travels by the shortest path.

Fermat’s explanation of why Snell’s law must necessarily be the law of refraction is supremely important, for two reasons. First, it helped establish Snell’s law as an important physical law, rather than merely an individual empirical relationship. And, more crucially, Fermat’s explanation shows that “shortest path” arguments are not just philosophy. They give you real, non-trivial physics.

We have now reached the end of the first part of our story. We have seen that explaining physical laws by showing that they maximize or minimize something comes from philosophy and theology, and long pre-dates the calculus. And we have seen such an explanation entering seventeenth-century optics in tandem with Fermat’s algebraic method of maxima and minima.³ The second part of our story will move even more deeply into mathematics.

2. Leibniz, Bernoulli, and Optimization

The invention of the calculus made it easier — and a lot more natural — to seek and find maxima and minima. Leibniz, for both philosophical and mathematical reasons, was especially interested in doing so. As a philosopher, Leibniz had philosophical reasons for wanting to maximize and minimize things, and, as an independent inventor of the calculus, he was able to do so very well. In fact, Leibniz called his first publication of his differential calculus a “new method of maxima and minima” ([17]; translated in [26, pages 273–280]).

Leibniz’s first non-trivial application of his method of maxima and minima was to derive the law of refraction using his new differential calculus. He assumed that the light follows the shortest path, defined path as distance times resistance, set the differential of the path function equal to zero, and obtained Snell’s Law. Figure 4 presents his derivation. The derivation is elegant, and Leibniz said that this shows how good his new calculus is.

But why, for Leibniz, should light follow the shortest path? Not just because Fermat said so. It is an example of something much more general that lies at the heart of Leibniz’s philosophy: the principle of sufficient reason.

³ For details of Fermat’s method of maxima and minima, see the selections from Fermat in [26, pages 222–227], and the detailed description in [12, pages 509–510].

Figure 4: Leibniz derives Snell's law from the "shortest path" assumption using calculus (page 278 from [26]).

"Nothing happens without a reason why it is so and not otherwise," Leibniz said ([16], [32, page 92]). Leibniz gave many examples of early instances of the use of the principle. For instance, Archimedes had used the principle of sufficient reason to prove that a balance with equal weights at equal distances balances, because there is no reason it should incline to one side rather than the other. The principle of sufficient reason leads naturally to ideas of symmetry and economy. But Leibniz pushed it much farther. "For every true proposition [about nature], ... a reason can be given," said Leibniz. He added that, "The first decree of God, [is] to do always what is most perfect" ([15], [32, page 94]). And he related these views to Plato's principle of maximal existence.

How many beings must this world contain? All possible kinds, said Leibniz. Leibniz famously stated that this is the best of all possible worlds, and what he meant by “best possible” was the world in which “the quantity of existence is as great as possible.” So God, from all logically possible worlds, chooses the best, the one with the greatest number of things in it [18, page 172]. According to Leibniz, that explains why the laws of nature are as simple as possible. This allows God to find room for the most possible things. Leibniz wrote, “If God had made use of other, less simple laws, it would be like constructing a building of round stones, which leave more space unoccupied than that which they fill.” [18, page 179]

Thus, Leibniz’s philosophy demands that natural laws will be the simplest, with the maximum existence, the shortest paths. This principle may be easy for us to challenge today, but Leibniz’s philosophical justification for the simplest laws, the maximum existence, the shortest paths, made his contemporaries much more interested in, and excited about, finding such laws.

For example — a very influential example — Leibniz’s disciple Johann Bernoulli, in 1696, challenged the mathematical world with an interesting minimum problem. Let a body fall under gravitation. What is the curve along which its descent will be quickest? One might first guess that it would be a straight line, but, fifty years before Bernoulli, Galileo showed that it was not. Galileo himself thought it was a circle, but it is not that either. Bernoulli solved the problem by using Fermat’s and Leibniz’s “path with resistance” idea, and by that approach, he found that the path of quickest descent is a cycloid ([12, pages 586–588],[26, pages 391–399]).⁴

Bernoulli modeled the curve of quickest descent by imagining the body “falling” through a medium made up of many infinitesimally thin layers (illustrated by four layers in my sketch in Figure 5). Each layer has a different density and thus each layer has a different resistance. The path of least-time descent is modelled in the same way as the curved path of a light ray. The path is bent as it passes from one layer to the next. In each layer, the ve-

⁴ Though it is not part of the present story, it is worth knowing that Newton solved the problem also, on the night that he received it [31, page 582]. See also [12, page 586].

Figure 5: Bernoulli “many levels” Snell’s law.

locity varies inversely as the resistance, so, as it enters each layer, it moves according to Snell’s law, where the sine of each new angle depends on the new velocity. Working this out for the interface between each infinitesimally thin pair of layers, and using Leibniz’s calculus, Bernoulli got a differential equation (where t is his notation for the velocity; see Figure 6):

$$dy = t dx : \sqrt{(aa - tt)}$$

or in modern notation:

$$dy = \frac{t dx}{\sqrt{a^2 - t^2}}$$

For the case of falling bodies, as Galileo had already shown without calculus in the 1630s, the velocity squared of a falling body varies as the distance fallen. Using this fact, the differential equation becomes this one:

$$dy = dx \sqrt{\frac{x}{a - x}},$$

which, Bernoulli recognized, defines the curve known as the cycloid. See Figure 7.

Figure 6: Bernoulli differential equation.

So the cycloid is the curve of quickest descent. Bernoulli thought this fact was important for both philosophical and mathematical reasons. The method he used here, he said, “solves at one stroke two important problems — an optical one and a mechanical one.” These two vastly different problems “have the same character” [26, page 394]. That is already economical. Furthermore, Bernoulli added, the cycloid is also the curve in which, as Huygens found, pendulums of fixed length and arbitrary amplitude oscillate in the same time. So both the shortest-descent curve and the curve for same-time pendulum motion is the cycloid. This cannot be an accident. In answering the question of why the same curve can serve two separate purposes, Bernoulli explained, and helped establish, the general principle: “Nature *always* tends to act in the simplest way.” [26, page 395; my italics]

Figure 7: Bernoulli cycloid: the details of Bernoulli's solution from [26].

3. From Maupertuis to Hamilton

Now that we have described this background, the third and fourth parts of this paper will address what happened to these ideas in the eighteenth and nineteenth centuries. The first line of development occurred in physics, and stems from Pierre-Louis Maupertuis. The second is scientific but is also philosophical and theological, and stems from Colin Maclaurin.

We begin with Maupertuis, who was an admirer of Newton and Newtonian physics. It was Maupertuis who showed that Newton was right in predicting the flattened-spheroid shape of the earth. Maupertuis travelled north to Lapland and actually measured the length of a degree of arc of the meridian there.

But Maupertuis, though his physics was Newtonian, philosophically was a Leibnizian. In 1744 he wrote a paper on the refraction of light. His goal was philosophical: to use “those laws to which Nature herself seems to have been subjected by a superior Intelligence, who, in the production of His effects, makes Nature always proceed in the simplest manner” [27, page 176]. Like Leibniz, Maupertuis believed that individual laws of nature exemplify general rational principles, and he wanted to find what those rational principles are.

When Maupertuis tried to find the rational principle that explains the reflection and refraction of light, he appealed to Leibniz’s principle of sufficient reason, saying that there is no reason for light to choose time, rather than distance, to minimize. So there must be something more general than either distance or time, and this more general thing will be minimized both in the reflection of light and in the refraction of light, and also by unimpeded light moving in a straight line. What is it that is minimized? Maupertuis invented something that is: he called it “action.” He initially defined “action” as the product of speed and distance. And he showed that minimizing this “action” gives you all three laws of light. Here we have the first instance of something being called the principle of least action. The mathematics was simple, and the paper captured many people’s imaginations.

In 1746, Maupertuis added mass to his definition of “action” so it became mvs . He was able to use this improved principle to describe the motion of two bodies colliding on the plane, with the collision either elastic or inelastic. He explained the laws of collision via least action by saying that, since nature acts as simply as possible, “whenever there is any change in nature, the quantity of action necessary for that change is the smallest possible” [27, page 272]. For collisions, this is mathematically equivalent to what we now call the conservation of momentum and energy.

Maupertuis saw the success of his approach as a mathematically-based proof for the existence of God. He said that he had discovered a universal principle, uniting all phenomena in the universe with “maximal efficiency,” and that this demonstrates God’s infinite wisdom in planning it all [27, page 276]. Again, this may be too theological for us, but Maupertuis’s work influenced d’Alembert, Euler, Lagrange, and Hamilton. These men refined the concept of “action” and used it in their various formulations of what we now call classical mechanics. Let us briefly look at what they did.

D'Alembert realized that the principle of least action means that one no longer needed to talk about force. Forces, said d'Alembert, are “obscure and metaphysical entities.” Instead, he said, we should consider motion by means of geometrical paths in space, without speculating about causes. He played down the metaphysics of the principle of least action, instead calling it a “geometrical truth.” Nonetheless, he said, it is an important truth, since it works, not only for light and for collisions, but also for planetary orbits, as Euler had shown [27, pages 290–292].

Euler, unlike d'Alembert, did accept Maupertuis's metaphysics. Euler wrote, “Since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear” ([13, page 573], [27, page 278]). And, as a pioneer of the calculus of variations, Euler developed the mathematics to back this statement up.⁵ His work on the calculus of variations grew out of studying many problems, each of which had as its goal to find a curve that maximizes or minimizes a particular integral [26, pages 399–406]. See Figure 8.

In the general theory Euler gave in his *Methodus inveniendi* (1744), the key idea was to use a polygonal approximation to an integral to develop a necessary condition for an extremal value, a condition eventually expressed as a differential equation. In modern notation,

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0.$$

From this equation he gets many special cases, including those we have already discussed. For instance, minimizing the distance function gives $y' = 0$ and thus the path is a straight line (see Figure 9). Minimizing the time of fall under gravitation yields Johann Bernoulli's equations for the cycloid (see Figure 10). And so on.

Euler saw that his derivation needed some more work. It was geometric, thus relying on intuition instead of pure analysis [26, pages 399–400, 407]. Also, it did not address finding a sufficient condition for the extremum. The 19-year-old Joseph-Louis Lagrange, beginning in 1755, took the next steps [3].

⁵ For a thorough account of the history of the calculus of variations, see [4].

Figure 8: Euler minimizes an integral.

To be sure, Lagrange began with Euler's mathematics, but Lagrange also paid homage to Maupertuis's philosophy. Lagrange wrote to Maupertuis in 1756 that he, Lagrange, was working on a paper which would "demonstrate with the greatest possible universality how your principle always supplies with marvelous facility the solution to all cases most complicated and hard to solve otherwise, in dynamics as well as hydrodynamics... I am fortunate... to contribute... to the universal application of such a principle... the most beautiful and important discovery of mechanics" [27, page 355]. Lagrange used the principle of least action and the method of the calculus of variations to get his equations of motion. Nor was this all. He also applied a minimum principle in fluid dynamics, for both compressible and incompressible fluids, and was able to derive Euler's equations for fluid dynamics. Thus, Lagrange concluded, a minimum principle governs this subject just as it does the motion of particles and rigid bodies [13, page 739]. Three cheers for the principle of least action!

Figure 9: Euler integral necessary condition.

Or perhaps only two cheers. Although William Rowan Hamilton wanted to continue the tradition of Lagrange's deductive mathematical approach to mechanics, Hamilton decided that the time had come to throw out the metaphysics. Writing in 1833, Hamilton said, "Though the law of least action has attained a rank among the highest theorems of physics, yet its pretensions to a cosmological necessity, on the ground of economy in the universe, should be generally rejected" [13, pages 740–741].

In other words, those philosophical and theological arguments got us here, but now it's time to kick away the ladder. Of course, we still use these principles. Philosophers appreciate such broad formulations — consider Ernst Mach's phrase "the economy of thought" — perhaps for philosophical, perhaps for aesthetic reasons. But physicists since Hamilton no longer cite such

Figure 10: Euler gets the cycloid.

principles as evidence of God's wisdom or efficiency. Nevertheless — and this is my key point — believing in God's and Nature's rationality and efficiency is what led people to these formulations in the first place. Traces of the metaphysics remain in popular scientific culture. Witness, for instance, the title of Hildebrandt and Tromba's recent book on optimal principles in science, *The Parsimonious Universe* [9], or the title of Paul Nahin's more mathematical book on the same topic, *When Least Is Best* [24].

4. From Maclaurin to Classical Economics

The general point that we still use the philosophical arguments in modern thought will become clearer as we look at the last of the stories I want to tell: how the philosophical, theological, and mathematical ideas moved once more, this time from science into the larger society. Here, we begin with Colin Maclaurin.

When Maclaurin, a minister's son, was sixteen, he tried to build a calculus-based mathematical model for ethics. In a Latin essay whose title translates into "On the good-seeking forces of mind" [19], Maclaurin analyzed mathematically the forces by which our minds are attracted to different morally good things. Using language borrowed from Newton's physics, Maclaurin said that the "forces with which our minds are carried towards different good things are, other things being equal, proportional to the quantity of good in these good things." Also, the attractive force of a good one hour in the future exceeds the attractive force of the same good several hours in the future. Maclaurin represented the total quantity of good as the area under a curve whose x -coordinate was time and whose y -coordinate was the intensity of the good at that time. Maclaurin said one could find the maximum and minimum intensities of any good or evil using calculus, and that one could integrate and find the total good over any finite or infinite time. One conclusion supported by his mathematical models was that good men need not complain "about the miseries of this life" since "their whole future happiness" (eternal life after death) will be greater. Thus the young Maclaurin had proved mathematically, at least to his own satisfaction, that the Christian doctrine of salvation maximizes the happiness of good men.

Of course, Maclaurin grew up. The mature Maclaurin was a mathematician of stature and a respected contemporary of Euler. Maximizing and minimizing are important to Maclaurin's mature work in mathematics and mathematical physics. It was Maclaurin who gave the first complete theory of maxima and minima in terms of whether derivatives of all orders are positive, negative, or zero. He did this using the Maclaurin series to study maxima, minima, and points of inflexion of curves ([7, pages 216–217], [21, §261, §§858–859]). Maclaurin's work on extrema was highly influential. He also applied the techniques of finding maxima and minima in many physical situations and then compared his results to the best data. He studied solids of least resistance, and the best designs for the hulls of ships, waterwheels, and windmills. He used these techniques also in his studies of the shape of rotating bodies under gravitation and thus of the shape of the earth ([6], [7, pages 217–218]).

And, remember Pappus and the bees from Section 1? Maclaurin added the third dimension. He wrote a paper [20] telling how bees, in constructing the

cells of their honeycombs, use the shape that, for a given amount of material, maximizes the volume of honey that a cell can contain; see Figure 11.

Figure 11: Maclaurin's bee cell diagram [23, page 387]: maximum volume when the angle is $2 \arctan \sqrt{2}$, about $109^\circ 28''$.

The pyramidal base of a cell of a honeycomb, whose side is hexagonal, is bounded by three rhombuses. In this situation, when does one get the maximal volume for a given surface area? The maximal volume is obtained, Maclaurin showed in an elegant geometric argument, when the angle of the rhombus is $2 \arctan \sqrt{2}$, which is about 109 degrees 28 minutes ([20], [23, pages 386–387]). A few years earlier, Maraldi and de Réaumur had actually measured the angles of honeycomb cells — this is not easy — and got 110 degrees. Good job of optimization, bees.

Why did Maraldi and de Réaumur want to measure that angle? Whether on the Continent or in Britain, the motivation was the same. Maraldi and de Réaumur both hoped to be able to show that the bees chose the most economical angle. They themselves were unable to do this. Although Maclaurin did it purely geometrically, it obviously could be done by calculus. Samuel Koenig, who was a follower of Leibniz, did do it using calculus. The average bee doesn't know much calculus, of course, so Koenig turned to theology, saying that divine guidance had been given to the bees [28, page 23].

Maclaurin, too, included philosophical observations with his argument. He began by saying that the fact that the bees' design uses the least wax confirms the honeycomb's "Regularity and Beauty, connected of Necessity with its Frugality" [23, page 386]. And he concluded, "By following what is Best in One Respect, unforeseen Advantages are often obtained, and what is most Beautifull and Regular, is also found to be most Useful and Excellent." [23, page 391]

Charles Darwin was impressed with this result, which, as with other adaptations in nature, he did not ascribe to divine guidance. In Chapter VI of his *Origin of Species*, Darwin stated that not only traits, but also instincts, can be acquired and modified through natural selection. This applies even, wrote Darwin, to "so marvelous an instinct as that which leads the bee to make cells, which have practically anticipated the discoveries of profound mathematicians."

Interest in this question has continued into recent times. Tom Hales has proved what is called the Honeycomb Conjecture: Any tiling of the plane into regions of equal area must have total perimeter at least that of the regular hexagonal honeycomb tiling [8]. So Pappus was right; the bees did know it. Still, for three dimensions, the mathematician Fejes Toth showed in 1964 that if one constructed honeycomb cells bounded by two rhombuses and two hexagons instead, slightly less wax would be used than in Maclaurin's solution, so the bees' choice differs from the optimal by .35% [29].

The influence of Maclaurin's philosophical and theological ideas on maxima and minima, though, goes far beyond science. Maclaurin's early interest in maximizing goodness helped promote some other well-known and influential ideas. One of Maclaurin's classmates at Glasgow was the moral philosopher Francis Hutcheson. Hutcheson used mathematical principles — and language like Newton's and Maclaurin's — to describe and to demonstrate his supposed laws of virtue. Consider Hutcheson's essay of 1728, "The manner of computing the morality of actions." Hutcheson wrote, "the Virtue is in proportion to the Number of Persons to whom the Happiness shall extend... and in equal Numbers, the Virtue [varies] as the Quantity of the Happiness." So, he concluded, "That Action is best, which procures the greatest Happiness for the greatest Numbers." Later on, a similar approach to the study of society is found in the utilitarianism of Jeremy Bentham and John Stuart Mill,

who saw the best action as the one that produces the greatest good for the greatest number. People like Hutcheson, Bentham, and Mill thought that they were importing the methods of natural science into the social realm.

The economist Adam Smith will provide my last example. The resemblance between some of Smith's ideas and Maclaurin's and Hutcheson's is no accident. Before going on to Oxford, Smith was Hutcheson's student at Glasgow. And Smith not only learned science, he actually wrote a long essay, full of admiration for Newtonian science, about the history of astronomy.⁶

Smith's great 1776 classic in economics, *The Wealth of Nations* [25], states, in words resembling Hutcheson's: "Upon *equal*... profits... every individual naturally inclines to employ his capital in the manner in which it is likely to afford the *greatest* support to domestic industry, and to give revenue and employment to the *greatest* number of people of his own country." (my italics)

This is no minor point for Adam Smith. The most famous passage from *The Wealth of Nations* immediately follows the one I just quoted, a fact which shows how central this point is for Smith's economics. The individual, said Smith, "generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it... he intends his own gain, and he is thus ... led by an invisible hand to promote an end which was no part of his intention" [25, Book IV, Chapter II]. That is, individuals try to maximize their personal gain, and as a result, the outcome is the best of all possible economic worlds. Whether readers agree or disagree with Adam Smith, I am sure they will agree that the influence of these ideas is still with us today.⁷

⁶ Smith's essay, entitled *The Principles which Lead and Direct Philosophical Enquiries Illustrated by the History of Astronomy*, was not published until 1880. Note that the term "philosophical" in Smith's time often meant what we would today call "scientific". Compare Newton's great *Principia*, whose full title in English is *Mathematical Principles of Natural Philosophy*.

⁷ Among examples of the influence of Smith's analysis of economic efficiency, especially the key idea of the division of labor, see any economics textbook. F. W. Taylor's idea of "scientific management," which involved detailed decomposition of industrial labor tasks in order to organize them into an optimal production method, is also related to the history of the ideas we have been tracing. See (and note the main title of) [11].

5. Conclusion

Let us return to our original questions. Why do explanations presupposing maxima and minima work so well and in so many scientific fields? That is a philosophical question. Saying “It’s because nature is governed by conservation laws, or by partial differential equations” just changes the terms; it doesn’t answer the question. So why is nature like that? Maybe that is how God chose to design the world. Maybe the universe itself is parsimonious. Maybe it is an artifact of the way we think. But it does work.

But the historical question, “How did we come to learn to expect, seek, and therefore find such explanations?” has, I think, been answered. Philosophical ideas about the economy of nature and about God as a rational economist, ideas powerfully reinforced by examples from geometrical optics and the geometrical insights of honeybees, and vastly accelerated by the techniques of the calculus — these ideas have led to Hamiltonian mechanics, to the idea of the greatest good for the greatest number, to the invention of the calculus of variations, and to the theories of free-market economics.

Science has high standards for proof. But for discovery, it looks as though anything goes, from beekeeping to theology. Once you’ve made a discovery, though, it’s mathematics that lets you rigorously work out the consequences, and then test those consequences against nature.

Science did not have to develop in the way that it did. It did so as the result of a number of contingent historical and cultural events. I believe that science’s successful search for the best, and most, and most economical embodies and validates the idea — an idea now so embedded in our teaching and practice that we cannot imagine that it ever was otherwise — that mathematics in general, and the calculus in particular, is the best way to model the world.⁸

However, whether the objects of our study are *necessarily* optimal has not been demonstrated, even if assuming them to be so has yielded many successes. That the economic marketplace always produces the best possible result is a generalization, not an observation, and is historically based on theological and philosophical ideas.⁹ So, at the end, a caution:

⁸ For an interesting and influential discussion of this topic, see [33].

⁹ This generalization has been criticized from a theological point of view, with worship

We were lucky that theology and philosophy led us to ask certain scientific questions, and that mathematical analysis allows us answer them rigorously. Because using optimization has been so successful, it is tempting to conclude that proceeding this way was “all for the best.” But, as the computer scientist Joseph Weizenbaum observed long ago, we need to remain aware of the limitations of our tools as well as their power [30, page 277].

References

- [1] Cohen, Morris R., and Drabkin, I. E., *A Source Book in Greek Science*, Harvard University Press, Cambridge, MA, 1958.
- [2] Cox, Harvey, “The Market as God,” *The Atlantic Monthly*, March 1999, 18. Available at <https://www.theatlantic.com/magazine/archive/1999/03/the-market-as-god/306397/>, last accessed on January 21, 2021.
- [3] Fraser, Craig, “J.-L. Lagrange’s Changing Approach to the Foundations of the Calculus of Variations,” *Archive for History of Exact Sciences*, Volume **32** (1985), pages 151–191.
- [4] Goldstine, Herman, *History of the Calculus of Variations from the 17th through the 19th Century*, Springer, New York, Heidelberg, Berlin, 1977.
- [5] Grabiner, Judith V., “Descartes and Problem-Solving,” *Mathematics Magazine*, Volume **68** (1995), pages 83–97. Reprinted in [7, pages 175–190].
- [6] Grabiner, Judith V., “Was Newton’s Calculus a Dead End? The Continental Influence of Maclaurin’s Treatise of Fluxions,” *American Mathematical Monthly*, Volume **104** Number 5 (1997), pages 393–410. Reprinted in [7, pages 209–228].
- [7] Grabiner, Judith V., *A Historian Looks Back: The Calculus as Algebra and Selected Writings*, Mathematical Association of America, 2010.
- [8] Hales, Thomas, 2002. “The Honeycomb Conjecture,” arXiv preprint available at <https://arxiv.org/abs/math/9906042>, last accessed on January 21, 2021.

of the market mechanism being compared to idolatry. For one of many examples, see [2].

- [9] Hildebrandt, S., and Tromba, A., *The Parsimonious Universe: Shape and Form in the Natural World*. Springer, New York, 1995.
- [10] Hutcheson, Francis, “On the Manner of Computing the Morality of Actions,” 1728. Reprinted in Francis Hutcheson, *An Inquiry into the Original of Our Ideas of Beauty and Virtue*, Revised ed. Liberty Fund, 2012. Project MUSE: <https://muse.jhu.edu/book/18165>.
- [11] Kanigel, Robert, *The One Best Way: Frederick Winslow Taylor and the Enigma of Efficiency*, Viking Books, New York, 1997.
- [12] Katz, Victor J., *A History of Mathematics: An Introduction*, third edition, Addison-Wesley, Boston, 2009.
- [13] Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972.
- [14] Koyré, Alexandre, *From the Closed World to the Infinite Universe*, Johns Hopkins, Baltimore, 1957.
- [15] Leibniz, G. W., “The Principle of Sufficient Reason,” from *Theoria motus abstracti* (1671), in [32, pages 93–96].
- [16] Leibniz, G. W., “The Exigency to Exist in Essences: Principle of Plenitude” [n.d.] in [32, pages 91–93]; [18, pages 177ff].
- [17] Leibniz, G. W., “A New Method for Maxima and Minima... and a Remarkable Type of Calculus for Them,” 1684, translated and annotated in [26, pages 271–280].
- [18] Lovejoy, Arthur, *The Great Chain of Being: The Study of the History of an Idea*, Harvard University Press, Cambridge, MA, 1961.
- [19] Maclaurin, Colin, “De viribus mentium bonipetis,” 1714, MS 3099.15.6, The Colin Campbell Collection, Edinburgh University Library.
- [20] Maclaurin, Colin, “Letter to Sir Martin Folkes, November 3, 1732”. Reprinted in [23, pages 386–391]. Published as Colin Maclaurin “On the Bases of the Cells wherein the Bees deposite their Honey,” *Philosophical Transactions of the Royal Society*, **42** (1742–1743).
- [21] Maclaurin, Colin, *A Treatise of Fluxions in Two Books*, T. Ruddimans, Edinburgh, 1742.

- [22] Mahoney, Michael S., *The Mathematical Career of Pierre de Fermat, 1601–1665*, Princeton University Press, Princeton, 1973.
- [23] Mills, Stella, ed., *The Collected Letters of Colin MacLaurin*, Shiva Publishing, Nantwich, 1982.
- [24] Nahin, Paul, *When Least is Best: How Mathematicians Discovered Many Clever Ways to Make Things as Small (or as Large) as Possible*, Princeton University Press, Princeton, 2004.
- [25] Smith, Adam, *An Inquiry into the Nature and Causes of the Wealth of Nations*, W. Strahan and T. Cadell, London, 1776.
- [26] Struik, D. J., ed., *A Source Book in Mathematics, 1200–1800*, Harvard University Press, Cambridge, MA, 1969.
- [27] Terrall, Mary, *The Man Who Flattened the Earth: Maupertuis and the Sciences in the Enlightenment*, University of Chicago Press, Chicago, 2002.
- [28] Thompson, D’Arcy, *On Growth and Form*, Cambridge University Press, Cambridge, 1917.
- [29] Toth, Fejes, “What the bees know and what they do not know,” *Bulletin of the American Mathematical Society*, Volume **70** (1964), pages 467–481.
- [30] Weizenbaum, Joseph, *Computer Power and Human Reason: From Judgment to Calculation*, Freeman, San Francisco, 1976.
- [31] Westfall, Richard S., *Never At Rest: A Biography of Isaac Newton*, Cambridge University Press, Cambridge, 1981.
- [32] Wiener, Philip, ed., *Leibniz: Selections*, Scribner’s, New York, 1951.
- [33] Wigner, Eugene, “The Unreasonable Effectiveness of Mathematics in the Physical Sciences,” *Communications in Pure and Applied Mathematics*, Volume **13** Number I (February 1960), pages 1–14. Reprint available online at <https://www.maths.ed.ac.uk/~v1ranick/papers/wigner.pdf>, last accessed on January 21, 2021.