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Intersection Cographs and Aesthetics

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Abstract

Cographs are complete graphs with colored lines (edges); in an intersection cograph, the points (vertices) and lines (edges) are labeled by sets, and the line between each pair of points is (or represents) their intersection. This article first presents the elementary theory of intersection cographs: 15 are possible on 4 points; constraints on the triangles and quadrilaterals; some forbidden configurations; and how, under suitable constraints, to generate the points from the lines alone. The mathematical theory is then applied to aesthetics, using set cographs to describe the experience of a person enjoying a picture (Mu Qi), poem (Dickinson), play (Shakespeare), or piece of music (Notebook of Anna Magdalena Bach).

Keywords: *art, finite geometry, graph labeling, graph theory, literature, mathematical model, music, poetry*

1. Introduction

Cographs are complete graphs with colored lines. Their study can partly generalize ordinary graph theory, or may be seen as an intersection of graph theory with finite geometries. The type of cographs called intersection cographs, which promise a mathematical approach to aesthetics, are the focus of this article.

"Cograph" is an acronym for "complete graph with colored lines (edges)."[1] Figure 1 shows an example: It consists of the four points, labeled P, Q,

[1]The cographs explored in this article are quite distinct from complement-reducible graphs, which are certain combinatorial objects that some graph theorists call cographs.

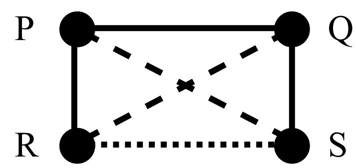


Figure 1: A cograph.

R, and S; and, joining each pair of distinct points, a line segment, colored red, blue, green, etc. (denoted in black-and-white print by patterning: solid, dashed, dotted, etc.). A cograph is a combinatorial object, in the sense that no distinctions are made based on the placement of the points on the page, or permutations of either the point names or the line colors. The specification "complete" (a *complete* graph has a line drawn between each pair of its points) is harmless here, because if a line were missing, one could simply draw it in, giving it its own unique color, without changing the information content of the picture. Big catalogue figures, like Figure 3 below, omit all such "single-copy" lines simply to reduce clutter.

Formally, we can define a **cograph** $\mathcal{C}$ **of** n **vertices** as a graph labeling of the complete graph $\mathbf{K}_n = (\mathbf{V}_n, \mathbf{E}_n)$, where the set $\mathbf{V}_n = \{v_i \,|\, i \leq n\}$ is the set of n vertices, the set $\mathbf{E}_n$ is the set $\{E_{i,j} \,|\, i, j \leq n\}$ of $\frac{n(n-1)}{2}$ edges, and there is a **color function** $F_{\mathcal{C}} : \mathbf{E}_n \to C$ mapping the elements of the edge set $\mathbf{E}_n$ to a non-empty set C of **colors**. That is, a cograph $\mathcal{C}$ is completely determined by n, C, and $F_{\mathcal{C}}$. In the following, we use this formal approach interchangeably with the informal one using points and lines in place of vertices and edges.

Cographs arise naturally in many branches of mathematics. The graph on the left of Figure 2 below shows the algebraic "sum cograph" arising from the four integers $0, 1, 2$, and 3 under the rule that the line between points P and Q in cograph $\mathcal{C}$, denoted $\mathcal{C}(P, Q)$, is their sum $P + Q$. Formally, $n = 4$, the vertex set $\mathbf{V}_4$ is identified with the set $\{0, 1, 2, 3\}$, and the function $F_{\mathcal{C}}$ maps the edge $E_{i,j}$ between any two vertices v_i and v_j to the sum $v_i + v_j$.

The graph in the middle of Figure 2 shows the one-dimensional geometric "difference cograph," where $\mathcal{C}(P, Q) = |P - Q|$. In the formal approach, n is once again 4, we still keep the vertex set the same, but this time the labeling function $F_{\mathcal{C}}$ assigns to each edge $E_{i,j}$ the difference $|v_i - v_j|$ between the labels of the two vertices connected by the edge.

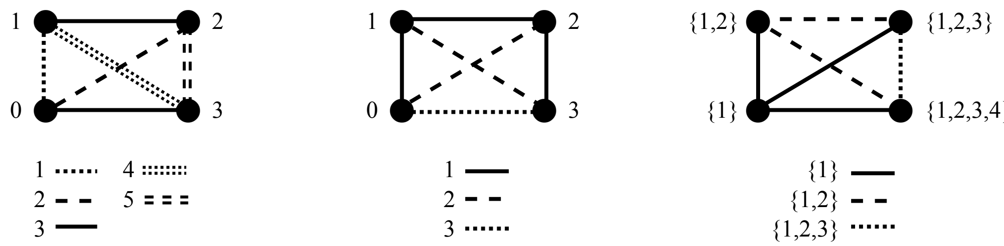


Figure 2: Sum, difference, and intersection cographs.

The graph on the right of Figure 2 shows the set-theoretic "intersection cograph" arising from the points $\{1\}, \{1,2\}, \{1,2,3\}$, and $\{1,2,3,4\}$ under the rule $\mathcal{C}(P,Q) = P \cap Q$. Formally, our n is yet again 4, but this time the vertex set $\mathbf{V}_4$ is identified with a set of sets:

$$\mathbf{V}_4 = \{\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\},$$

and the function $F_{\mathcal{C}}$ assigns to each edge $E_{i,j}$ the intersection $v_i \cap v_j$ of the sets labeling the two vertices connected by the edge.

One might equally well study the "union cograph" from the rule $\mathcal{C}(P,Q) = P \cup Q$ — except that, by taking complements and using DeMorgan's laws, one can show it is equivalent to an intersection cograph.

The first published work bearing on cographs that I am aware of is Harary and Palmer's 1973 study [8], which used an extended version of the Polya counting theorem to enumerate them. Cographs are very numerous: 1 on 2 points, 3 on 3 points, 25 on 4 points, 1299 already on 5 points [5, 6]. But the sequence 1, 3, 25, 1299... is not currently in the Sloane online encyclopedia of integer sequences [11], suggesting that cographs have not been much studied before under any name.

Why not? The answer seems an accident of mathematical fashion. Graph theory grew remarkably in the 20^{th} century under the stimulus of the four-color conjecture, which in 1976 became a computer-proved theorem; the conjecture, for well over a hundred years, had enjoyed "the distinction of being both the simplest and most fascinating unsolved problem in mathematics."[2]

[2] K. O. May, quoted in Harary [7, page 5]; see also page 126 therein.

The four-color theorem deals with map colorings, in which adjacent countries must be colored differently. In consequence, the traditionally accepted "right" definition for a graph coloring was that adjacent points (or lines) must be colored differently.

But this constraint massively limits the number of allowed colorings, converting every coloring question to an intricately restricted combinatorial problem. It shifts the problem to a restricted focus; it excludes fine earlier work like Harary and Palmer's; it adds a constraint that, one finally sees, is restrictive and unnatural. It declares the second graph in Figure 2 above, for instance, not a legitimate line coloring, since the two lines $\mathcal{C}(0,1)$ and $\mathcal{C}(2,1)$ incident at 1 are both colored 1 (and similarly for the third graph therein). But why shouldn't one encode distances by colors? And why shouldn't two equal distances then meet at a point? The present article shows the rich consequences of removing this unnatural restriction.

Section 2 below develops the elementary theory of intersection cographs. It opens with Figure 3, cataloguing the 15 possible intersection cographs on 4 points. Proposition 2.1 and its several corollaries, including Proposition 2.5, then itemize elementary constraints on their triangles and quadrilaterals, and several types of configurations that are forbidden to occur. The concluding Proposition 2.6 presents a UIE ("union of incident edges") construction that generates the point sets of a full intersection cograph from its lines alone, provided that they satisfy the triangle and quadrilateral constraints.

Section 3 applies the mathematical theory to aesthetics, by using set cographs to describe the experience of a person enjoying a picture, poem, or piece of music. It views such an experience as a complex, *active* process, in which the percipient is constantly shifting attention from one to another *aspect* of the object or performance, such as: line, shape, and color of the objects in a picture; sound, rhythm, and sense of the words in a poem; or pitch, timbre, rhythm, harmony, and melodic line of the notes in a piece of music. The viewer or auditor's perception is *multidimensional*, each dimension corresponding to one such aspect; and the aesthetic experience can be described by a cograph cataloguing the commonalities (intersections) in each dimension between each pair of its elements (points).

Section 4 suggests that the insights and techniques developed here, using cographs to analyze the complexities of aesthetic perception, might be likewise helpful in studying many other branches of art and science.

This article is one of a series on the mathematical theory of cographs. Others currently in preparation cover cographs arising from: sums; differences; linear spaces; groups; and in the abstract. My article [5] on arXiv contains preliminary writeups for all of them.

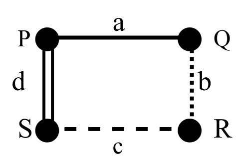
2. Intersection Cographs

In an intersection cograph, the points and edges are sets, and each edge is labeled by the intersection of the sets making up its endpoints. Equivalently, and more formally, we label each vertex of a complete graph by a set, and then the associated labeling for the edges assigns to each edge the set that is the intersection of the two sets labeling the vertices connected by the edge. Figure 3 gives the fifteen four-point intersection cographs (showing only the multiple-copy edges), together with choices of sets for their points to produce them.

That this listing is complete follows from the simple Proposition 2.1 (see the remark to its Corollary 2.4):

Proposition 2.1 (Quadrilateral rule)**.** *In an intersection cograph, any quadrilateral abcd satisfies* $a \bigcap c = b \bigcap d$.

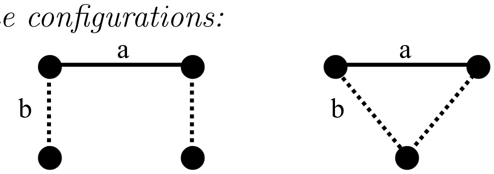
Proof. Let *abcd* be a quadrilateral, and label its points P, Q, R, and S:



Then $a \cap c = P \cap Q \cap R \cap S = b \cap d$. □

Some consequences are listed below as Corollaries 2.2-2.4.

Corollary 2.2. *The configurations:*



in an intersection cograph each imply $a \supset b$.

Proof. Letting $d = b$ in Proposition 2.1, $a \supset a \cap c = b \cap d = b \cap b = b$. □

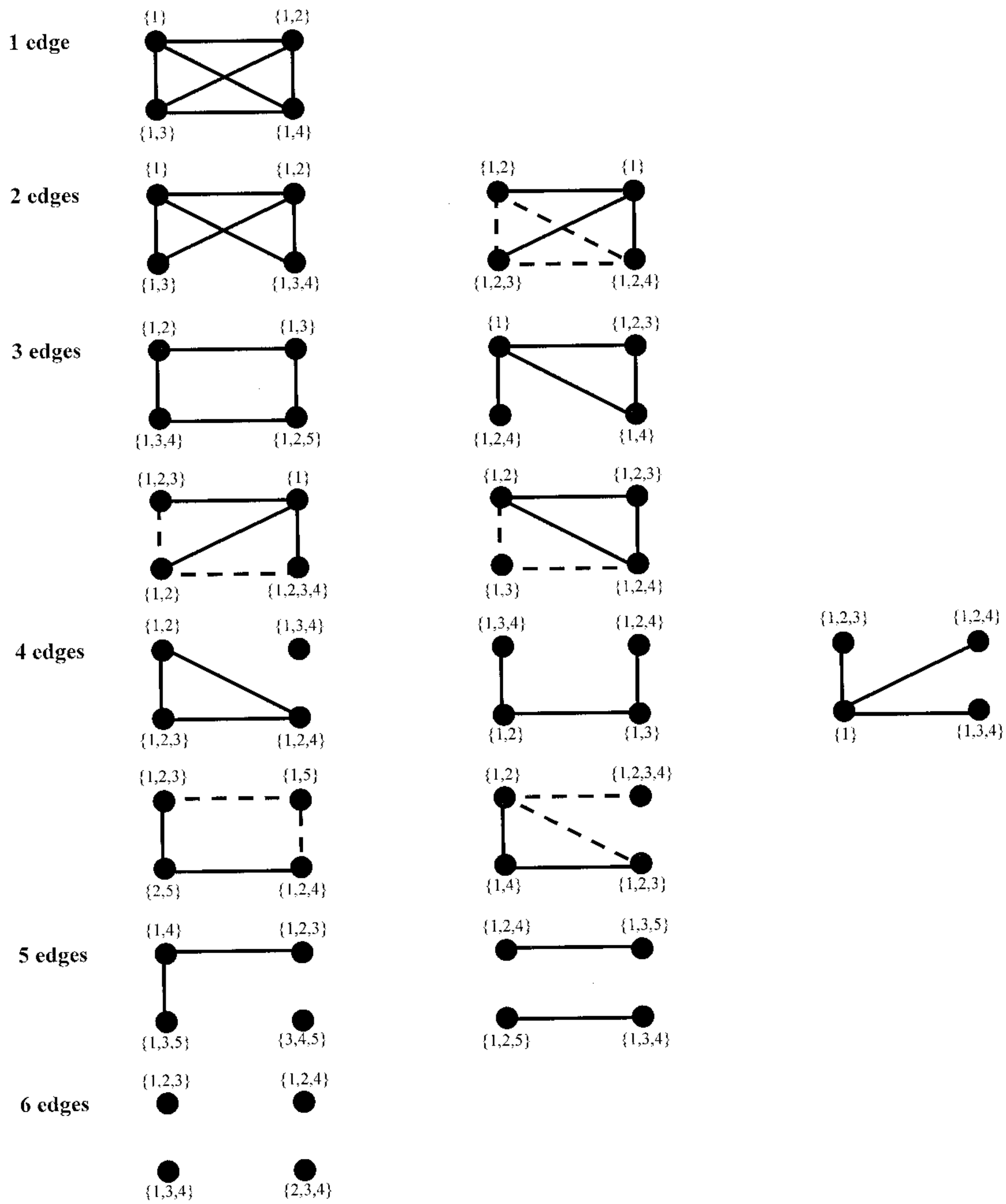
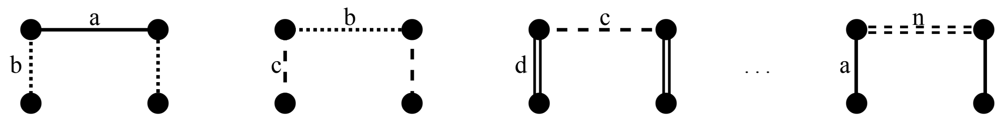


Figure 3: Catalogue of four-point intersection cographs (showing repeated edges only) with choices of sets for their points to produce them.

Corollary 2.3. *Any "inclusion cycle" of distinct edges, as in*



is forbidden in an intersection cograph.

Proof. Such an inclusion cycle would entail $a \supset b \supset c \supset d \supset ...n \supset a$, forcing a $=$ b $=$ c $=$ d $=$ ... $=$ n, which contradicts the assumption that the edges are distinct. □

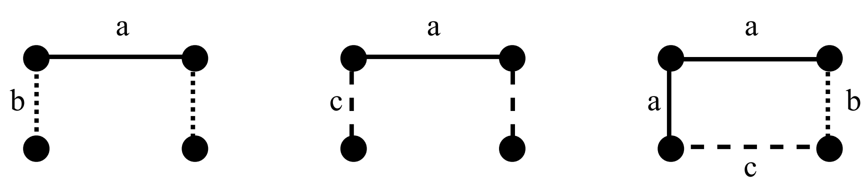
Corollary 2.3 directly implies the following:

Corollary 2.4. *"Inclusion cycles" of length two and three, as in*

are forbidden in an intersection cograph.

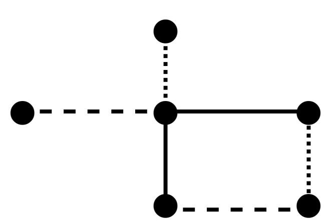
Remark. Checking the catalogue of all 25 four-point cographs [5, 6], every one omitted in Figure 3 contains one of the inclusion cycles described in the last two corollaries.

Another type of obstruction, as in



$a \supset b$ $a \supset c$ $a \cap b = a \cap c$

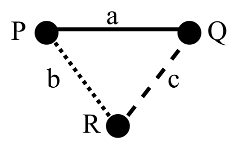
yields the contradiction $b = a \cap b = a \cap c = c$, and can actually be packed into just six points:



The following, though easy to prove, deserves its own name:

Proposition 2.5 (Triangle rule)**.** *Any triangle abc in an intersection cograph satisfies* $a \cap b = b \cap c = a \cap c$*.*

Proof. Let abc be a triangle, and label its points P, Q, and R:



Then convert edge intersections to point intersections as in the proof of Proposition 2.1. □

Remark. It is easy to verify that the triangle rule (Proposition 2.5) has two further equivalents: (1) $a \cap b = b \cap c = a \cap c = a \cap b \cap c$, and (2) $a \cap b \subset c$, $b \cap c \subset a$, and $a \cap c \subset b$.

We summarize the elementary properties of and constraints for intersection cographs we have presented so far in Figure 4 below.

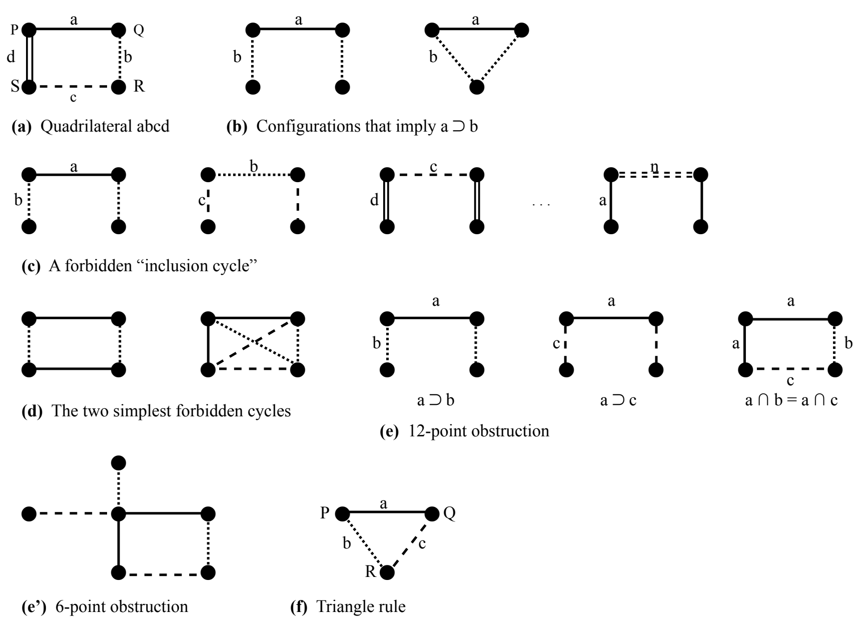


Figure 4: Elementary properties of and constraints for intersection cographs.

Intersection cographs are a special type of **set cographs**, where the color set C is a set of sets. The UIE ("union of incident edges") construction of Proposition 2.6 below shows how to generate a fully compatible intersection cograph from the edge set of any given set cograph, provided the quadrilateral and triangle rules, that is, the main conditions of Proposition 2.1 and Proposition 2.5, respectively, are both satisfied by that cograph:

Proposition 2.6 (UIE construction). *Let every quadrilateral in the edge set of set cograph $\mathcal{C}$ satisfy the quadrilateral rule, and every triangle the triangle rule. Then the set of points $\{P' : P \in \mathcal{C}\}$, where $P' = \{P_\circ\} \cup (\cup\{\mathcal{C}(P,Q), Q \in \mathcal{C}\})$, yields an intersection cograph with the edges of $\mathcal{C}$.*

Note that the singleton $P_\circ$ included in each P' is just a label, for the purpose of distinguishing points which may happen to have identical sets of incident edges.

Before we move on to prove this proposition let us state it more formally: Let $\mathcal{C} = (\mathbf{V}_n, \mathbf{E}_n, F_\mathcal{C})$ be a set cograph, and suppose every quadrilateral in the edge set $\mathbf{E}_n$ satisfies the quadrilateral rule. That is, for any quadrilateral $E_{i_1,i_2}E_{i_2,i_3}E_{i_3,i_4}E_{i_4,i_1}$ in the edge set $\mathbf{E}_n$ of $\mathcal{C}$, we assume:

$$F_\mathcal{C}(E_{i_1,i_2}) \bigcap F_\mathcal{C}(E_{i_3,i_4}) = F_\mathcal{C}(E_{i_2,i_3}) \bigcap F_\mathcal{C}(E_{i_4,i_1}).$$

Similarly assume that every triangle in the edge set $\mathbf{E}_n$ satisfies the triangle rule. That is, for any triangle $E_{i_1,i_2}E_{i_2,i_3}E_{i_3,i_1}$ in the edge set $\mathbf{E}_n$, we assume:

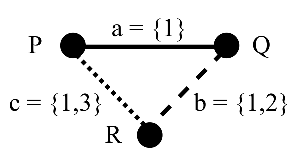
$$F_\mathcal{C}(E_{i_1,i_2}) \bigcap F_\mathcal{C}(E_{i_2,i_3}) = F_\mathcal{C}(E_{i_2,i_3}) \bigcap F_\mathcal{C}(E_{i_3,i_1}) = F_\mathcal{C}(E_{i_1,i_2}) \bigcap F_\mathcal{C}(E_{i_3,i_1}).$$

Then Proposition 2.6 asserts that set of points $\{v_i' : v_i \in \mathbf{V}_n\}$, where

$$v_i' = \{i_\circ\} \cup \left(\bigcup_{v_j \in \mathbf{V}_n} F_\mathcal{C}(E_{i,j}) \right),$$

yields an intersection cograph on $\mathbf{K}_n$ with the color function $F_\mathcal{C}$.

We illustrate the above with an example. Let us start with the set cograph given below:



Define the new labels for the three vertices as follows:

$$P' = \{P_\circ\} \cup \{1,3\}, \qquad Q' = \{Q_\circ\} \cup \{1,2\}, \qquad R' = \{R_\circ\} \cup \{1,2,3\}.$$

This gives us a intersection cograph on $\mathbf{K}_3$ where the labels (or equivalently, the colors) of the edges coincide with the original labels.

We are now ready for the proof, which we present in the informal approach; readers are encouraged to fill in the details to get to a formal version.

Proof. Start with any two points $P, S \in \mathbf{V}_n$ in the original set cograph. Then:

$$P' \cap S' = \left(\bigcup_{Q \in \mathbf{V}_n} \mathcal{C}(P,Q) \right) \cap \left(\bigcup_{R \in \mathbf{V}_n} \mathcal{C}(S,R) \right) = \bigcup_{Q,R \in \mathbf{V}_n} \left(\mathcal{C}(P,Q) \cap \mathcal{C}(S,R) \right).$$

The terms in this union fall into five cases:

Case 1. $Q = S, R = P$, in which case we have:

$$\mathcal{C}(P,Q) \cap \mathcal{C}(S,R) = \mathcal{C}(P,S) \cap \mathcal{C}(S,P) = \mathcal{C}(P,S);$$

Case 2. $Q = S$, in which case we have:

$$\mathcal{C}(P,Q) \cap \mathcal{C}(S,R) = \mathcal{C}(P,S) \cap \mathcal{C}(S,R) \subset \mathcal{C}(P,S);$$

Case 3. $R = P$, in which case we have:

$$\mathcal{C}(P,Q) \cap \mathcal{C}(S,R) = \mathcal{C}(P,Q) \cap \mathcal{C}(S,P) \subset \mathcal{C}(P,S);$$

Case 4. $Q \neq S, R \neq P, Q = R$, in which case we have:

$$\mathcal{C}(P,Q) \cap \mathcal{C}(S,R) = \mathcal{C}(P,R) \cap \mathcal{C}(S,R) = \mathcal{C}(P,S) \cap \mathcal{C}(S,R) \subset \mathcal{C}(P,S);$$

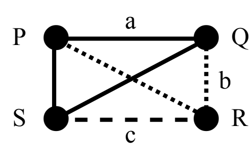
Case 5. $Q \neq S, R \neq P, Q \neq R$, in which case we have:

$$\mathcal{C}(P,Q) \cap \mathcal{C}(S,R) = \mathcal{C}(P,S) \cap \mathcal{C}(Q,R) \subset \mathcal{C}(P,S);$$

utilizing in case 4 that the edges of PRS satisfy the triangle rule, and in case 5 that the edges of PQRS satisfy the quadrilateral rule. The entire union is therefore $P' \cap S' = \mathcal{C}(P,S)$, as was to be proved. $\square$

Remark. The UIE-constructed cograph P' is not the only intersection cograph sharing the same edge labels as the original set cograph we started out with, but it is minimal, in the following sense: If P^* also has intersection cograph $\mathcal{C}$, then for each P, necessarily $P^* \supset P' - \{P_\circ\}$, and the elements of $P^* - P'$ are not contained in any other $Q^* - Q'$, else they would appear in the intersection $P^* \cap Q^*$. That is, if a collection of P^* make up a set of labels for $\mathbf{V}_n$ such that the labels the color function $F_{\mathcal{C}}$ assigns to each edge of $\mathbf{K}_n$ coincide with those in the intersection cograph defined by the P^*, then for each $P \in \mathbf{V}_n$, necessarily $P^* \supset P' - \{P_\circ\}$.

To see how we use Proposition 2.6, let us now represent the abstract cograph PQRS given as:



as an intersection cograph.

PRQ (edges bab, Corollary 2.2) forces $a \supset b$, and PQRS (Proposition 2.1 Quadrilateral rule) then forces $a \cap c = a \cap b = b$. Letting $a = \{1, 2\}$, $b = \{1\}$, and $c = \{1, 3\}$ yields UIE-constructed points $P = \{1, 2\}$, $Q = \{1, 2, 4\}$ (the 4 added to distinguish it from P), $R = \{1, 3\}$, and $S = \{1, 2, 3\}$.

3. Intersection Cographs and Aesthetics

Mathematics is such a beautiful subject it is not surprising that many mathematicians have been strongly moved by beauty, and a number of them have devised mathematical formulations of aesthetics. For instance, H. Weyl in his book *Symmetry* [13] traces the role of group theoretical symmetry in the visual arts, while Birkhoff in [2] offers a definition of beauty through a concept of “aesthetic measure”. To a degree, of course, “beauty is in the eye of the beholder”; that is, the beauty must arise not solely from the beautiful object itself, but rather in the interaction of that object with the percipient, in the act of perception. This section will suggest how intersection cographs might offer a mathematical model for that interaction.

We begin by reformulating intersection cographs into a product binary form, less compact but more transparent for generalization, as illustrated by an example; see Figure 5 below.

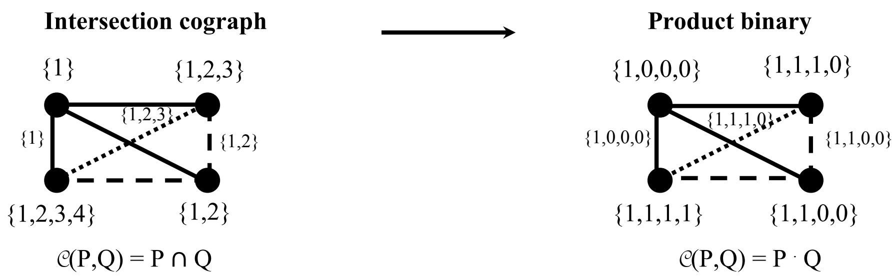


Figure 5: Intersection cograph reformulated into product binary form.

Here the cograph on the left is the ordinary intersection cograph on the four points (sets) {1}, {1,2}, {1,2,3}, and {1,2,3,4}, with its three edges {1}, {1,2}, and {1,2,3}, represented respectively by solid, dashed, and dotted lines, given by the rule $\mathcal{C}(P, Q) = P \cap Q$. On the right the same abstract cograph is represented by four elements in the algebraic product $Z_2 \times Z_2 \times Z_2 \times Z_2$, where Z_2 is the "binary" two-element ring of integers mod two. Each set of the intersection cograph (note that both vertices and edges are labeled by sets) is represented by its "characteristic function," and the cograph rule is $\mathcal{C}(P, Q) = P \cdot Q$. By this formulation it is evident how the notion might be generalized to products having many more, or even an infinite number, of "dimensions."

The approach here now is to view the perceived world abstractly as such an intersection cograph. Perceived "objects" are sets — the sets of perceptions (or "properties," or, in philosophy, "accidents") that an observer can ascertain from each of them. The focus of interest, for instance, in making the judgment "beauty," is to compare these sets among themselves, that is, to contemplate their intersections. Here, first, are three or four specific examples from the arts of painting, poetry, and music. They have been chosen for their extreme simplicity (and therefore rather unadorned, abstract character), to highlight the remarkable richness and complexity inherent in the judgment "beautiful."

The first example is the famous painting "Six Persimmons" by the thirteenth century Chinese painter Mu Qi displayed in Figure 6 (see [10] for more on this painting). This picture is art of the utmost simplicity: six stylized pieces of fruit painted in black ink, without color, background, shadows, pictorial details, or dramatic perspective. It can, nevertheless, arouse a powerful emotional response in a sensitive viewer: "passion ... congealed into a stupendous calm," is the reaction of one critic [12].

Figure 6: Mu Qi: "Six Persimmons," 13^{th} century, Southern Song (Chinese), Collected in Daitokuji, Kyoto, Japan. Public domain image obtained from https://en.wikipedia.org/wiki/Muqi, last accessed on January 8, 2022.

Intersection cographs can describe the process of perception as the appreciating eye plays over this picture. We go through this analysis step by step, and record our reactions in Figure 7.

Aspect \ Persimmon		1	2	3	4	5	6
a	Frontality	0	0	1	0	0	0
b	Frontality’	0	1	1	0	0	0
c	Frontality”	0	1	2	1	1	0
d	Color	0	1	1	3	2	0
e	Size	1	1	0	2	1	1
f	Shape	0	1	1	2	2	1
g	Stem	2	2	1	3	1	2

Figure 7: Aspects of Mu Qi’s “Six Persimmons”.

We begin with the six persimmons. The first line of Figure 7 shows the result of the first glance (Aspect a): Of the six persimmons in the picture, numbers 1, 2, and 4, 5, 6 share the characteristic of being in one row in back, while number 3 is further to the front. Aspect b gives the second closer glance: fruit 2 is slightly ahead of the rest of the back row, and thus is united in similarity with number 3. Aspect c gives the most detailed look: the persimmons at both ends are subtly overlapped, hence behind the adjacent fruits.

As one studies the painting, first one fruit and then another catches the eye, gaining prominence not only from position, but also by size, shape, or shading. Aspect d shows the cograph for color: the two fruits at the ends share the palest color, the fourth is darkest, the others are intermediate. Aspect e shows size; aspect f shape: oval, round, or squarish. Aspect g compares the lengths of the stems of the fruits.

All of us become amateur artists when we make photos, and triumph when we center our friends in a snapshot. In his picture Mu Qi has “balanced” all seven different aspects of position, shape, color, ... (and there are more) described in the cographs. The composition would be destroyed by omitting any one of the six fruits, and ruined by so much as shifting any position, size, shape, or color. It is this perfect equipoise of strong figural forces that produces the feeling of calm and passion noted by the critic; it is this exquisite balance that makes the painting a great work of art.

A similar balance characterizes the beauty of a poem. The example here is a verse from a lyric by Emily Dickinson (the first poem she valued highly enough to send to a critic [3, 9]), describing the noble repose of the redeemed dead awaiting their resurrection on Judgment Day:

> Safe in their Alabaster Chambers—
>
> Untouched by Morning—
>
> And untouched by Noon—
>
> Sleep the meek members of the Resurrection,
>
> Rafter of Satin—and Roof of Stone—

Some reactions to this verse are captured in Figure 8; here is a step by step analysis: The first impression one receives in reading this poem is perhaps the rhythm (Figure 8, Aspect a). The idiomatically mixed dactylic (–◡◡) and trochaic (–◡) pulse carries the words along to make them "verse" rather "prose," while placing special emphasis on emotionally important words like "safe," "untouched," and "sleep." The rhythm alone creates the powerful effect in the last line, where the drumbeat of the dactylic "Rafter of Satin" (–◡◡ – ◡) slows to the iambics "and Roof of Stone" (◡ – ◡–) to produce a feeling of unshakable solidity and firmness that will outlast the eons.

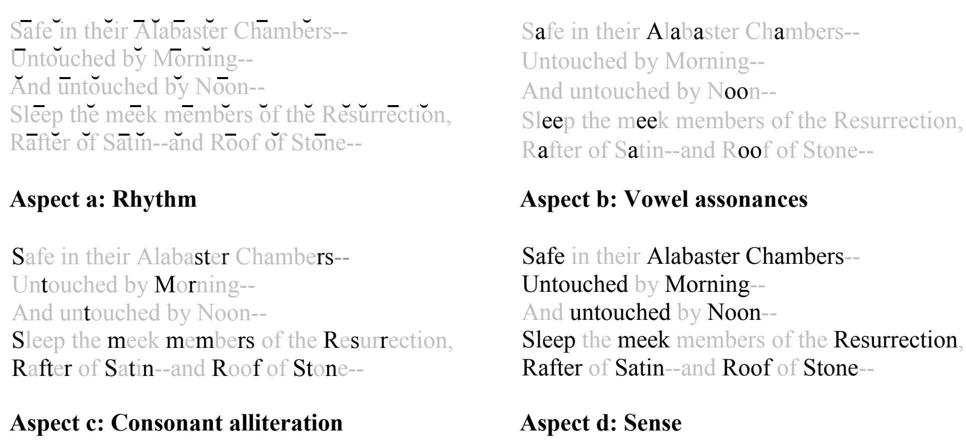


Figure 8: Aspects of the Dickinson poem "Safe in their Alabaster Chambers."

Reinforcing the poem's initial rhythmic pattern then is the music of the language itself. Aspect b, as it is labeled in Figure 8, summarizes the vowel rhymes and assonances; these, for example, link the sound of "Safe" to that of "Chambers" in the first line, "Sleep" to "meek" as an internal rhyme in the fourth, and "Alabaster" in the first to "Rafter" and "Satin" in the last.

The consonantal alliterations (Aspect c, as it is labeled in Figure 8) provide even more numerous linkages. For example, the "s" sound in the first word "Safe" is echoed in "Alabaster," "Chambers," "Sleep," "members," "Resurrection," "Satin," and the last word "Stone." The "m," "r," and "t" sounds recur similarly. The alliterations also contribute notably to the effect of the last line, where the "r," "f," "s," "t," and "n" of "Rafter of Satin" are echoed exactly by those in "Roof of Stone."

Poetry, finally, requires a harmony of sense mutually reinforcing that of sound. Aspect d, as it is labeled in Figure 8, indicates some of the sense patterns in this lyric: The central thought is how the physical environment ("Chambers," "Rafter," and "Roof"), charged with emotional connotations of protection and permanence ("Alabaster," "Satin," "Stone"), shields its inhabitants from time ("Morning," "Noon," and "Resurrection"). The great majority of words in the lyric express this protection: "Safe," "Alabaster," "Chambers," "Untouched," "Untouched," "Sleep," "Meek," "Rafter," and "Roof." As with the "Six Persimmons" painting, this lyric is created from only a few ingredients. The exquisite rightness and economy of its crafting, each word linked to the others in a balance of rhythm, sound, and sense, make it, too, a great work of art.

Though this example deals with the minutest elements of sound and meaning, intersection cographs also easily represent much larger literary structures. For example, the cograph of Figure 10, with solid, dashed, or invisible white line segments, shows the main characters and relationships in Shakespeare's play *King Lear*:

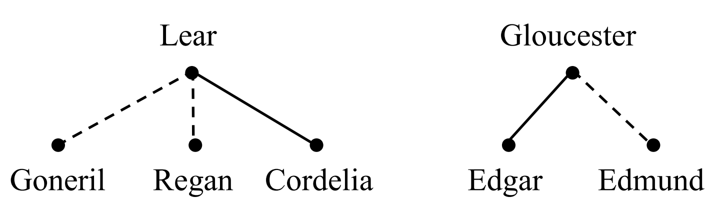


Figure 9: Cograph of characters in Shakespeare's *King Lear*.

The central issue in this play, announced already in its third line "Is not this your son, my lord?", is the nature of the relationship between parent and child. The cograph schematizes, graphically and instantaneously, the two forms occurring here: the false one, between Lear and Goneril, Lear and Regan, and Gloucester and Edmund, and the true one, between Lear and Cordelia, and Gloucester and Edgar.

The final example is a musical one. It is difficult to find a profound piece of music on as miniature a scale as the Mu Qi painting or the Dickinson lyric, and we will content ourselves with a fragment, presented in Figure 10, the first section of the familiar beginner's minuet in G from the Notebook of Anna Magdalena Bach [1].

Figure 10: Beginning of the Minuet in G, from the Notebook of Anna Magdalena Bach. The image obtained from a screenshot of the video recording at https://www.youtube.com/watch?v=p1gGxpitL08, last accessed on January 13, 2022.

Music, like poetry but unlike painting, is organized along a strictly linear pattern extended in time. Its first impression is therefore also the underlying rhythmical pattern. The rhythm is stricter for music than poetry, and the first rhythm cograph (not shown) simply records its steady 1-2-3 pattern of beats. This strict foundation, however, then permits the elaboration of more complex hierarchical structures: Figure 11a highlights the repeated figure of four eighth notes leading up to a quarter note.

Coincident with the rhythmic patterns are melodic and harmonic ones. Musical analysis (pioneered most formally by Schenker [4]) reveals these latter patterns most clearly by "rhythmic reduction" which omits ornamental filigree notes. The underlying pattern then stands out clearly: here, two simple scale passages, ascending, then descending to the tonic note G (Figure 11b - circled notes).

Figure 11: Minuet in G: a) a repeated figure; b) the underlying scale pattern. Hand-drawn on the image from Figure 10.

Other dimensions of musical expression include the shading of dynamics, ranging from soft to loud; progression of the underlying harmonies; small but important adjustments in tempo, such as ritards or accelerandos near musical climaxes; and, in ensemble music, use of the palette of colors of the different instruments. As with the other arts, an aesthetically satisfying musical composition or performance will be one in which the multiple dimensions of structure summarized schematically by the cographs are integrated into a convincing whole.

4. Other Possible Avenues of Exploration

This article has shown how the new mathematical concept of intersection cographs (§§1-2) provides a unifying theoretical framework to help describe and understand how people appreciate such art forms as painting, poetry, and music (§3). The UIE construction (Proposition 2.6) describes conditions under which the edges of a set cograph determine its points; but conditions guaranteeing that an abstract cograph can be realized as an intersection cograph (like those in the catalogue Figure 3) are not yet known. Other types of cographs, arising naturally in algebra and geometry (§1, [5], [6]), undoubtedly raise interesting new mathematical questions for investigation too. Intersection cographs themselves might then also prove useful for modeling the balancing of forces occurring in a variety of real-world contexts outside of art: for instance, the summation of attractive and repulsive electrostatic forces (the "Madelung constant") in an ionic crystal; the psychological forces of personality, family background, and "chemistry" sustaining a compatible couple in their marriage; or sociological forces like nationality, race, gender, and class needed for a stable society.

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