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Plane Figurate Number Proofs Without Words Explained With Pattern Blocks

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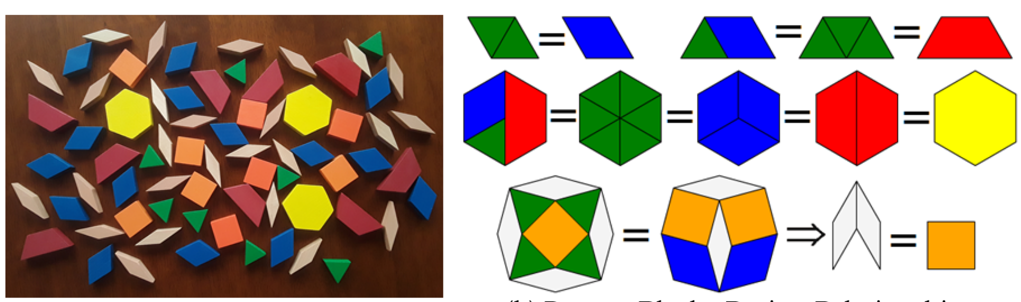
Abstract

This article focuses on an artistic interpretation of pattern block designs with primary focus on the connection between pattern blocks and plane figurate numbers. Through this interpretation, it tells the story behind a handful of proofs without words (PWWs) that are inspired by such pattern block designs.

Keywords: *figurate number, polygonal number, pattern blocks, proof without words, visualization.*

Pattern blocks consist of a green equilateral triangle, a blue rhombus, a red isosceles trapezoid, a yellow regular hexagon, an orange square, and a thin rhombus (Figure 1). All sides of all equilateral pattern blocks are congruent, which are considered to be of 1 unit length in this note. The short side of the red trapezoid is taken to be the same length.



(a) Pattern Blocks Math Manipulatives (b) Pattern Blocks Region Relationships

Figure 1: Pattern blocks math manipulatives and region relationships.

Region relationships among pattern blocks hold in the following manner (Figure 1b):

(i) The yellow hexagon is equal in area to two red trapezoids (or equivalently, three blue rhombuses; or six green triangles; or one trapezoid–rhombus–triangle combination);

(ii) One red trapezoid is equal in area to three green triangles (or equivalently, one rhombus–triangle combination);

(iii) One blue rhombus is equal in area to two green triangles.

(iv) One orange square is equal in area to two thin rhombuses.[1]

For an artistic interpretation of pattern blocks and some mathematically rich activities using them, see [1].

K-gonal numbers (also known as polygonal numbers) are determined by the number of dots placed on regular polygons following certain pattern arrangements (Figure 2). As such, they can be thought of as the intersection of geometry and number sense branches of mathematics In this note, we focus on an artistic interpretation of pattern block designs by revealing the connection between pattern blocks and plane figurate numbers. More specifically, with the help of a variety of examples, we explain the process of creating images that model figurate number relationships in accordance with pattern block designs. In that sense, this note introduces and tells the story behind a handful of proofs without words (PWWs) that are inspired from such pattern block designs.[2]

Section 1 focuses on polygonal numbers and their notations along with the connection between the polygonal number and the suggested pattern block design representation. Elementary proofs without words (PWWs) are considered in Section 2. Section 3 takes into account designs that model identities that include a linear function of n. Section 4 of the note focuses on the generation and establishment of more complicated figurate number relationships facilitated by further pattern block designs. Finally, Section 5 offers the reader a list of figurate number identities that do not yet have known PWWs using pattern blocks; some possible promising paths are also provided.

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[1] The orange square and the thin rhombus are not expressible as a whole number multiple (in area) of a green triangle. The present note focuses on the first four types of blocks.

[2] A proof without words (PWW) is a proof of a mathematical identity that only uses a visual representation to demonstrate the truth of the identity. Several examples can be found in [4, 5]. See [2] for more on PWWs.

1. Polygonal Numbers and Correspondence with Pattern Block Designs

The nth k-gonal number p_n^k can be defined in the most explicit form via

$$p_n^k = ((k-2)n^2 - (k-4)n)/2 \qquad \text{for } k \geq 3.$$

Equivalently, we can use triangular numbers

$$T_n = n(n+1)/2$$

and represent polygonal numbers via the equation

$$p_n^k = n + (k-2)T_{n-1},$$

see [3, Part III] and [6, Chapter 1]. Note that $p_n^3 = T_n$. Table 1 highlights the simplified notation that we adopt for the polygonal numbers p_n^k for $3 \leq k \leq 10$ that are the focus of this article.

Table 1: Notation Used for Polygonal Numbers

k	$p_n^k = ((k-2)n^2 - (k-4)n)/2$	Simplified Notation
3	$p_n^3 = (1n^2 + 1n)/2$	$T_n = n(n+1)/2$
4	$p_n^4 = (2n^2 - 0n)/2$	$S_n = n(2n)/2 = n^2$
5	$p_n^5 = (3n^2 - 1n)/2$	$P_n = n(3n-1)/2$
6	$p_n^6 = (4n^2 - 2n)/2$	$H_n = n(4n-2)/2 = n(2n-1)$
7	$p_n^7 = (5n^2 - 3n)/2$	$X_n = n(5n-3)/2$
8	$p_n^8 = (6n^2 - 4n)/2$	$O_n = n(6n-4)/2 = n(3n-2)$
9	$p_n^9 = (7n^2 - 5n)/2$	$N_n = n(7n-5)/2$
10	$p_n^{10} = (8n^2 - 6n)/2$	$D_n = n(8n-6)/2 = n(4n-3)$

Figure 2a introduces the first terms of the triangular, square, pentagonal, and hexagonal number sequences. The other figurate numbers can be obtained in a similar manner.

In what follows, we use the green triangle to model triangular number T_n or T_{n-1} or T_{n-2} or T_{n+1} depending on the context.

Perhaps surprisingly, we do not use the orange pattern block square to model the square number; instead, we choose to use the blue rhombus to represent the square number S_n or S_{n-1} depending on the context; see Figure 3.

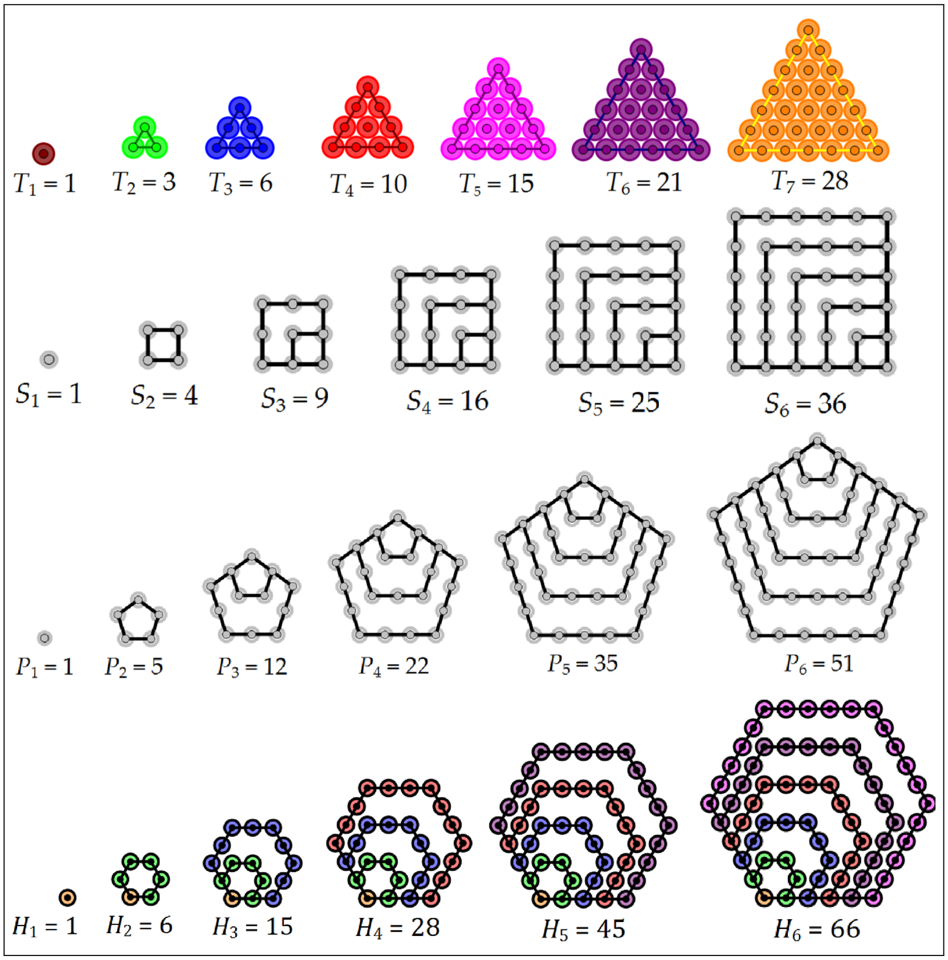


Figure 2: Polygonal numbers: triangular, square, pentagonal, hexagonal numbers.

The shear does not affect the value of S_n; the blue rhombus still has four sides, just like the square. Moreover, the blue rhombus is interesting in that, depending on the context, it can also serve to model a pronic (heteromecic) number. (A pronic number is one that is the product of two consecutive integers. The nth pronic number is denoted via E_n in this note.)

Once again surprisingly, as shown in the second and third rows of Figure 3, the red trapezoid will be taken into account for the modeling of the pentagonal number P_n.

Finally, as shown in the last two rows of Figure 3, we model the hexagonal number H_n in two possible ways: as one concave hexagon or as one large triangle. These can be made in three different ways: using four green triangles, or two blue rhombi, or one red trapezoid together with one green triangle.

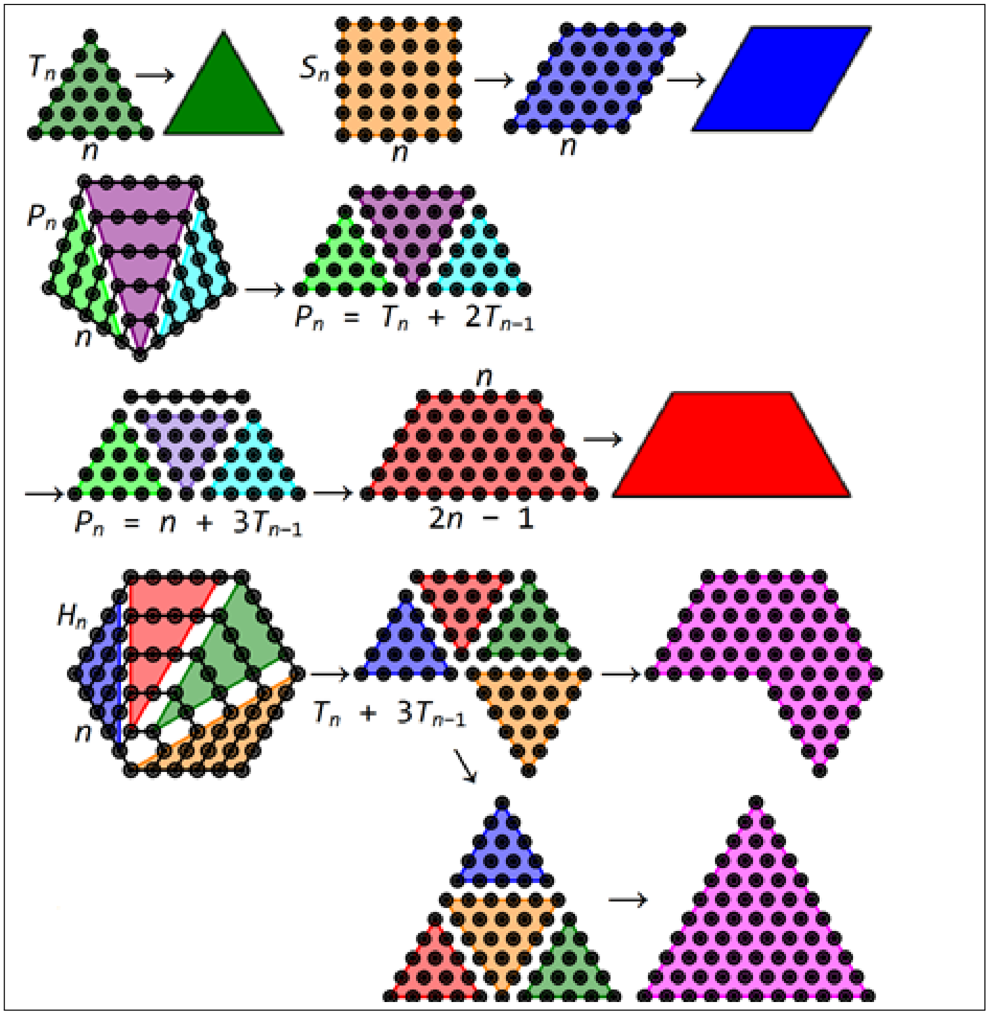


Figure 3: Polygonal numbers represented by pattern blocks.

To sum up, Figure 3 suggests a correspondence between polygonal numbers and their pattern block design counterparts for triangular, square, pentagonal, and hexagonal number sequences (for a specific value $n = 6$ without loss of generality).

The pattern block representations for $k = 7, 8, 9, 10$ (i.e., for heptagonal, octagonal, nonagonal, and decagonal number sequences) can be generated in a similar manner; see Figure 4 for figurate number representations of heptagonal, octagonal, nonagonal, and decagonal numbers for $n = 6$, in addition to the suggested pattern block representations. The main idea lies in the decomposition of the k-gonal number into triangular numbers guided by the expression $T_n + (k-3)T_{n-1}$. Note that we colored the two final representations of the hexagonal number from Figure 3 light pink.

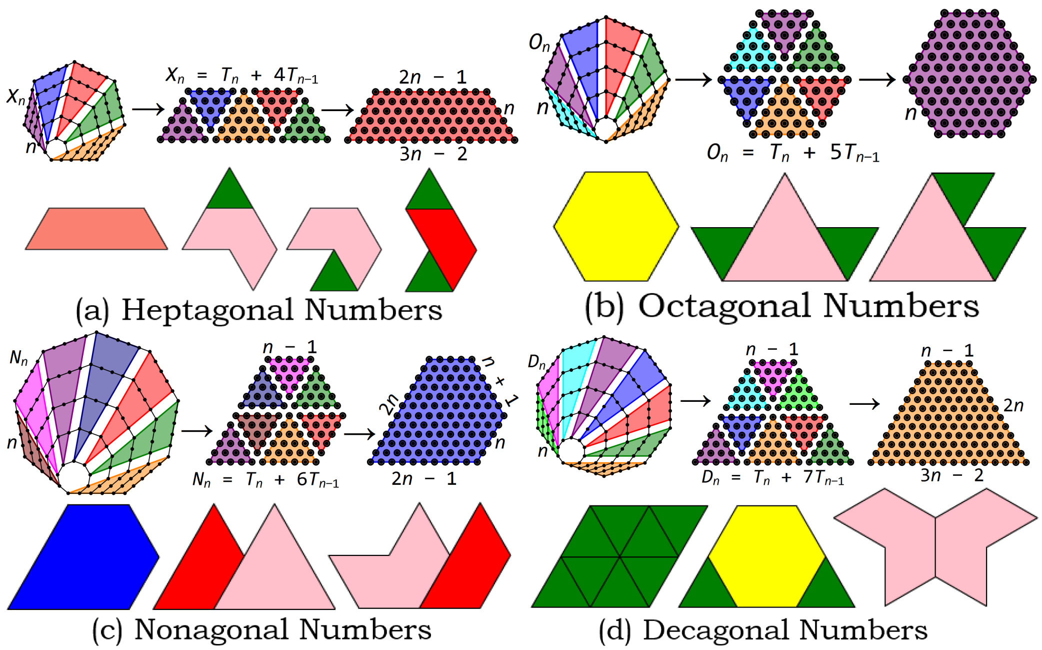


Figure 4: Pattern block representations for heptagonal, octagonal, nonagonal, decagonal numbers.

2. Elementary Proofs Without Words Involving Triangular, Square, Pentagonal, and Hexagonal Numbers

In this section I present a handful of elementary examples of PWWs that I have generated using pattern blocks. Each example that follows comprises three things:

(i) The polygonal number design (i.e., the proof without words).
(ii) The relevant identity of the form LHS = RHS.
(iii) The corresponding (mere geometric) pattern blocks design.

Example 1: Regular Triangle Pattern Block Design Made of 4 Green Triangles (or its Equivalent) Modeling a Triangular Number Identity Involving T_{2n-1}

Among many such possibilities, Figure 5 depicts four proofs without words (PWWs), each accompanied by the corresponding four pattern block designs. (The figure displays the case for a specific value, $n = 5$, without loss of generality).

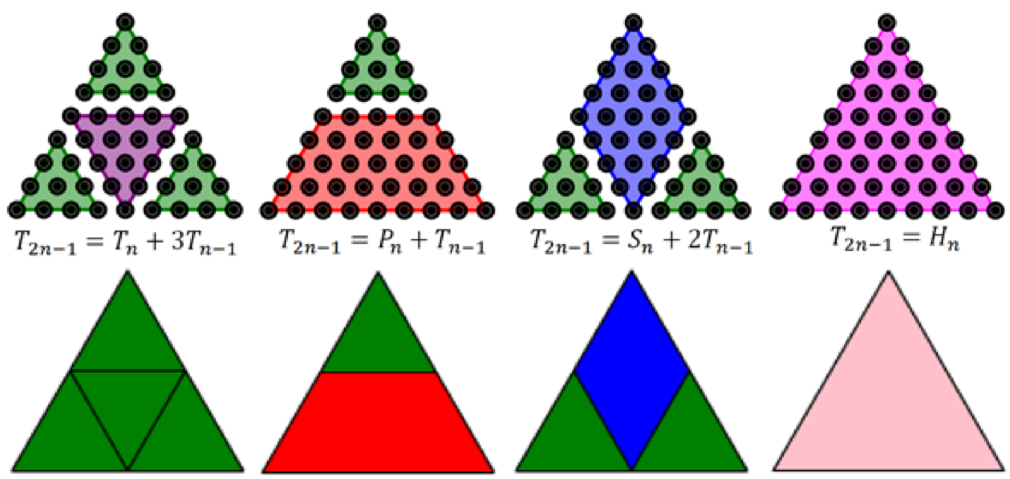


Figure 5: Modeling a triangular number identity involving T_{2n-1}.

Example 2: Regular Triangle Pattern Block Design Made of 9 Green Triangles (or its Equivalent) Modeling a Triangular Number Identity Involving T_{3n-1}

Among many such possibilities, Figure 6 depicts five proofs without words (PWWs), each accompanied by the corresponding five pattern block designs. (The figure displays the case for a specific value, $n = 5$, without loss of generality).

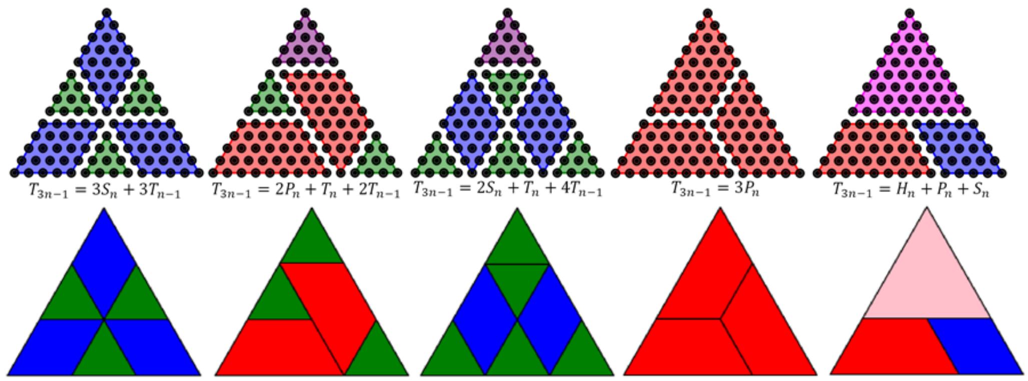


Figure 6: Modeling a triangular number identity involving T_{3n-1}.

3. Pattern Block Designs Modeling Identities Involving a Linear Function of n

I was not always very lucky when I was generating proofs without words (PWWs) by relying on pattern block designs and elementary algebra. At times, the pattern block design did not quite "match" the figurate number identity I proposed. This happened in a variety of ways:

(i) The indices (subscripts) of the general term of the sequence(s) needed to be adjusted;

(ii) The identity had to be revised by adding/subtracting a linear term;

(iii) The identity had to be revised by adding/subtracting a constant term.

In such instances, I made use of Pólya's "Guess and Check" strategy [7], in addition to drawings and algebraic approaches. The examples that follow present figurate number identities involving linear and/or constant terms that were obtained this way.

Example 3: An Identity Involving Hexagonal and Heptagonal Numbers

The guess and check strategy is illustrated as follows. Guided by Pólya's four-step approach to problem solving, suppose we conjecture that $X_n + H_n = T_{3n}$. Then we see that the LHS, $X_n + H_n$, equals

$$\frac{5n^2 - 3n}{2} + \frac{4n^2 - 2n}{2} = \frac{9n^2 - 5n}{2},$$

which is very close in amount to the familiar triangular number form

$$\frac{9n^2 - 3n}{2} = \frac{(3n)(3n-1)}{2} = T_{3n-1}.$$

Then, T_{3n-1} exceeds $X_n + H_n$ by the amount of a linear term, n, so we modify the initial conjecture as

$$X_n + H_n + n = T_{3n-1},$$

which is the correct figurate number identity (Figure 7b). Slightly modifying Figure 7b, we obtain $X_n = T_{3n-2} - T_{2n-2}$ as illustrated in Figure 7c.

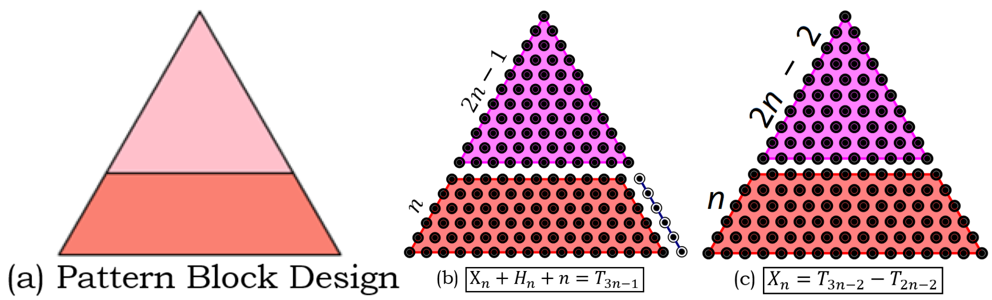


Figure 7: Designs modeling identities prone to excess / lack of linear term / constant term.

Example 4: An Identity Involving Triangular and Hexagonal Numbers

In a similar manner as before, we may claim that $4H_n = T_{4n}$ as in Figure 8a. The LHS is shown to be equal to $4H_n = 4n(2n-1)$, which is reminiscent of the triangular number form

$$\frac{(4n)(4n-1)}{2} = T_{4n-1}.$$

Comparing $4H_n = 8n^2 - 4n$ and $T_{4n-1} = 8n^2 - 2n$, we deduce that T_{4n-1} exceeds $4H_n$ by $2n$, which leads us to the correct identity

$$4H_n + 2n = T_{4n-1}$$

as in Figure 8b. Slightly modifying Figures 8a-b, we obtain the companion identity $3H_n + 2n = P_{2n}$ as illustrated in Figures 8c-d.

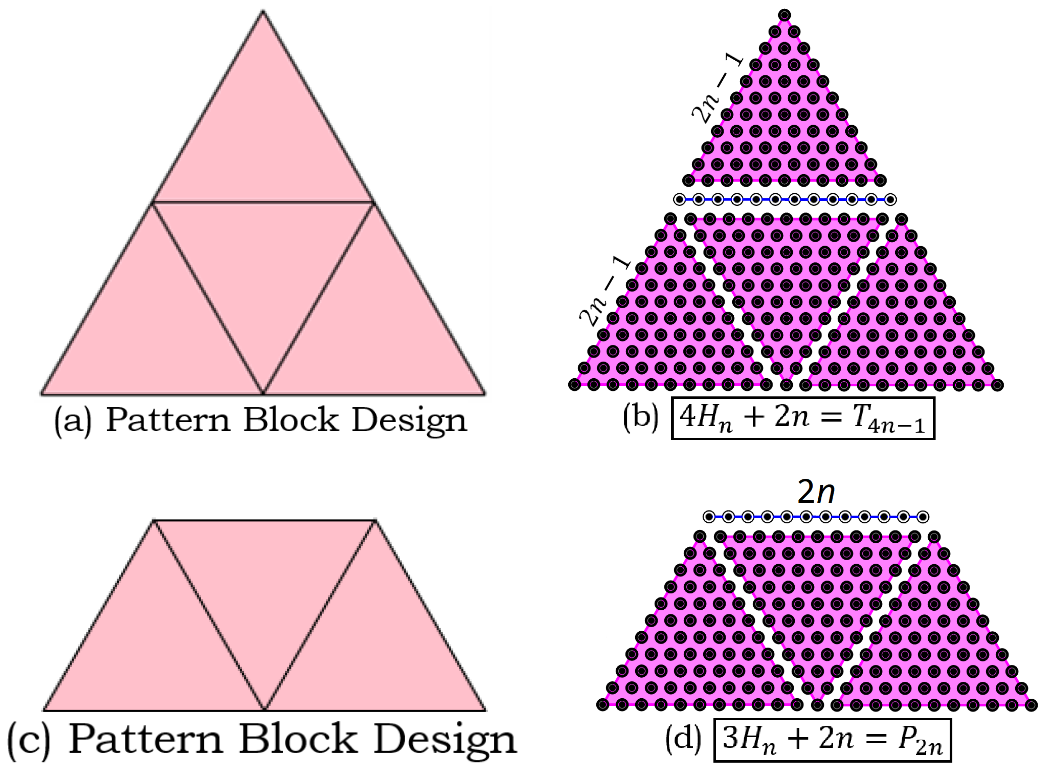


Figure 8: More designs modeling identities prone to excess / lack of linear term / constant term.

Example 5: An Identity Involving Pentagonal and Hexagonal Numbers

We begin with the conjecture:

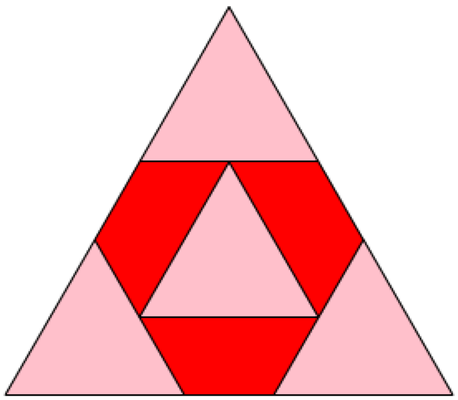
$$4H_n + 3P_n = T_{5n}$$

as in Figure 9a. The LHS, $4H_n + 3P_n$, can be calculated as

$$4n(2n-1) + \frac{3n(3n-1)}{2} = \left(\frac{25}{2}\right)n^2 - \left(\frac{11}{2}\right)n,$$

which reminds us of the triangular number form

$$\frac{(5n)(5n-1)}{2} = T_{5n-1}.$$



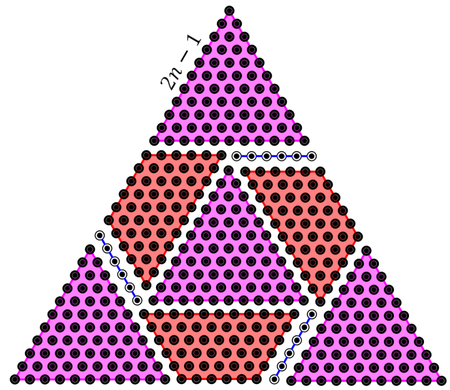


Figure 9: Even more designs modeling identities prone to excess / lack of linear term / constant term.

Expanding T_{5n-1} we obtain

$$\left(\frac{25}{2}\right)n^2 - \left(\frac{5}{2}\right)n,$$

which seems too far from what we want, so we try T_{5n-2} instead. Comparing $4H_n+3P_n$ and T_{5n-2}, we deduce that the former exceeds the latter by $(2n-1)$, which leads us to the identity

$$4H_n + 3P_n = T_{5n-2} + (2n-1).$$

This identity is algebraically correct. However, when I drew the proof without words (PWW) version (Figure 9b), it was easier to draw $4H_n + 3P_n + 3n = T_{5n-1}$ than this one.

Example 6: Octagonal Number in Terms of Hex Number

As I was working on expressing an octagonal number as a centered hexagonal number (also known as a hex number),[3] I was able to discover the following relationship between polygonal and centered polygonal numbers:

[3] Centered polygonal numbers and their connections to pattern blocks are explored in another article.

Let $P(n, k)$ and $C(n, k)$ denote the k-gonal number of n sides and centered k-gonal number of n sides, respectively. Then, we have:

$$P(n, k) = C(n, k) + (n - 1).$$

In other words, the k-gonal number of n sides exceeds the centered k-gonal number of n sides by $(n - 1)$. See Figure 10.

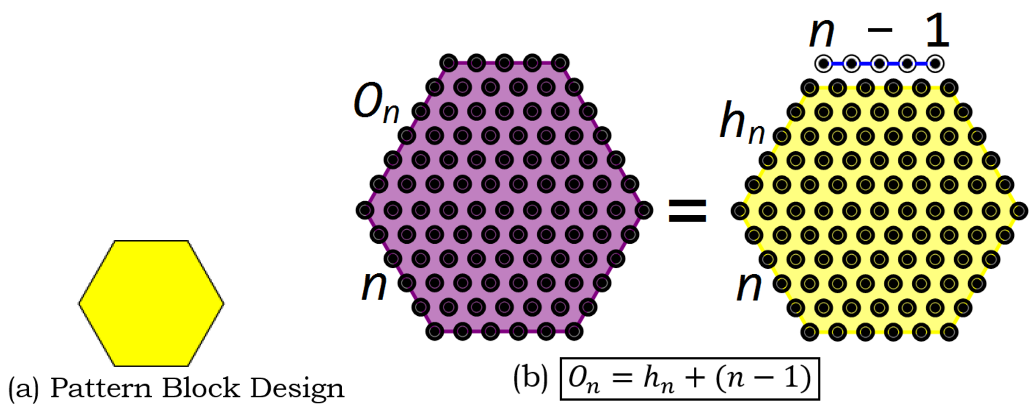


Figure 10: Yet another design modeling an identity prone to excess / lack of linear term / constant term. See Figure 4b for the representation used for the octagonal number.

4. Crowded Designs Modeling More Complex Figurate Number Identities

Each of the examples in this section describes a challenging figurate number relationship whose discovery was facilitated by crowded pattern block designs. The generation of the following identities typically involved Pólya's Four-Step Approach to Problem Solving along with the Guess and Check Strategy, and the corresponding PWW drawings could be carried out in a similar manner as explained in Section 3. As a result, we only present the identities and figures here, without further discussion.

Example 7: An Identity Involving Triangular and Heptagonal Numbers

Here is a PWW for the identity $3X_n + T_{n-2} = T_{4n-2}$:

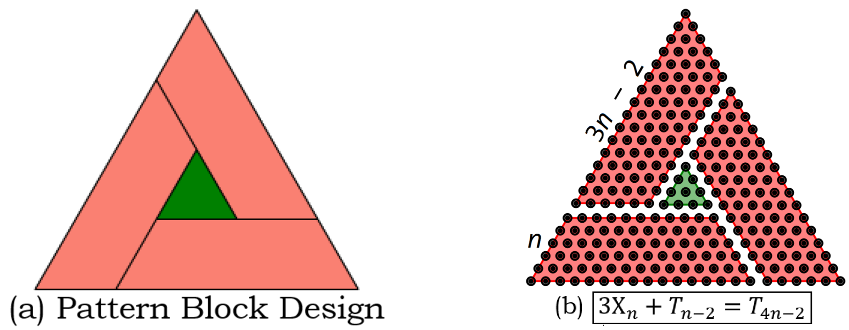


Figure 11: A crowded design modeling a more complex figurate number identity.

Example 8: Identities Involving Triangular and Hexagonal Numbers

Here are two identities involving triangular and hexagonal numbers:

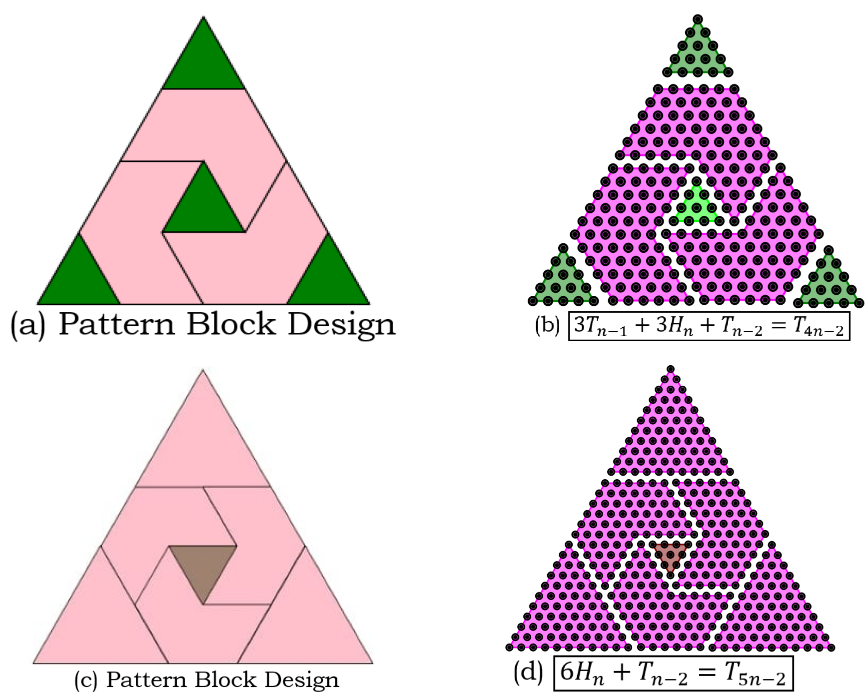


Figure 12: More crowded designs modeling more complex figurate number identities.

Example 9: An Identity Involving Triangular, Hexagonal and Pronic Numbers

Here is a PWW for the identity $3X_n + H_n + 3E_n = T_{5n-1}$:

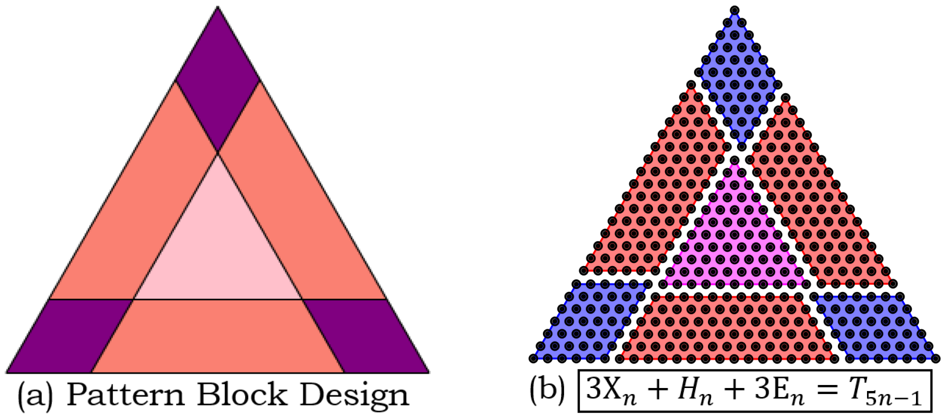


Figure 13: Yet another crowded design modeling a more complex figurate number identity.

Example 10: An Identity Involving Triangular and Nonagonal Numbers

Here is a PWW for the identity $T_{n+1} + 3N_n + 3T_{n-1} = T_{5n-2}$:

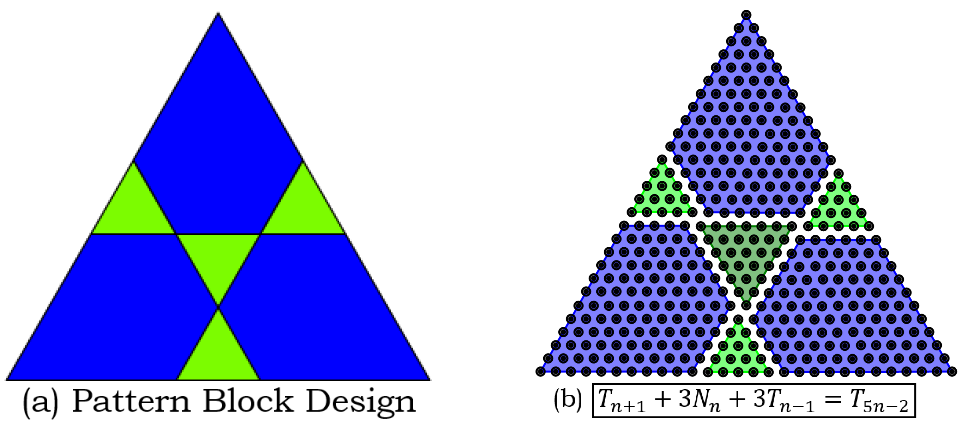


Figure 14: One more crowded design modeling a more complex figurate number identity.

5. Open Questions

We conclude this note with a list of figurate number identities currently lacking polygonal number designs (that is, proof without words drawings) see Figure 15. We only list suggested pattern block designs that are accompanied by algebraic identities.

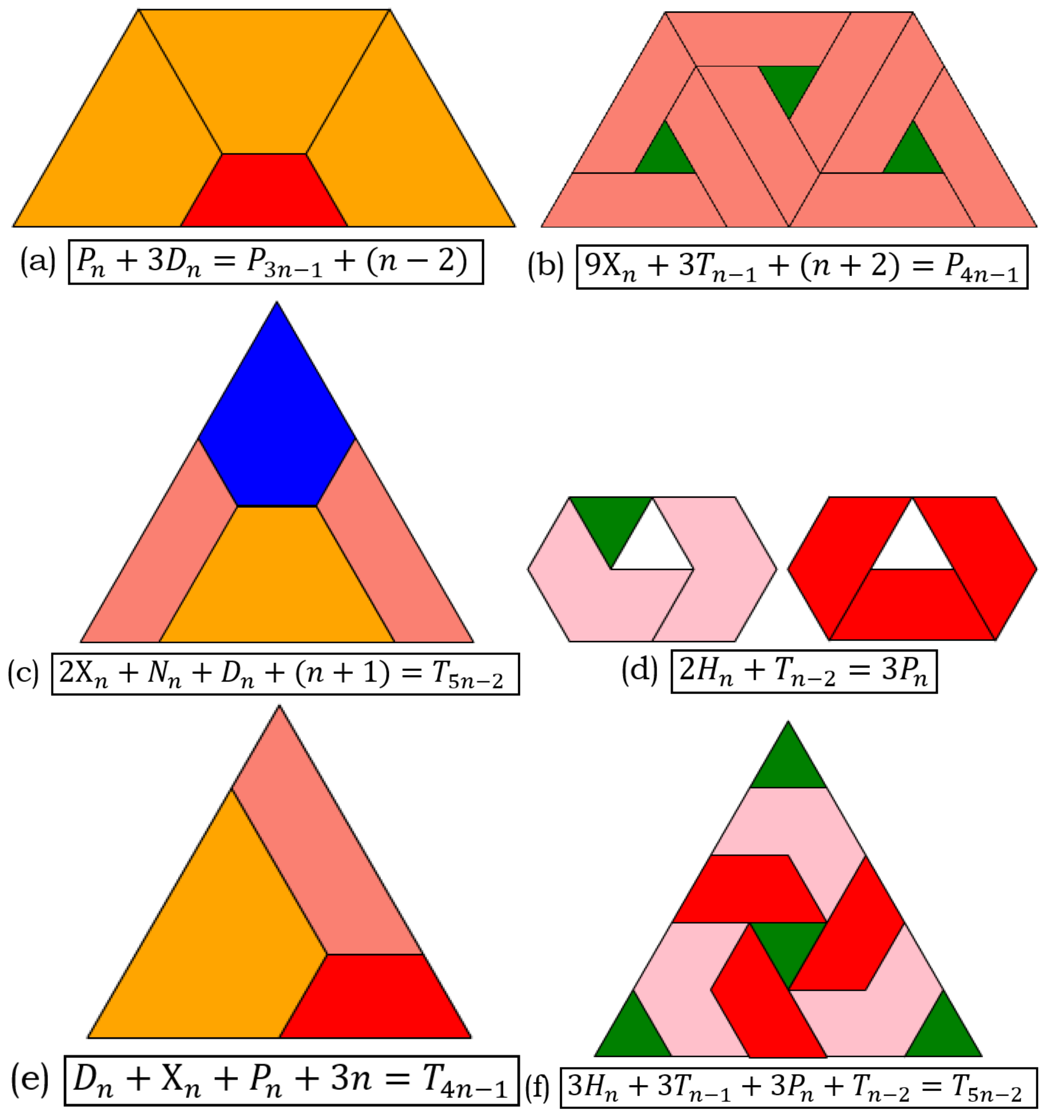


Figure 15: Some figurate number identities currently lacking polygonal number designs.

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