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Decisions, Decisions How Should The Votes Be Counted?

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Synopsis

It is a simple matter for the members of a group to decide among two options. When there are three or more options among which to choose, the situation is much more complicated. This is precisely what faces the electorate each time there are more than two candidates running for a single office. And while there is debate over which voting method should be used, there is wide agreement over the method that should not be used: plurality, the most common approach taken in the United States. This article presents a simple classroom activity which provides students the opportunity to explore this issue in the context of a group of friends deciding which movie to watch. The methods of plurality, Borda count, approval, Condorcet, and Instant Runoff are discussed, along with advantages and disadvantages of each. All theoretical discussions are illustrated with voting data collected from fifty-eight students; an accompanying spreadsheet contains the data and the tabulations corresponding to each method.

Sometimes making a choice is a simple thing to do. If I am picking a pizza topping, it's Italian sausage, hands down. On Sunday afternoon I may ask my family if we should play corn hole or croquet. Someone may be disappointed, but if the majority would rather play croquet, then croquet it is.

Things obviously get messier when there is a group of people making a choice and there are several options among which to choose. We see this played out every four years when it is time to elect the President of the United States. But it also happens in such everyday situations as a family deciding where to go for vacation or when a group of friends decides to watch a movie. How should the group make its choice?

A common answer is the Plurality Method of voting: the option that gets the most votes wins. For example, in 1998 former professional wrestler Jesse “The Body” Ventura was elected governor of Minnesota. He won about 37% of the vote as the Reform Party candidate, better than Republican Norm Coleman’s 34% and Democrat Skip Humphrey’s 28%. Ventura became governor even though about 63% of the voters chose not to vote for him. Does the Plurality voting system, forcing people to choose only one option out of three or more candidates, accurately reflect the preferences of the voters?

Mathematicians say no. In fact, there is a strong consensus that the Plurality method is in fact the worst voting method to use when three or more options are in play [4]. There is very little consensus, however, about what voting method should replace it.

In this article we explore a simple classroom activity that helps students to think about the difficulties involved in determining the choice of a group. It seems most appropriate for a “math for liberal arts” course but can also be used with math majors who have not yet explored the topic of voting methods. The results should be eye-opening, since the group’s preference (the “winner”) will vary depending on the voting method in place, even when exactly the same ballot results are used.

Our students do not have enough opportunities to explore truly open problems such as the one posed by this activity. Almost all of their work in mathematics leads them to “the answer”. But this time “the answer” is not so easily established!

1. The Voting Activity

When I most recently introduced this activity in August 2019, I did not explain to the students that it would illustrate a variety of voting methods. I just handed out half-sheets of paper on which I had listed ten films:

1. *Avatar*;
2. *Toy Story 3*;
3. *Harry Potter and the Deathly Hallows: Part 2*;
4. *The Avengers* (2012);
5. *Frozen*;

6. *Transformers: Age of Extinction*;
7. *Star Wars VII – The Force Awakens*;
8. *Captain America: Civil War*;
9. *Star Wars VIII – The Last Jedi*;
10. *Avengers: Infinity War*;

In choosing these ten films for my list, I simply picked what I thought to be a popular movie from each of the years 2009 through 2018. Most of them are sci-fi/action films. But I liked having *Toy Story 3* and *Frozen* in the mix to try and appeal to people who are not Marvel or Star Wars fanatics!

Before distributing the sheets, I told the students my premise. You are with a group of friends in a house with no available internet connection. There are, however, DVDs of the ten films listed above. The group must decide which DVD to watch. Each person in the group is to complete a ranking of the ten films, with #10 going to the film you would MOST like to watch, down to #1, the LEAST desirable film. I told them to rank ALL of the films, even if there are some on the list that you would refuse to watch. But I allowed them to express this by having a line at the bottom of the sheet that read “I ranked all of these movies, but I really wouldn’t want to watch any of them beginning with number ____.” I said that, for example, if you hate the last three movies on your ranking, put a 3 in the blank. If you would be happy to watch any of the films, then leave the blank, well, blank! I then gave them a few minutes to complete their rankings and collected the sheets. In all, fifty-eight students completed a ranking.

After collecting the ballots, I entered each student’s ranking as a column in a spreadsheet; a sample of that sheet is shown in Table 1 below. You are reading that first column correctly: the student did not want to watch any film except *Avatar*! (For the full results, see the “Data and First-Place Votes” tab of the spreadsheet which accompanies this article.)

If you want to use this activity in your classroom, you might choose to use other movies — it makes sense to update the list as our students will have new ones to prefer. Or if you like, feel free to move away from movies, to, say, snacks, or other recreational activities. In any case, start with a list of comparable items for the students to rank. Also see some related comments at the beginning of Section 7.

Table 1: A Sample of Student Rankings from August 2019.

<i>Avatar</i>	10	8	5	6	2	5	4
<i>Toy Story 3</i>	9	2	2	3	7	9	10
<i>Harry Potter and the Deathly Hallows: Part 2</i>	2	9	1	2	8	4	8
<i>The Avengers</i> (2012)	3	5	7	9	10	10	7
<i>Frozen</i>	1	1	3	1	1	6	9
<i>Transformers: Age of Extinction</i>	8	6	4	7	3	3	3
<i>Star Wars VII - The Force Awakens</i>	4	10	8	4	5	2	1
<i>Captain America: Civil War</i>	7	4	10	8	6	8	6
<i>Star Wars VIII: The Last Jedi</i>	5	3	6	5	4	1	2
<i>Avengers: Infinity War</i>	6	7	9	10	9	7	5
Disapprove	9	1	0	1	3	4	0

2. Plurality Voting

The spreadsheet being ready, make it available for each student to look at individually, or, if this is not possible, project the data on a screen. Now ask your students: what are we to do with this information? It won't be long before someone suggests counting the number of "10" votes for each film. In Excel, this can be done quickly with the "COUNTIF" command. Table 2 lists our results from August 2019.

Table 2: First Place Votes from August 2019.

<i>Infinity War</i>	13
<i>Toy Story 3</i>	9
<i>Harry Potter</i>	8
<i>Avatar</i>	7
<i>Avengers</i> (2012)	6
<i>The Force Awakens</i>	6
<i>Civil War</i>	3
<i>Frozen</i>	2
<i>Transformers</i>	2
<i>The Last Jedi</i>	2

Infinity War is the winner! That was perhaps not a surprising result, given that it was the most recent film on my list and was a hugely popular film in a hugely popular franchise. But only about 22% of the students named it as their top choice. 78% of them wanted to watch something else instead. Is there any other way to organize the data?

3. Borda Count

In my experience it doesn't take long for a new suggestion to be raised: tally the total score that each film received. The highest total would reflect the film with the highest overall ranking for the group. Table 3 lists our results from August 2019.

Table 3: Total Ranking Scores from August 2019.

<i>Avengers (2012)</i>	420
<i>Infinity War</i>	402
<i>Civil War</i>	355
<i>Toy Story 3</i>	346
<i>Harry Potter</i>	333
<i>Avatar</i>	317
<i>The Force Awakens</i>	279
<i>The Last Jedi</i>	254
<i>Frozen</i>	252
<i>Transformers</i>	232

Now we have a different winner, the first of the *Avengers* ensemble films (“Avengers Ensemble!?”). *Infinity War* is now in second place and, in general, the results are rather jumbled compared to the table of first-place votes. (See the “Borda Count” tab of the spreadsheet.) Notice that we would have gotten the same ordering if a student had suggested that we use average rank instead of total points. In this case *Avengers* had an average rank of about 7.2. *Transformers*, in last place, had an average rank of 4.

This is the “Borda” count or Borda method. This method assigns the most points to the voter’s top choice, the second-most points to the second choice, and so on. The option with the most total points is the winner. The method has an obvious advantage: voters are able to express their preferences of ALL of the candidates in an election, not just one. On the other hand, it is possible that a candidate could be the top choice on a majority of ballots and still lose the Borda Count [1, page 176]. This is clearly an undesirable feature!

4. Approval Voting

With prompting, the students will recognize another approach: how can we make use of the “...I really wouldn’t want to watch any of them beginning

with number ____” part of the ballot? If the “voters” filled in that blank, then they “approved” of all of the films from 10 down to that number and “disapproved” of any of the films at the specified rank or lower.

The corresponding voting method is “Approval Voting.” It allows the voters to vote to approve any candidate on the ballot whom they would be satisfied to see as the winner, with no ranking involved. The candidate with the most approval votes wins.

Recall that, when coding the data into Excel, I put a “Disapprove” row under each ballot’s ranking. In this row I put the number the voter had used to complete the blank. If no number was filled in, I put a “0” on my spreadsheet. I then copied nine more rows of these “disapproval” numbers, and under that, put a command like `IF(K1>K11,1,0)`. (Here “K” represents one column of the spreadsheet, which holds the rankings of one voter.) I then copied this command line for nine rows. This has the effect of replicating each ballot, but only with 1s and 0s: 1 for each “approved” film and 0 for each “disapproved” film. I then tallied the number of 1s for each film. (See the “Approval Votes” tab of the spreadsheet.) Table 4 lists our results from August 2019.

Table 4: Approval Votes from August 2019.

<i>Avengers</i> (2012)	54
<i>Toy Story 3</i>	54
<i>Infinity War</i>	53
<i>Avatar</i>	51
<i>Civil War</i>	51
<i>Harry Potter</i>	46
<i>Transformers</i>	43
<i>The Force Awakens</i>	43
<i>Frozen</i>	42
<i>The Last Jedi</i>	42

We see some ties, in particular at the top spot: *Avengers* and *Toy Story 3* garnered the most approval votes of any of the films and are co-winners. Perhaps to honor that tie, the group should indulge in a mini movie marathon and watch a double feature!

A significant advantage of Approval Voting is that it eliminates the need to choose between preferred candidates. If a voter likes two or more candidates, they can vote for them all! This also would eliminate concerns that a third-

party candidate has siphoned votes away from a major party candidate. A disadvantage of the method is that it does not allow the voter to distinguish between acceptable (or unacceptable) options: all approvals are created equal.

5. The Condorcet (Head-to Head) Method

We will now examine two other ways to analyze the data. In my experience, however, students do not think of these ways on their own. Should you choose to make the activity largely student-directed, you could then omit the following methods. As enrichment, though, and as further ways to explore data tabulation, they are valuable.

The first of these is the “Condorcet” method, in which every possible head-to-head matchup between two films is tallied. The film that has a higher ranking on a ballot wins that ballot. For example, if we compare *Toy Story 3* to *Frozen*, *Toy Story* had a higher ranking (and was therefore preferred) to *Frozen* on forty-three of my fifty-eight ballots. *Toy Story 3* would have been a clear winner had those been the only two DVDs available. But, for example, *Infinity War* had a higher ranking than *Toy Story* on thirty-seven of the ballots.

This takes a bit longer to determine in Excel. For example, the rankings for *Toy Story 3* are recorded in row 5 of my spreadsheet and those of *Captain America: Civil War* are in row 11. To compare the two films, use a command like “IF(K5>K11,1,0)”. If K5 is more than K11, that means the voter prefers *Toy Story* to *Civil War*. In this case, a “1” is returned. Otherwise a “0” is returned. When this is tabulated for each ballot, meaning each column of the spreadsheet, sum the results. The total number of “1”s will indicate the number of our voters who prefer *Toy Story* to *Civil War*. At the same time, of course, we will also know how many voters prefer *Civil War* to *Toy Story*. Then repeat this procedure for all possible head-to-head matchups. (See the “Condorcet” tab of the spreadsheet.)

If there is a single option that wins every possible head-to-head matchup, that option is the Condorcet winner. There may, however, be no such winner for a given set of ballots. For example, consider just three of our films, *Toy Story*, *Transformers*, and *Frozen*, and three of my student voters whose ballots are in columns B, D and J of my spreadsheet. Voter B preferred these three films

in the order *Toy Story*, then *Transformers*, then *Frozen*. Voter D preferred *Transformers* to *Frozen* to *Toy Story*. Finally, voter J ranked *Frozen* the highest of these three, followed by *Toy Story*, followed by *Transformers*. If these had been my only three voters, we would have been left with an irresolvable cycle: two of our voters prefer *Transformers* to *Frozen*, two prefer *Frozen* to *Toy Story*, and two prefer *Toy Story* to *Transformers*. The fact that this may happen means that an election held under the Condorcet method may lead to no winner being declared. That is a big disadvantage!

In our case, however, this did not happen. *Avengers* (2012) was the Condorcet winner. A Condorcet winner makes a strong case: it beats every other challenger in a hypothetical, two-candidate race. Table 5 lists the number of times *Avengers* was preferred to the other films on each of the fifty-eight ballots from August 2019.

Table 5: Number of Ballots on Which *Avengers* was Preferred to the Listed Film from August 2019.

<i>Transformers</i>	50
<i>The Last Jedi</i>	43
<i>Frozen</i>	43
<i>The Force Awakens</i>	42
<i>Civil War</i>	41
<i>Avatar</i>	40
<i>Toy Story 3</i>	37
<i>Harry Potter</i>	36
<i>Infinity War</i>	30

The closest call was between *Avengers* and *Infinity War*. Thirty voters preferred *Avengers*, meaning that twenty-eight preferred *Infinity War*. A close call for *Avengers*, but still a majority.

Some patterns are emerging. *Avengers*, although in the middle of the pack with only six first-place votes, has won or tied under three different voting methods. And *Transformers: Age of Extinction* is almost unanimously the least-favorite film of the ten!

6. Instant Runoff

Our last voting method is “Instant Runoff.” Instant Runoff will not declare a winner until a candidate has earned a majority of first-place votes. If the top

candidate only has 49% of the votes, then there is no winner. Georgia, for example, famously employed a runoff election for its two U.S. Senate seats in January 2021. No candidate won 50% or more of the votes in November 2020, so the top two candidates for each seat faced off in a runoff election. This procedure has the advantage of requiring the winning candidate to attain majority support from the voters, even if it takes two rounds of voting to achieve this. The disadvantage is the expense to the state of holding a second round of voting.

The Instant Runoff method achieves the effect of a runoff election with a single ballot. Immediately after the election in which the voters rank the candidates, as we did with our films, first-place votes are tallied. If a candidate has a majority of first-place votes, that is, if more than half of the voters voted that candidate as their top choice, the election is over and we have a winner. But if not, then the candidates with the least number of these votes are eliminated.

Take another look at Table 2, where the first-place votes were listed. Even though it has the most number of first-place votes, *Infinity War* is far from having a majority of them, while *Frozen*, *Transformers*, and *Jedi* are tied for last place. Those three are now eliminated from the competition. But so as not to disenfranchise the six voters who selected one of those films as their top choice, we go back to those six ballots and move those voters' second choice to the top position. Then we retabulate the first place votes. (See the "Instant Runoff" tab of the spreadsheet.) Table 6 shows us what happens.

Table 6: Instant Runoff #1 for Rankings from August 2019.

<i>Infinity War</i>	13
<i>Toy Story 3</i>	12
<i>Harry Potter</i>	9
<i>Avatar</i>	8
<i>Avengers (2012)</i>	7
<i>The Force Awakens</i>	6
<i>Civil War</i>	3

No film has a majority of the first-place votes, so we repeat the process. We eliminate the film in the last place (*Civil War*) and assign new first-place votes on the three ballots that had *Civil War* in the first place. The updated results are presented in Table 7.

Table 7: Instant Runoff #2 for Rankings from August 2019.

<i>Infinity War</i>		15
<i>Toy Story 3</i>		12
<i>Harry Potter</i>		9
<i>Avatar</i>		8
<i>Avengers (2012)</i>		8
<i>The Force Awakens</i>		6

We still do not have a winner, that is, a film which has the majority of first-rank votes. So we repeat the process a few more times. The results are presented in Table 8.

Table 8: Instant Runoff #3, #4, and #5 for Rankings from August 2019.

Instant Runoff #3			Instant Runoff #4			Instant Runoff #5	
<i>Infinity War</i>	17		<i>Infinity War</i>	29		<i>Infinity War</i>	37
<i>Toy Story 3</i>	12	→	<i>Toy Story 3</i>	17	→	<i>Toy Story 3</i>	21
<i>Harry Potter</i>	11		<i>Harry Potter</i>	12			
<i>Avatar</i>	9						
<i>Avengers (2012)</i>	9						

Infinity War is our Instant Runoff winner. We determined this after an election and five runoff elections, but all done with the information found on the original ballots. No re-voting was necessary.

On Excel, this procedure may be carried out by deleting the rows that correspond to last-place films and then assigning the highest remaining number on each voter's ballot the value of "10". The film represented in that row becomes their new first-place choice; the number of "10"s is then recounted using the COUNTIF command.

7. Closing Remarks

Perhaps you imagine that it might be hard to get a set of student preferences that lead to different winners according to different voting mechanisms. Here I should note that I did not "doctor" the ballots in any way. I asked the students to complete their ratings, I checked the results, and I reported them as they are. I have done a similar exercise a few other times in the past and have mostly been likewise rewarded with different winners depending on the different methods.

Once I was not so fortunate: I included *The Dark Knight* a few years ago, and it was the runaway winner, regardless of method. When you set up your ballots, you may wish to avoid an option that would dominate the field. You could pick a popular film from each of the last ten years, as I did here, or perhaps two films from each of the last five years. In the ballots I used with my students, I listed the films in chronological order, with the oldest of the films (*Avatar*) listed first. (Disclaimer: There is research on how ballot order may impact voter choice [7]. Being listed first is advantageous, and this effect is more pronounced “down ballot” when many offices are being selected.)

As you conclude this activity, it is important to reinforce the point that there are many ways to conduct an election. You might like to lead a class discussion: Which way (or ways) is most fair? Which way best captures the intent of the voting group as a whole? What are the downsides to employing one or another of these methods?

I find there is a great advantage in engaging in this activity. Students (and others) often think that mathematics has all the answers, even if they personally are not able to do the math very well. It may not occur to them that there are numerical questions that mathematics cannot answer. Yet Kenneth Arrow’s famous theorem, for which he won the Nobel Prize in 1972, proves that there is no voting method involving three or more candidates which does not violate some very reasonable expectations [3]. It is a good thing for students to realize that sometimes mathematics cannot take us as far as we would like.

Your students may also be surprised to find that these methods are not just hypothetical. The Borda Count is used to elect the winner of college football’s Heisman Trophy. Many U.S. cities, including San Francisco, Minneapolis, and Santa Fe use Instant Runoff for their municipal elections. Maine has recently begun using it for all federal and statewide offices. And city officials in Fargo, ND, the United Nations Secretary General, and the officers of the Mathematical Association of America are all elected using Approval Voting. In fact, asking your students to find actual municipalities or organizations that do use these methods will be a helpful addition to the exercise. I suggest not telling them about the examples above and seeing if they can find them (and others) for themselves.

For additional information on election methods, see [2, 5, 6]. Your students will also enjoy exploring [FairVote](#), a site dedicated to promoting the use of Instant Runoff in elections.¹

Let's conclude by asking our question one more time: What movie should the group watch? *Infinity War*, if we decide by the Plurality or Instant Runoff methods. But it's the *Avengers* (2012), if we decide using the Borda Count or Condorcet methods. And if we use Approval Voting, it's a tie between *Avengers* and *Toy Story 3*. So what movie should the group watch that night? Let the debate begin!

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¹ FairVote is a group founded in 1992 to promote the use of instant runoff voting. As of September 2021 they are based in Takoma Park MD.