Journal of Humanistic Mathematics

Volume 12 | Issue 1 January 2022

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Lingguo Bu

Southern Illinois University Carbondale

Recommended Citation

Lingguo Bu, "Twisting the Cube: Art-Inspired Mathematical Explorations," *Journal of Humanistic Mathematics*, Volume 12 Issue 1 (January 2022), pages 352-365. DOI: 10.5642/jhummath.202201.27. Available at: https://scholarship.claremont.edu/jhm/vol12/iss1/27

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Twisting the Cube: Art-Inspired Mathematical Explorations

Lingguo Bu

School of Education, Southern Illinois University Carbondale, Illinois, USA
lgbu@siu.edu

Synopsis

A cube can be twisted in a playful manner for visual and algebraic insights. The twisting process and the resulting ruled surfaces can be demonstrated using 3D modeling tools (e.g., GeoGebra® and Autodesk Fusion 360®) or elastic cords on a 3D-printable scaffold. The twisted cube is aesthetically appealing, posing interesting questions that are worthwhile at multiple levels. Algebraically, the volume of the twisted cube is shown to be two-thirds of the reference cube. The twisted faces are parts of hyperbolic paraboloids, whose implicit and parametric equations can be established from diverse perspectives in support of further dynamic explorations and discussions about the surface area.

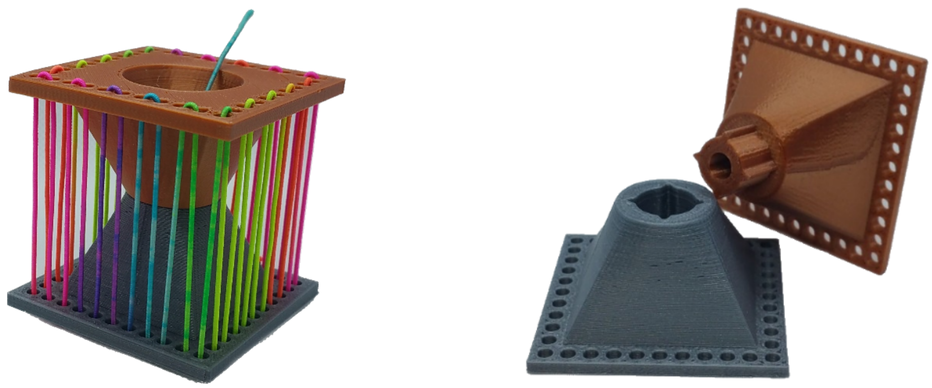
Keywords: *cube, quadric surface, ruled surface, twisting, algebraic analysis, 3D modeling.*

1. Introduction

The cube affords an intriguing world of art and mathematics. In everyday life, it is the base for numerous mathematical puzzles and toys. In school mathematics, the cube also serves as a fundamental reference for three-dimensional geometry, supporting the development of a variety of mathematical structures. A cube can be dissected, flattened, or spun for aesthetic and mathematical explorations.[1]

[1] I have written on various interesting ways one can manipulate the cube in other papers. In [1], I explored a historical cube dissection. Three other papers complement this current one in various ways: [2] on using technology to visualize cube rotations; [3] on a "volume invariant twist", and [4] on the diagonal spin and its algebraic analysis.

In a mathematical art project, I decided to give the cube a twist and thus came to know both the geometric and algebraic nature of the twisted cube. In the literature, however, there is very little discussion about twisting the cube and its artistic and mathematical consequences. In a related context decades ago, Steinhaus briefly described surfaces made of straight lines across a skew quadrilateral and the resulting saddle-shaped hyperbolic paraboloid [10]. Today, we can take advantage of emerging 3D design and dynamic modeling technologies, such as Autodesk Fusion 360® and GeoGebra®, to approach the processes and outcomes of cube twisting and further delve into the art and mathematics around the twisted cube.

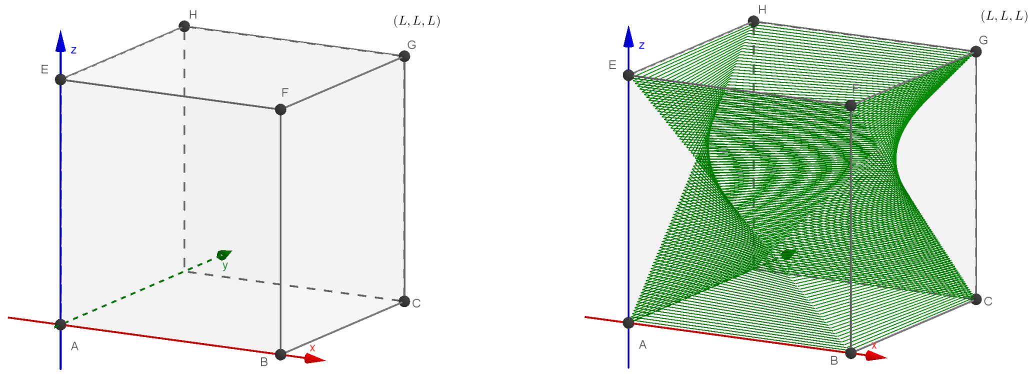


(a) (b)

Figure 1: Twistable cube model made of rigid top and bottom and four elastic faces around. 3D printable models are accessible at `https://www.thingiverse.com/thing:5195134`, last accessed on January 22, 2022.

To start with, let us imagine a cube that has a rigid top and a rigid bottom with four elastic faces around, as shown in Figure 1a. We then hold the bottom steady and twist the top 90 degrees clockwise or counterclockwise, without changing the distance between the top and bottom faces. The twisting process can be conveniently approached using 3D printable models and elastic cords (Figure 1) or modeled virtually using GeoGebra (Figure 2). The physical model can be 3D designed and printed for hands-on demonstrations alongside the elastic cord model (Figure 3). Both are visually appealing and mathematically inviting.

Among a host of questions about the twisted cube, we pose and aim to answer the following that are particularly interesting:



(a) (b)

Figure 2: A cube in the first octant is twisted 90 degrees clockwise (created with GeoGebra®).

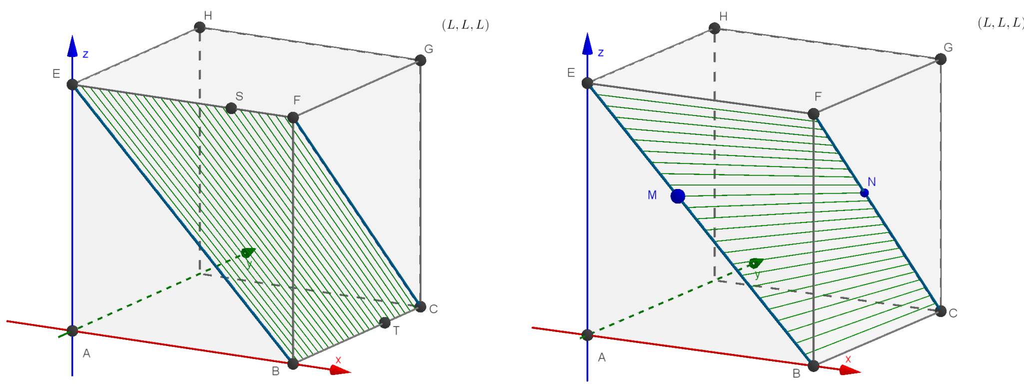
(i) What has happened to the four square faces around the cube?
(ii) What is the volume of the twisted cube?
(iii) What is the area of each of the four twisted faces?
(iv) What is the algebraic nature of the twisted faces?

(a) (b)

Figure 3: Two models for the twisted cube: A 3D printed shell and an elastic-cord cage.

2. Twisted Faces As Ruled Surfaces

Let us first look at the four twisted surfaces, which are flat square faces before the 90-degree twist. There are two ways to conceptualize the twisting if we do not make any distinction between clockwise and counterclockwise twists. As shown in Figure 4a, a twisted face is a ruled surface [6] formed by a face diagonal (CF or BE), namely, the ruling. Alternatively, we can view the twisted face as a ruled surface swept out by a cube edge (BC or EF) moving from the bottom to the top or vice versa (Figure 4b).



(a) (b)

Figure 4: Either a face diagonal or an edge can be used as a ruling for the twisted surface (created with GeoGebra®).

In the following sections, we develop algebraic descriptions of the ruled surface using both perspectives and eventually calculate its area.

3. Volume of the Twisted Cube

The twisted cube does not retain the original volume of the cube. It is much smaller, as seen in the virtual and physical models. To find the volume of the twisted cube, we position a cube of edge length L in the first octant of the 3D Cartesian System, as shown in Figure 5, and consider the square intersection MNM_1N_1 between a horizontal plane $z = z_0$ ($0 \leq z_0 \leq L$) and a family of four face diagonals. On line BE in the xz-plane, $M = (L - z_0, 0, z_0)$. Also, the center of square MNM_1N_1, $O = (\frac{L}{2}, \frac{L}{2}, z_0)$. Thus, $|OM|^2 = (\frac{L}{2} - z_0)^2 + (\frac{L}{2})^2$. Further, since $\triangle MNO$ is a right triangle with $OM \cong ON$, the area of square MNM_1N_1 is $|MN|^2 = 2|OM|^2 = 2\left((\frac{L}{2} - z_0)^2 + (\frac{L}{2})^2\right)$.

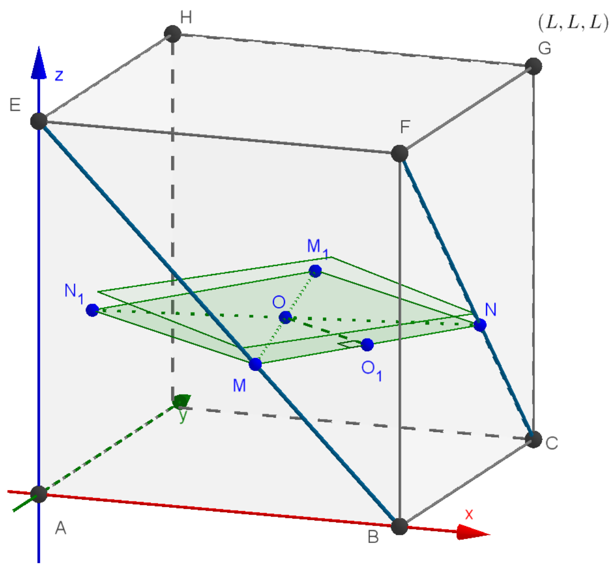


Figure 5: The volume of the twisted cube is the integration of the square area over the height of the cube (created with GeoGebra®).

Therefore, the volume of the twisted cube is

$$\begin{aligned} V &= 2\int_0^L \left(\left(\frac{L}{2} - z\right)^2 + \left(\frac{L}{2}\right)^2\right) \mathrm{d}z \\ &= \frac{2}{3}L^3, \end{aligned} \tag{3.1}$$

which means, surprisingly, that the twisted cube has a volume that is two-thirds of the original. Specifically, if the cube edge is $L = 5$, the volume of the twisted cube is approximately 83.33 cubic units.

4. Implicit Equation for the Twisted Face

Let us again position a cube of edge length L in the first octant of the 3D Cartesian System, as shown in Figure 6, where vertex A is at the system origin, with $B = (L, 0, 0)$, $E = (0, 0, L)$, $C = (L, L, 0)$, and $G = (L, L, L)$. In the xz-plane, the face diagonal BE is algebraically

$$z = -x + L, \tag{4.1}$$

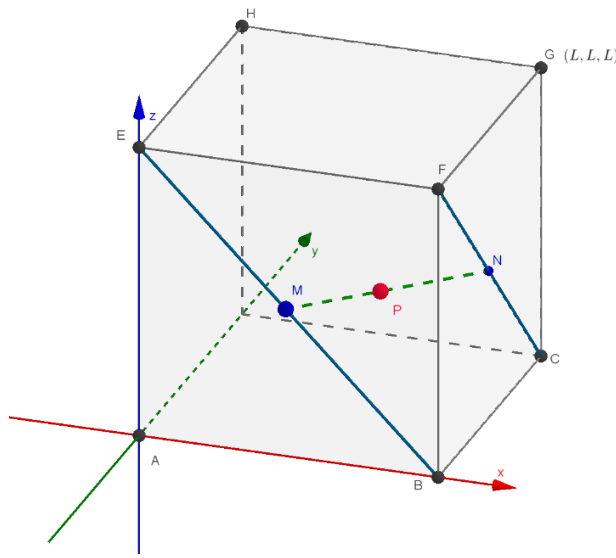


Figure 6: Seeking an implicit equation for the twisted face of the cube (created with GeoGebra®).

and in the plane $x = L$ (that is, the plane $BCGF$), the face diagonal CF is

$$z = -y + L. \tag{4.2}$$

We now consider a horizontal plane $z = z_0$ with $0 \leq z_0 \leq L$, which intersects with BE and CF at M and N, respectively. Hence, $M = (L - z_0, 0, z_0)$ and $N = (L, L - z_0, z_0)$.

In the horizontal plane $z = z_0$, line MN has a slope of $m = \frac{L - z_0}{z_0}$ and a y-intercept of $b = \frac{-z_0^2 + 2Lz_0 - L^2}{z_0}$. Thus, line MN has the algebraic form of

$$y = \left(\frac{L - z_0}{z_0}\right) x + \frac{-z_0^2 + 2Lz_0 - L^2}{z_0}, \tag{4.3}$$

which, when z_0 is replaced with z for the general case, yields

$$yz = Lx - xz - z^2 + 2Lz - L^2, \tag{4.4}$$

or

$$xz + yz + z^2 - Lx - 2Lz + L^2 = 0. \tag{4.5}$$

For example, when $L = 5$, Equation (4.5) becomes

$$xz + yz + z^2 - 5x - 10z + 25 = 0, \tag{4.6}$$

which can be readily plotted and visually manipulated in GeoGebra 3D Graphics (or any other similar 3D graphing environments).

5. Parametric Equation for the Twisted Face: Cube Edge As the Ruling

To establish a parametric function for the twisted cube face and thus calculate its area, we consider the vector $\overrightarrow{AP}$ in Figure 7, where P is on the ruling MN. It follows from the previous discussion in Section 4 that

$$\overrightarrow{AP} = \langle x, \frac{L-z}{z}x + \frac{-z^2 + 2Lz - L^2}{z}, z \rangle, \tag{5.1}$$

where $0 \le z \le L$, $L - z \le x \le L$, as shown in Equations (4.1) and (4.3).

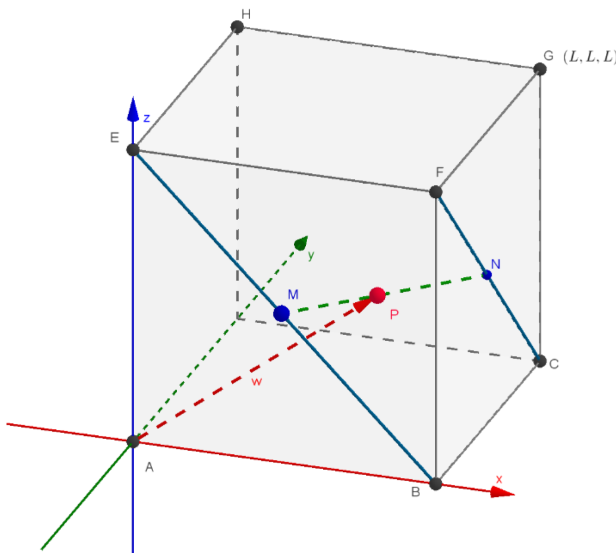


Figure 7: Seeking a parametric function for the twisted face of the cube (created with GeoGebra®).

Following the convention of parametric surfaces, we use u for x, v for z, and rewrite Equation (5.1) as

$$r(u, v) = \overrightarrow{AP} = u\hat{i} + \left(\frac{-uv - v^2 + Lu + 2Lv - L^2}{v} \right) \hat{j} + v\hat{k}, \tag{5.2}$$

where $0 \le v \le L$, $L - v \le u \le L$.

When $L = 5$, we have $r(u, v) = u\hat{i} + \left(\frac{-uv-v^2+5u+10v-25}{v}\right)\hat{j} + v\hat{k}$, which can be graphed in GeoGebra® using the following `Surface` command:

```
Surface((u, (-uv-v^2+5u+10v-25)/v, v), u, 0, 5, v, 0, 5)
```

Taking the partial derivatives of $r(u, v)$ with respect to u and v, respectively, we get

$$r_u(u, v) = \hat{i} + \left(\frac{L - v}{v}\right)\hat{j} + 0\hat{k}, \tag{5.3}$$

and

$$r_v(u, v) = 0\hat{i} + \left(\frac{-v^2 - Lu + L^2}{v^2}\right)\hat{j} + \hat{k}. \tag{5.4}$$

The magnitude of the cross product $r_u \times r_v$ is thus

$$\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{L-v}{v} & 0 \\ 0 & \frac{-v^2-Lu+L^2}{v^2} & 1 \end{matrix}\right\| = \sqrt{1 + \left(\frac{-Lu}{v^2} + \frac{L^2}{v^2} - 1\right)^2 + \left(\frac{L}{v} - 1\right)^2}. \tag{5.5}$$

Therefore, the surface area of one twisted cube face is

$$\begin{aligned} \text{SA} &= \iint_R \|r_u \times r_v\| \; \mathrm{d}u\mathrm{d}v \\ &= \int_0^L \int_{L-v}^L \sqrt{1 + \left(\frac{-Lu}{v^2} + \frac{L^2}{v^2} - 1\right)^2 + \left(\frac{L}{v} - 1\right)^2} \, \mathrm{d}u\mathrm{d}v. \end{aligned} \tag{5.6}$$

When $L = 5$, for example, the integral in (5.6) becomes

$$\text{SA} = \int_0^5 \int_{5-v}^5 \sqrt{1 + \left(\frac{-5u}{v^2} + \frac{25}{v^2} - 1\right)^2 + \left(\frac{5}{v} - 1\right)^2} \, \mathrm{d}u\mathrm{d}v, \tag{5.7}$$

which is approximately 22.5178 square units.

6. Parametric Equation for the Twisted Face: Face Diagonal As the Ruling

A parametric equation for the twisted cube face can also be established on the basis of a dynamic face diagonal. As shown in Figure 8, Q is a point on the moving face diagonal ST.

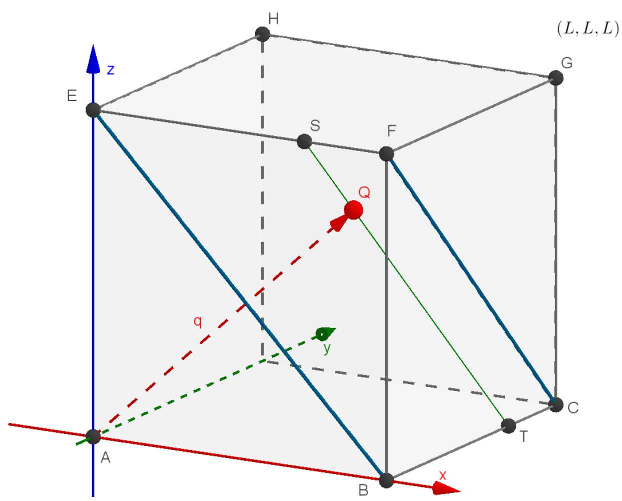


Figure 8: Parametric equation based on a dynamic face diagonal (created with GeoGebra®).

Let $t = |SF| = |TC|$, where $0 \leq t \leq L$, be the first parameter. Then, $S = (L - t, 0, L)$, and $T = (L, L - t, 0)$ in the context of Figure 8, and vector $\overrightarrow{ST} = \langle t, L - t, -L \rangle$.

The parametric equation of line ST, using S as a starting point and $\overrightarrow{ST}$ as the direction, is

$$\begin{aligned} \overrightarrow{r_{ST}} &= \langle L - t, 0, L \rangle + s\langle t, L - t, -L \rangle \\ &= \langle L - t + st, Ls - st, L - Ls \rangle, \quad \text{where} \quad 0 \leq s \leq 1,\ 0 \leq t \leq L. \end{aligned} \tag{6.1}$$

The parametric equation for the surface swept by ST is therefore

$$q(s,t) = \overrightarrow{AQ} = (L - t + st)\hat{\imath} + (Ls - st)\hat{\jmath} + (L - Ls)\hat{k}, \tag{6.2}$$

where $0 \leq s \leq 1$, $0 \leq t \leq L$.

When $L = 5$, for example, $q(s,t) = (5 - t + st)\hat{\imath} + (5s - st)\hat{\jmath} + (5 - 5s)\hat{k}$, which can be graphed in GeoGebra® using the following `Surface` command:

```
Surface((5 - t + s t, 5s - s t, 5 - 5s), s, 0, 1, t, 0, 5)}
```

as shown in Figure 9.

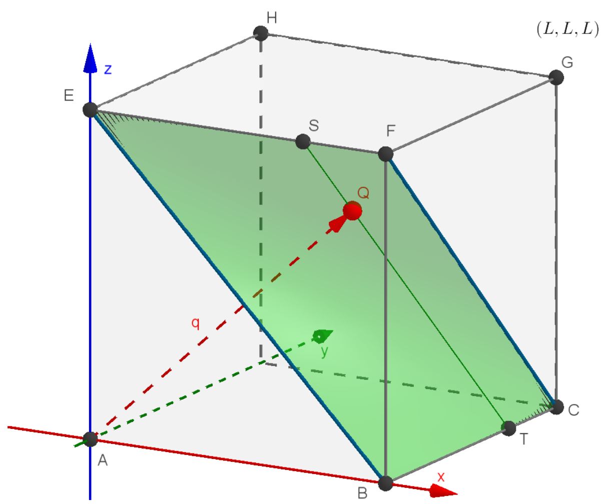


Figure 9: Parametric surface based on a sweeping face diagonal (created with GeoGebra®).

Using Equation (6.2), the surface area of one twisted face is thus

$$\begin{aligned} \mathrm{SA} &= \iint_R \|q_s \times q_t\| \; \mathrm{d}s\mathrm{d}t \\ &= \int_0^L \int_0^1 \sqrt{(L - Ls - t)^2 + (L - Ls)^2 + L^2 s^2} \, \mathrm{d}s\mathrm{d}t. \end{aligned} \tag{6.3}$$

Note that Equation (6.2) is equivalent to Equation (5.2), although they use different parameters. As can be proven, when they have the same $\hat{i}$ and $\hat{k}$ components, they also have the same $\hat{j}$ component. Specifically, if $u = L - t + st$ and $v = L - Ls$, then $t = \frac{L^2 - Lu}{v}$ and $s = \frac{L-v}{L}$. Thus, the $\hat{j}$ component of Equation (6.2) is $Ls - st = \frac{-uv - v^2 + Lu + 2Lv - L^2}{v}$, which is the $\hat{j}$ component in Equation (5.2).

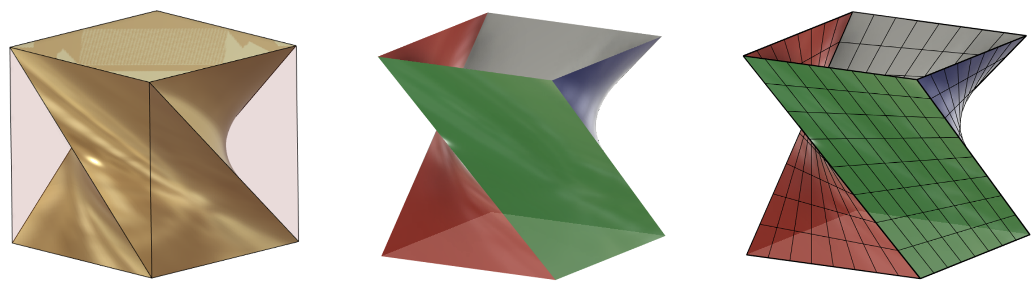
Similarly, an implicit equation for the twisted cube face can be derived from Equation (6.2). Let $x = L - t + st$ and $z = L - Ls$, then $s = \frac{L-z}{L}$ and $t = \frac{L(L-x)}{z}$, and

$$\begin{aligned} y &= Ls - st \\ &= L - z - \frac{(L - z)(L - x)}{z} \\ &= \frac{-xz - z^2 + Lx + 2Lz - L^2}{z}, \end{aligned} \tag{6.4}$$

which is equivalent to equation (4.5).

7. Dynamic Modeling and 3D Designs

The twisted cube can be dynamically modeled in GeoGebra® 3D Graphics using the `Trace` tool, as shown in Figures 2b, 4a, and 4b. There are also multiple 3D design environments where the cube can be twisted for hands-on modeling. In Autodesk Fusion 360®, the cube can be twisted using the `Loft` function in the `Surface` workspace. Starting with a cube, we further sketch the face diagonals. According to our previous discussions, there are two ways to create the twisted cube faces. We can `loft` two corresponding top and bottom edges, using the face diagonals as `rails` or, alternatively, `loft` two adjacent face diagonals while defining the bottom and top edges as `rails`. Figure 10 shows three different views of the resulting surfaces.



(a) (b) (c)

Figure 10: Three different views of the twisted cube (created with Autodesk Fusion 360®).

Further 3D manipulations can be performed to create a twisted solid or a hollow cup. The two-piece twistable model can be designed from a cube with a connector at the center allowing 90-degree rotations (Figure 1b). Both the 3D solid and the elastic cord models are visually appealing and playful, inviting rich mathematical conversations and offering problem posing opportunities.

8. Conclusion

The cube is one of the most common mathematical objects in everyday life and school mathematics. Yet it provides endless opportunities for generative mathematical conversations about shapes and geometric transformations [7]. In twisting the cube, we put into practice the art of problem posing and mathematical play in the context of dynamic modeling and 3D design, "coming to know" [5, page 2] both the visual appeal of the resulting solid and a rich and motivating world of mathematical connections [8, 9, 11]. As is shown by the deceptively simple example of the cube and its myriad manipulations, the integration of emergent modeling technologies holds a fresh promise for operationalizing mathematical aesthetics, allowing educators to be responsive to students diverse needs for hands-on manipulation, intellectual inquiry, and expressive facility.

As I already mentioned in Footnote 1, over the years, I have revisited the cube from various perspectives, inspired by the curiosity, often spontaneous in nature, of both young children and college students alike. In the open manipulations of the cube, my students and I have created surprisingly attractive physical artifacts, which subsequently led us to the re-discovery of elegant mathematical ideas and structures. All throughout, the cube has served as a tool of student engagement and a mathematical scaffold, affording a sense of playfulness, ownership, and accessibility regarding some foundational mathematical processes.

In a classroom, the cube can be twisted at multiple levels, where a *what-if* thought experiment might be an engaging starting point. With elementary students, the two-piece model can be printed and used with elastic cords for students to feel and see the surface taking shape. Interestingly, the length of the cord can be a great question to pose, too. If the reference cube is known to be 50mm in all three dimensions and each top or bottom edge has eleven holes, how much cord do we need to loop all round the cube while taking into account all the twists and turns? The answer is another surprise — it is approximately eight feet!

In the middle grades, GeoGebra® and 3D design environments can be utilized to construct a cube for virtual simulation and solid generation, where the visual processes and artifacts allow rich mathematical talks. At the secondary level and above, all the multimodal aspects of the twisted cube can be examined for geometric and algebraic insight, using GeoGebra® for problem setups and computational confirmations. Surprisingly, the twisted cube loses one-third of its volume; each twisted face is part of a hyperbolic paraboloid across a quadrilateral made of two cube edges and two face diagonals, losing about 10% of the area of a cube face. There are two rulings for each twisted surface and two equivalent parametric equations.

There are certainly many more questions we can pose about the twisted cube, and, more importantly, there are many ways we can re-imagine mathematics teaching and learning as we reflect on the traditions and further experiment with new modeling technologies.

Acknowledgments

The author would like to thank the Southern Illinois University STEM Education Research Center and the University Foundation for providing 3D printers and printing materials in support of the present and related projects in exploring mathematical arts in STEM teacher education and K–12 outreach. All views, findings, and potential mistakes belong to the author and do not necessarily represent those of the project sponsors.

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