Journal of Humanistic Mathematics

Volume 12 | Issue 1 January 2022

# Nilpotents Leave No Trace: A Matrix Mystery for Pandemic Times

Eric L. Grinberg
*University of Massachusetts Boston*



Recommended Citation
Eric L. Grinberg, "Nilpotents Leave No Trace: A Matrix Mystery for Pandemic Times," *Journal of Humanistic Mathematics*, Volume 12 Issue 1 (January 2022), pages 139-147. DOI: 10.5642/jhummath.202201.10. Available at: https://scholarship.claremont.edu/jhm/vol12/iss1/10


JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | http://scholarship.claremont.edu/jhm/

# Nilpotents Leave No Trace: A Matrix Mystery for Pandemic Times


Eric L. Grinberg

*Department of Mathematics, University of Massachusetts Boston, USA*
eric.grinberg@umb.edu



**Synopsis**

Reopening a cold case, Inspector Echelon, high-ranking in the Row Operations Center, is searching for a lost linear map, known to be nilpotent. When a partially decomposed matrix is unearthed, he reconstructs its reduced form, finding it singular. But were its origins nilpotent?



**Keywords:** *nilpotent matrix, singular matrix, row reduced echelon form, RREF, null space, kernel, Jorge Luis Borges, mystery.*


## 1. Early in the Investigation

In teaching Linear Algebra, the first topic frequently is row reduction [1, 5, 7], including *Row Reduced Echelon Form* (RREF); its applicability is broad and growing. Another topic, surprisingly popular with beginning students, is *nilpotent matrices*. One naturally wonders about their intersection. For instance, one would expect to find a book exercise asking:

> *What can be said about the row reduced echelon form of a nilpotent matrix?*

In the early days of the COVID-19 pandemic, as test delivery went remote, demand grew for new, Internet-resistant problems. A limited literature search for the Nilpotent-RREF connection came up short, suggesting potential for take-home final exam questions, hence the note at hand. We'll first explore examples sufficient to settle the $3 \times 3$ case, then consider the general situation. The upshot is that row reduction eliminates all traces of nilpotence.


Figure 1: "Very Early in The Investigation," by Frank Cotham, *New Yorker*, 2007. Permission for use was obtained by arrangement with Condé Nast.

## 2. Stumbling On Evidence

We refer to [1, 2] for general background on RREF and rank. (Most other linear algebra texts will also do, of course.) Recall that a matrix $M$ is *nilpotent* if it is square and if some power of $M$, say $M^k$, is the zero matrix; the smallest such $k$ is called the *nilpotent index* or just *index* of $M$. For instance, the rightmost matrix in (2.2) below is nilpotent, of index 3. Indeed, every *strictly upper-triangular matrix* (i.e., square, with zeros on and below the diagonal) is nilpotent. Every nilpotent matrix $N$ is singular and has additional standard properties. For instance, $N$ has trace zero, as do all its powers. Examining each general type of *singular* $3 \times 3$ matrix of RREF, we'll try to find those that are row equivalent to nilpotents.

A nilpotent $3 \times 3$ matrix, being singular, can have rank 0, 1, or 2. We'll begin by considering general $3 \times 3$ matrices of rank 1 in RREF. There are three types of such matrices:

$$\begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

where the entries $a, b, c$ are fixed but unspecified and unrestricted constants.

When working with matrices and their components we'll follow a *left to right* and *top to bottom* convention. Thus *first* means *leftmost*, etc. We enumerate the rows of a matrix using the notation $R_i$. Hence $R_2$ is the second row from the top.

The second and third matrices in (2.1) are strictly upper triangular, hence nilpotent. Call the first matrix $F$. It is not traceless, hence not nilpotent, but we can try to row reduce it into nilpotence. In $F$, if $b = 0$, interchange rows $R_1$ and $R_3$ (i.e., perform $R_1 \longleftrightarrow R_3$) to obtain a strictly lower triangular matrix, hence a nilpotent matrix. If $b \neq 0$, perform $R_3 \longrightarrow R_3 - \frac{1}{b}R_1$ (subtract $\frac{1}{b}$ times row $R_1$ from row $R_3$ and make that the new row $R_3$) to obtain

$$\begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ -\frac{1}{b} & -\frac{a}{b} & -1 \end{pmatrix}.$$

This matrix squares to zero, hence is nilpotent.

Next, we present the RREFs of $3 \times 3$ matrices with rank 2:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

The third matrix is strictly upper triangular, hence nilpotent. For the second matrix, exchange rows $R_1$ and $R_3$, then perform $R_1 \longrightarrow R_1 - (a)R_2$ to obtain

$$\begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 1 \\ 1 & a & 0 \end{pmatrix}.$$

This matrix cubes to zero, hence is nilpotent. As with all $3 \times 3$ matrices of rank 2, its square does not vanish (exercise), so it is nilpotent of index 3.

Up till now we have been far from systematic, and the leftmost matrix in (2.2), call it $T$, takes even more doing. The procedure we offer is no more systematic, and admittedly less than pleasant to parse, but rest assured that systematic relief cometh.

We can row reduce $T$ to the following matrix, which is nilpotent of index 3:

$$\begin{pmatrix} -1 & 0 & -a \\ -\frac{b}{a} & 0 & -b \\ -\frac{b-1}{a} & 1 & 1 \end{pmatrix}. \tag{2.3}$$

We can get from $T$, the leftmost matrix in (2.2), to (2.3) by the following row operations:

$$\begin{aligned} &R_2 \longleftrightarrow R_3; \\ &R_2 \longrightarrow R_2 - \left(\tfrac{b}{a}\right) R_1; \\ &R_1 \longrightarrow (-1) R_1; \\ &R_3 \longrightarrow R_3 - \left(\tfrac{b-1}{b}\right) R_2. \end{aligned} \tag{2.4}$$

This tacitly assumes that $a, b$ are both nonzero. If both $a$ and $b$ are zero, row swaps turn $T$ into a strictly lower triangular, and hence nilpotent matrix.

If $b = 0$ and $a \neq 0$ then $T$ is still nilpotent and row equivalent to (2.3), even though the steps we took to get there involve a zero denominator. (For this case, replace the last step of (2.4) by $R_3 \longrightarrow R_3 - \left(\frac{1}{a}\right) R_1$.)

In case $a = 0$, perform row reduction steps

$$R_2 \leftrightarrow R_3;\ R_1 \leftrightarrow R_2;\ R_2 \longrightarrow R_2 - bR_3,$$

obtaining the following nilpotent matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -b & -b^2 \\ 0 & 1 & b \end{pmatrix}.$$

But (2.3) and (2.4) and all the row manipulations beg the question: how did we come up with these constructs?

## 3. No Basis For An Investigation

> The facts which you have brought me are so indefinite that we have no basis for an investigation.
>
> –Sherlock Holmes
> in *The Adventure of the Dancing Men.*

> We have a good working basis, however, on which to start.
>
> –Sherlock Holmes
> in *A Study In Scarlet.*

Consider again the matrix

$$T = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Recall that the (right) *null space* of $T$ is the solution set of the linear system $T\mathbf{v} = \mathbf{0}$. One can check that the span of the vector $\mathbf{w} \equiv \begin{pmatrix} -a & -b & 1 \end{pmatrix}^t$ gives all solutions of this linear system.

We form a basis for $\mathbb{R}^3$ by extending the one element set $\{\mathbf{w}\}$, and use that to build a nilpotent matrix whose RREF is $T$. Using the familiar notation $\mathbf{e}_2 \equiv \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^t$ and $\mathbf{e}_3 \equiv \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t$, we write $\mathbf{u} \equiv \mathbf{e}_2$ and $\mathbf{v} = \mathbf{e}_3$. Then, if $a$ is a nonzero scalar, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$.

There is a unique linear transformation $H$ on $\mathbb{R}^3$ with the properties $H\mathbf{u} = \mathbf{v}$; $H\mathbf{v} = \mathbf{w}$; $H\mathbf{w} = 0$; we summarize these as follows:

$$\mathbf{u} \longrightarrow \mathbf{v} \longrightarrow \begin{pmatrix} -a & -b & 1 \end{pmatrix}^t \longrightarrow \mathbf{0}.$$

(This is not an exact sequence, and not even trying to be one.) Let's find $M$, the matrix representation of $H$ in the standard basis.

The second column of $M$ is the vector $M\mathbf{e}_2$, which is already prescribed: it is $\mathbf{e}_3$. The third column of $M$ is the vector $M\mathbf{e}_3$, prescribed as $\mathbf{w}$. What about the first column of $M$? It is $M\mathbf{e}_1$, but what's that? We can express $\mathbf{e}_1$ in the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ as follows:

$$(-a)\mathbf{e}_1 = \begin{pmatrix} -a \\ -b \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

which we can rewrite as $\mathbf{e}_1 = \frac{-1}{a}(\mathbf{w} - \mathbf{v} + b\mathbf{u})$. Thus

$$M\mathbf{e}_1 = \frac{-1}{a}(M\mathbf{w} - M\mathbf{v} + bM\mathbf{u}) = \frac{-1}{a}(\mathbf{0} - \mathbf{w} + b\mathbf{v}) = \begin{pmatrix} -1 & -\frac{b}{a} & \frac{1-b}{a} \end{pmatrix}^t,$$

which matches the first column of (2.3), and thereby reproduces (2.3).

This approach un-begs one question while begging another. The vector space basis procedure here is guaranteed to produce a nilpotent matrix, but how did we know that this matrix will have the requisite RREF, namely $T$? We know that $M$ and $T$ have the same null space: span$\{\mathbf{w}\}$. We now quote [5, Chapter 2, page 58]:

**Corollary** (Row Space–Row Equivalence)**.** *Let $A$ and $B$ be $m \times n$ matrices over the field $F$. Then $A$ and $B$ are row-equivalent if and only if they have the same row space.*

**Remark.** Everyone knows many famous theorems and some famous lemmas, but there is a dearth of famous (or just named) corollaries. In fact, the two best known to us are not mathematical, emanating from the Monroe Doctrine.[1] To address this dearth, I labeled the corollary above the "Row Space–Row Equivalence Corollary".

In our context, relating row equivalence to the null space is needed, and such a relation is implicit (though perhaps not salient) in the literature, e.g., [5] again, or [2, Theorem VFSLS: Vector Form of Solutions to Linear Systems]. In a recent paper [4], I stated the following as a corollary:

**Corollary.** *The null space of a matrix $M$ determines the RREF and the row space of $M$. Hence if two matrices of the same size have the same null space, they are row equivalent.*

Note that this is a corollary of our Row Space–Row Equivalence Corollary.

Thus, since the nilpotent matrix $M$ has the correct (right) null space, it has the correct RREF as well.

## 4. General Impressions

> Never trust to general impressions, ... but concentrate yourself upon details.
>
> –Sherlock Holmes
> in *A Case of Identity.*

> I have had no proof yet of the existence of this ...
>
> –Sherlock Holmes
> in *The Sign Of The Four.*

Here is the general result we were looking for:

[1] See https://en.wikipedia.org/wiki/Roosevelt_Corollary and https://en.wikipedia.org/wiki/Johnson_Doctrine.

**Theorem.** *Every singular matrix is row equivalent to a nilpotent matrix.*

*Proof.* Let $M$ be a singular $n \times n$ matrix, and take a basis of the (right) null space of $M$, $\{\mathbf{k}_1, \dots, \mathbf{k}_\ell\}$, where $\ell$ is the nullity of $M$. (Here and below, though we use the set notation $\{\cdots\}$, we are, in fact, working with *ordered sets* or *lists*.) As $M$ is singular, $\ell$ is greater than or equal to 1.

If $\ell = n$ then $M$ is the zero matrix, which is nilpotent, and we are done; so we assume $\ell$ is smaller than $n$. Extend $\{\mathbf{k}_1, \dots, \mathbf{k}_\ell\}$ to $\{\mathbf{z}_1, \dots \mathbf{z}_{n-\ell}, \mathbf{k}_1, \dots, \mathbf{k}_\ell\}$, a basis of the vector space of all $n \times 1$ columns. We consider a linear map that annihilates the basis vectors $\mathbf{k}_j$ and "shifts" each basis vector $\mathbf{z}_i$ to the next one, except for $\mathbf{z}_{n-\ell}$, which is shifted to $\mathbf{k}_1$.

This corresponds to a matrix $N$ with the following properties:

$$N\mathbf{z}_{i-1} = \mathbf{z}_i; \qquad N\mathbf{z}_{n-\ell} = \mathbf{k}_1,$$

for all suitable values of $i$. The construction of such a matrix is a standard extension of the process we used with the matrix $T$ in the previous section; see, for example, [6, Sections 4.1.1–4.1.2] for more details. Applying $N$ $n - \ell + 1$ times to a vector $\mathbf{v}$ shifts to zero the coordinates of that vector in the basis above, so the matrix $N$ is nilpotent of index $n - \ell + 1$, which is $\operatorname{rank}(M) + 1$, with null space spanned by $\{\mathbf{k}_1, \dots, \mathbf{k}_\ell\}$ (see, for example, [6, Section 3.5.1]). Thus $M$ and $N$ share a null space. Hence, by the companion corollary of the *Row Space–Row Equivalence Corollary*, they have the same RREF and are thereby row equivalent. □

**Remark.** We have, in fact, shown that a singular matrix of rank $r$ is row equivalent to a nilpotent matrix of index $r + 1$; alternative constructions can yield other indices. Of course, a nonsingular matrix is not row equivalent to a nilpotent matrix, since row reduction preserves nonsingularity, and every nilpotent matrix is singular.

## 5. What's It All About? The Aftermath

Nilpotency figures in the deepest moments of a first course in linear algebra [8]. It is particularly accessible to beginning students. Experience indicates that they latch onto to the subject, with curiosity and enthusiasm; ditto for RREF. Yet, in the literature, the two seldom interact. Why? The theorem above may give a clue. In fact, our discussion shows that it's not about nilpotency at all. It's about the null space.

## 6. Late Inspiration

In the course of the Spring 2020 academic-pandemic semester, this author developed the habit of staying up late and delving into the literature, mathematical, fictional, and non-fictional. Trying to compose Internet-lookup-resistant take-home final exam questions, he stumbled on the nilpotency-RREF pairing. At the same (late) time, he was reminded of the stories of Jorge Luis Borges, where mathematical ideas and constructs figure into detective stories, (see for example, [3], also see [9]). Could the nilpotency of a matrix serve a similar purpose? More generally, could a mathematical theorem give rise to a mystery story? (More generally still, is there a functor from the category of mathematics to the category of mystery stories?) That led to the present note.

Since the beginning of the pandemic, teachers have been asked to exercise particular understanding and accommodation with students. In the same vein one hopes that the reader will do similarly with the would-be *lockdown literato* responsible for this pandemic-produced essay.

*Acknowledgements*

The author thanks all who kindly provided helpful comments and suggestions; in particular, Mehmet Orhon, Géza Schay, Dennis Wortman, the referees, and the editors.

## References

[1] Beezer, Robert A., "Extended echelon form and four subspaces," *The American Mathematical Monthly*, Volume **121** Number 7 (2014), pages 644–647.

[2] Beezer, Robert A., *A First Course in Linear Algebra*, Open source book, 2016. Available at http://linear.ups.edu, last accessed on January 17, 2022.

[3] Borges, Jorge Luis, *The Aleph and Other Stories, 1933-1969*, edited and translated by Norman Thomas di Giovanni in collaboration with the author, E.P. Dutton, New York, 1970.

[4] Grinberg, Eric Liviu, *A gauche perspective on row reduced echelon form and its uniqueness*, forthcoming in *The American Mathematical Monthly*, 2022. Current version at arXiv/2005.06275.

[5] Hoffman, Kenneth M., Kunze, Ray Alden, *Linear Algebra*, 2nd edition, Prentice-Hall, Upper Saddle River, NJ, 1971.

[6] Schay, Géza, *A Concise Introduction to Linear Algebra*, 2nd edition, Birkhäuser, Boston, 2012.

[7] Strang, Gilbert, "The Core Ideas in Our Teaching," *Notices of the American Mathematical Society*, Volume **61** Number 10 (November 2014), pages 1243–1245. doi:10.1090/noti1174

[8] Trotter, Hale F., "A canonical basis for nilpotent transformations," *The American Mathematical Monthly*, Volume **68** Issue 8 (1961), pages 779–780.

[9] Zalcman, Lawrence, "Death and the calendar," *Hebrew University Studies in Literature and the Arts*, Volume **16** (1988), pages 97–112.

---

**Author bio:**

Eric L. Grinberg (ELG) began his study of linear algebra in the summer before his freshman year at Cornell, when Oscar S. Rothaus challenged him to read Halmos's *Finite Dimensional Vector Spaces* and face an oral exam. A reading course with R. Keith Dennis followed, using Hoffman and Kunze's *Linear Algebra*. ELG went on to write a thesis on Radon transforms in compact symmetric spaces, under the direction of Victor Guillemin and Shlomo Sternberg, from whom he continues to draw inspiration. His research interests are in analysis and geometry, especially in the context of group symmetry, with a focus on integral geometry. He taught at the University of Michigan, Temple University, Brooklyn Poly, the University of New Hampshire and, since 2010, at the University of Massachusetts Boston. In his career, he has served as associate dean, and as department chair with multiplicity.

---