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Conic Diagrams

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Synopsis

Textbooks may say that the so-called conic sections can be obtained from cones, but this is rarely proved. However, diagrams of the proof require no intuition for solids and can be read as flat. We construct the diagrams with ruler and compass and derive from them basic properties of conic sections as established by Apollonius of Perga, though again in a way that does not require a third dimension. The construction inevitably involves choices that give play to one's aesthetic sense.

1. Introduction

This article is about using ruler and compass to construct points on *conic sections.* These curves are usually approached analytically, with algebraic equations; our interest will be in *perceiving* them, thus perhaps even appreciating them aesthetically, the word *aesthetic* itself being derived from the Greek for perception.

The **conic sections** are three, called **ellipse, parabola,** and **hyperbola** respectively, for reasons not usually explained, although there are cognate literary terms *ellipsis, parable,* and *hyperbole.* By what I call the high-school definition, the three conic sections can be given respectively in a rectangular coordinate system by equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad y^2=4ax, \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{1}$$

The coordinate axes are also the **axes** of the hyperbola and ellipse, and the origin of coordinates is the **center** of each of these curves. The hyperbola and ellipse then are **central conics,** while the parabola has no center and one axis, the horizontal or x-axis.

Each point of intersection of a conic section with an axis can be counted as a **vertex** of the section. If (s,t) lies on the section, then the line segment between this point and $(s,0)$ is the associated **ordinate,** and the segment between $(s,0)$ and a vertex on the horizontal axis is an **abscissa.** That vertex is $(0,0)$ for the parabola, $(-a,0)$ or $(a,0)$ for a central conic. Thus a point on the parabola has one abscissa; on a central conic, two. In the ellipse, one can also make these definitions with respect to $(0,t)$ and the vertical axis.

It may be more usual to refer to the coordinates s and t themselves as abscissa and ordinate respectively. The reason for the present definition is that now the square on the ordinate varies as

- the abscissa, in the parabola;
- the product of the abscissas, in a central conic.

We shall use these *proportionality* conditions as an alternative definition of the conic sections. If we relax the condition that the ordinates be orthogonal to the abscissas, we obtain nothing new; the curves can still be given by the original equations in the appropriate rectangular coordinate system.

The equations (1) can be derived from a *focus* and a *directrix* in the coordinate plane; also from two foci, in the case of the central conics. However, our concern here is mainly how the conic sections arise in *solid* geometry.

If a point and a circle are given in Euclidean space, and the point does not lie in a plane of the circle, and an infinite straight line through the point intersects the circumference of the circle, and we move the point of intersection around the circumference, then the line traces out a **conic surface** whose **apex** is the original point. The conic surface has two parts or **nappes,** separated by the apex. The solid bounded by one nappe and the original circle is the **cone** with the same apex and whose **base** is the circle. A plane that does not contain the apex cuts the conic surface in a conic section. There are three possibilities.

1. If the conic section has two parts, one from each nappe, it is an hyperbola.
2. If it has one part, but is infinite, it is a parabola.
3. If it is finite, it is an ellipse.

We are going to make the connections between the proportionality conditions, the equations (1), and the solid geometry.

Let us then draw a circle having diameter BC as in Figure 1(a).

On this diameter, at a chosen point M, we erect a perpendicular, meeting the circle at D. We draw a segment VW that contains both M and a new point X, as in Figure 1(b), so that the lines BV and CW are not parallel, but intersect at a point A. From these data, we obtain the point P shown in Figure 1(c) by letting

1) AX meet BC at N,
2) J lie on the circle so that $NJ \parallel MD$,
3) P lie on AJ so that $XP \parallel MD$.

Using the additional lines shown in Figure 1(d), we shall show that XP is an ordinate, parallel to the ordinate MD, of an ellipse of which VW is called a **diameter,** because it bisects the chords that are parallel to ordinates. By letting X range along the diameter, we can obtain as many points P on the ellipse as we like.

We can have let the diameter VW lie in the plane of the original circle, so that everything else is in that plane. However, nothing has required this. If VW does not lie in the plane of the circle, then neither does A, and in this case, A serves as the apex of a cone whose base is the circle, and our ellipse is indeed a section of the corresponding conic surface. In this case, we have the option of letting MD be at right angles to VW and thus to the plane of VW and BC. In this case, VW is not just a diameter, but an axis of the ellipse.

We shall show how all of the foregoing agrees with the high-school definition of an ellipse. Meanwhile, let us note two variations of the construction.

- Of the segment VW that contains M and X, we can send the endpoint W off to infinity as in Figure 2, so that A still lies on BV, but now $CA \parallel VM$. Then P lies on a parabola.
- We let M lie on the extension of the segment VW, but not the segment itself, as in Figure 3. When X meets the same condition, then P traces out an hyperbola.

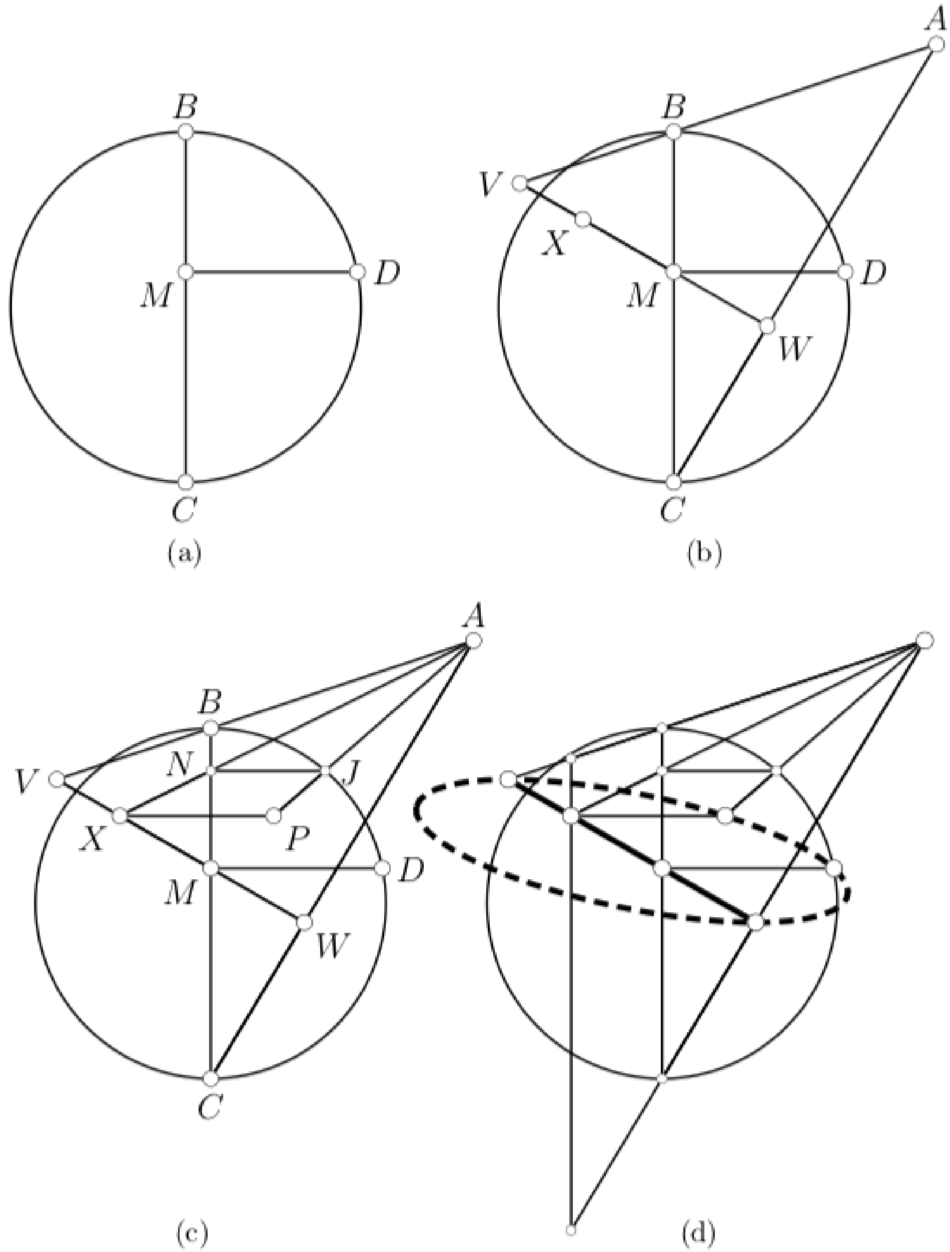

Figure 1: Construction of points of an ellipse

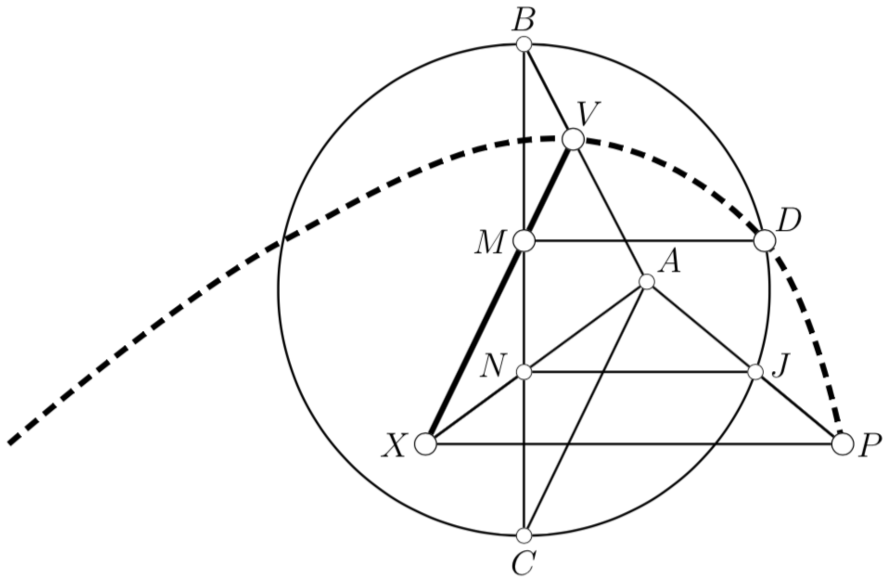

Figure 2: Points of a parabola

2. Aesthetics

This section is not required for what follows it.

Given just a diagram as in Figure 1(d), 2, or 3, one may treat it as a puzzle: *how* has it has been filled out from the points B, C, M, D, V, W, and X? One may also just contemplate the diagram as it is. I wonder then how well the diagram can satisfy the account of Kant in *The Critique of Judgement,* whereby

> *Beauty* is the form of *finality* in an object, so far as perceived in it *apart from the representation of an end.* [5, §17, page 80]

Beauty is finality without end — or in an alternative translation, purposiveness without purpose [4]. As Kant explains in a footnote, a tulip is beautiful, but utilitarian artefacts are not, even if we do not actually know their purpose; for the artefacts still obviously *have* a purpose.

Have our diagrams a purpose then? The artefacts that Kant mentions as having a purpose are "the stone implements frequently obtained from sepulchral tumuli and supplied with a hole, as if for [inserting] a handle." Presumably the location of the hole is chosen to serve a purpose.

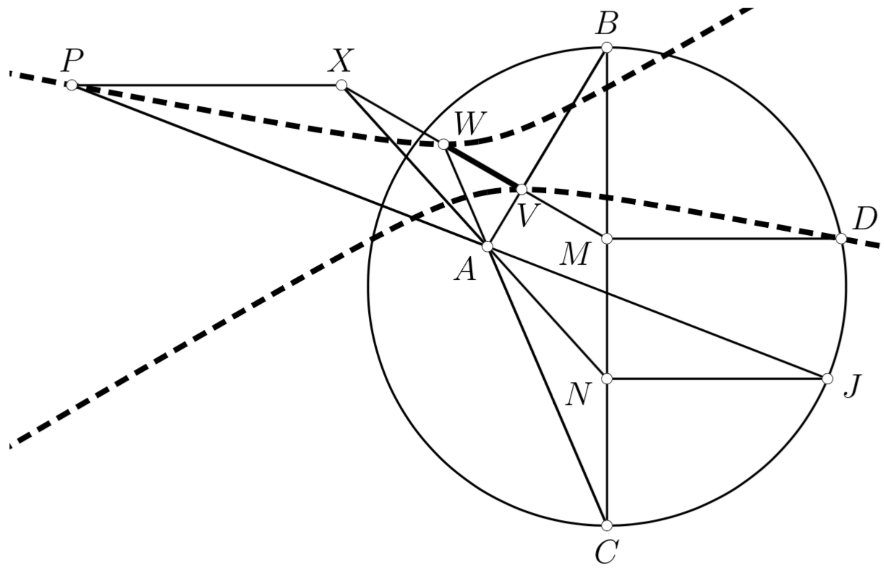

Figure 3: An hyperbola

Our diagrams will in fact serve a mathematical purpose. However, the choices

- in Figure 1(a) of the location of M on BC;
- in Figure 1(b) of the angle and the endpoints of the segment VW that contains M, and of the position of X on that segment

— these choices serve no purpose, except that the construction arising from them should please the eye. Still, part of that pleasure may come from following the mathematical argument to come.

Since reading selections from Kant's third *Critique* in college, I have remembered its example of a Sumatran pepper garden. The regularity of this garden ought to be irksome, Kant thinks, though it may first seem refreshing after time spent in the jungle. Kant takes the example from William Marsden, who says,

> A pepper garden cultivated in England would not . . . be considered as an object of extraordinary beauty, and would be particularly found fault with for its uniformity; yet, in Sumatra, I never entered one, after travelling many miles . . . through the woods, that I did not find myself affected with a strong sensation of pleasure. [6, page 113]

If one tries to consider a mathematical diagram as a work of art, how well does Kant's own criticism apply? He says generally,

> All stiff regularity (such as borders on mathematical regularity) is inherently repugnant to taste, in that the contemplation of it affords no lasting entertainment. Indeed, where it has neither cognition nor some definite practical end expressly in view, we get heartily tired of it. On the other hand, anything that gives the imagination scope for unstudied and final play is always fresh to us. [5, §22, page 88]

In Kant's terms, the diagrams of Figure 1 are going to "have cognition in view"; however, they do not yet "expressly" have this. They do exhibit regularity, but only in the sense that individual lines and circles are regular. These are regular, because produced mechanically, with ruler and compass. However, one may have no idea that there is a reason why those tools have used to produce the particular lines and circles of the diagrams.

3. My personal journey

Conic sections became a fascination for me in adolescence, as I started learning about curves and how they could be encoded in equations. Using graph paper and a pocket calculator, I plotted families of conic sections on the same axes, having looked up their equations in the analytic geometry textbook that my mother had used in college [7, pages 100–123]. About the historical origin of the parabola, ellipse, and hyperbola, the book says only,

> Because of the possibility of identifying these curves (including circles) with plane sections of right circular cones, they are called *conic sections.*

The book itself derives the equations of the conics from *planar* geometric definitions, whereby

- the points of the parabola are equidistant from a point called a *focus* and a line called a *directrix*;
- the points of the ellipse and hyperbola maintain, from two foci, distances whose sums and differences are respectively constant.

From these definitions, in a suitable rectangular coordinate system, the authers obtain the equations (1). Two years later, when I was in high school, the textbook used for algebra 2 gave the same geometric definitions and the same equations, but without fully proving the connection. The book stated in more detail how the conics came from cones, but still without proof [15, pages 397–399]. There followed the alternative characterization of a conic section as the locus of points whose distances from a focus and directrix bore a common ratio. Pappus had proved it, sixteen hundred years earlier [13, pages 494–503]; but for the readers of *Second Course in Algebra,*

> The technical difficulties are too great to allow us to make a general examination of the above definition.

Weeks and Adkins may have been correct not to undertake such an examination; but I did not like being told that it was too difficult for me.

The authors do take up a special case of the focus-directrix definition of a conic. When the focus is $(2, 0)$, and the directrix is given by $x = 8$, and the common ratio of distances from focus and directrix is $1/2$, the authors derive for the locus of points the equation $3x^2 + 4y^2 = 48$. Even though it is for a *second* course in algebra, the textbook does not then generalize, as by letting the focus be $(a, 0)$, the directrix be given by $(r^2 a, 0)$, and the common ratio be $1/r$, so that the locus is given by

$$(r^2 - 1)x^2 + r^2 y^2 = r^2(r^2 - 1)a^2.$$

The authors omit consideration of such a general equation, even though, by their own account, it would define an ellipse or hyperbola, depending on whether r^2 was greater or less than 1.

Weeks and Adkins must have known what they were doing. Evidently the abstraction of algebra has to be learned in stages. And yet the authors do give so-called

> Sidelights. These are, in essence, brief essays that introduce the reader to topics of a more advanced nature or provide somewhat deeper insights. [15, page *v*]

Deeper insight into the conic sections might not have been out of place. Moreover, a geometrical diagram is the opposite of abstract. It is right there on the paper, or (these days) on the screen.

4. Mathematics ancient and modern

A planar diagram is still challenging when it represents a solid object. One would have to face the challenge, if one wanted to learn the conic sections in the old days, when the diagrams and words of Apollonius of Perga were practically the only source about these sections.

We need not actually face the challenge of three-dimensional visualization, in order to establish the basic results of Apollonius. We shall use diagrams as in §1 to do this, even if we read them as planar. We can decline the option of reading the diagrams as parallel projections of solid figures.

We shall take advantage of what in some countries is taught as Thales's Theorem [9]; this is Proposition VI.2 of Euclid's *Elements* [3], that a line dividing two sides of a triangle divides them proportionally if and only if the line is parallel to the third side. This has the corollary that the ratio of two parallel segments is unaffected by parallel projection.

In our attempt at recovering something of the ancient understanding of conic sections, we may be instantiating the teaching of Diotima of Mantinea, as recalled by Socrates in the *Symposium* of Plato [12, 207E–8A]:

> not only do bits of knowledge come and go for us (we are never the same even in terms of our knowledge), but each single bit of knowledge also undergoes the same experience. What is called studying exists because knowledge goes away. Forgetting is the departure of knowledge, and study saves the knowledge by reimplanting a new memory in place of what has gone away, so that it seems to be the same knowledge.

Our diagrams are *like* those of Apollonius, but not the same. For one thing, ours have a precision that the diagrams of Apollonius himself must not have had.

Those diagrams had their own kind of precision, in the sense of showing the location of points that were undefined in the accompanying text. Reviel Netz has an example [8, page 23]: in Proposition I.11 of Apollonius [1, page 20], when, from a point K, a line KL is drawn, parallel to a line DE, readers are not told in words that L will be on a line FG; one must see this in the diagram.

Clarity can sometimes be achieved when straight lines are drawn curved.

When Archimedes inscribes a dodecagon in a circle, Netz remarks on how the sides of the polygon are not straight in some manuscripts, but curve inward [2, page 115]. Then he says,

> It is, I think, somewhat improbable that a scribe would invent such a practice, in defiance of his sources. If so, we may have in this practice a hint of Archimedes' own diagrammatic practices.

The diagrams of the present essay are drawn by computer. I happen to use `pstricks` and `pst-eucl` with LaTeX. In Figure 1, adjustments are possible in the choices of the ratio $BM : MC$, the position of A, the angle BMV, and the ratio $VX : VW$; but everything else is derived mechanically from these choices.

By the account of Eutocius [14, page 279], the Greeks first obtained a cone by revolving a right triangle about a leg; and they sectioned the cone only with a plane perpendicular to the hypotenuse of the triangle. This gives three possibilities, depending on whether the angle of the cone at its apex is acute, right, or obtuse. Only later did Apollonius show that a conic section of any of the three kinds could be obtained from any cone.

The cone need not even be right. It just has a circular base, and its apex can be any point not in the plane of the base. In a sense, we shall remove even this last condition, and what may be *seen* as the apex of a cone need only be a point not lying on the line of a selected diameter of the base of the cone.

5. Ellipse in Cartesian terms

From a point on the circumference of a circle of radius a, suppose we drop a perpendicular to a diameter. If the perpendicular has length y, and its foot is x from the midpoint of the diameter, then $y^2 = a^2 - x^2$ by the Pythagorean Theorem [3, I.47], and so

$$y^2 = (a + x)(a - x). \tag{2}$$

Let us call the perpendicular by the Latin term **ordinate.** The ordinates to the same diameter will all be parallel to one another, like columns in one of the ancient *orders* of architecture, the Doric, Ionic, and Corinthian [10].

An ordinate divides the diameter into two segments. For either of these, since it is something cut out, that is, excised — or rather cut off, that is, "abscised" — we use the Latin term **abscissa.** By equation (2) then, the square of the ordinate is the product of the abscissas.

We can weaken this condition. We shall still understand ordinates to be drawn to a given diameter, each ordinate cutting the diameter into two abscissas. Let us now require only that

- ordinates be parallel to one another;
- the proportion
  $$y^2 \propto (a+x)(a-x)$$
  be satisfied, so that the square of the ordinate will vary jointly as the abscissas.

Each ordinate still has one endpoint on the diameter. The other endpoints now trace out an **ellipse.**

Thus, if a segment VW is given as in Figure 4, and a point M on the segment, and a point D not on the segment, then that curve is an ellipse which is the locus of points P such that, when X is the point on VW such that

$$XP \parallel MD, \tag{3}$$

then

$$(XP : MD)^2 :: (VX : VM)(XW : MW). \tag{4}$$

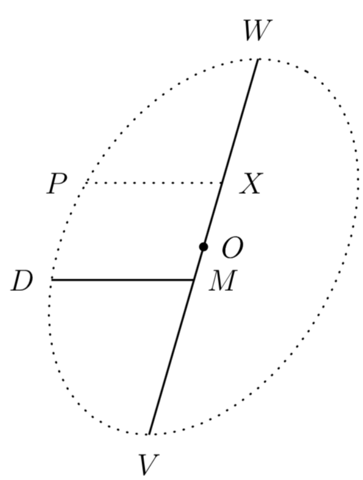

Figure 4: Ellipse determined by diameter and an ordinate

Note that an expression like XP has three or four related meanings:

- the infinite line through X and P;
- the segment of that line bounded by X and P;
- the directed segment from X to P, or the vector represented by this directed segment.

Context should make it clear what is meant. For example, ratios as in (4) are always ratios of parallel vectors. There are no ratios of nonparallel vectors. There could be ratios of the *magnitudes* of nonparallel vectors; but for now, our vectors need not have magnitudes, and our points need lie only in an affine plane.

We let O be the midpoint of VW, this being understood as a segment. We define

$$OV = \boldsymbol{a}, \qquad OM = \lambda \cdot \boldsymbol{a}, \qquad OX = s \cdot \boldsymbol{a},$$

and

$$MD = \boldsymbol{c}, \qquad XP = u \cdot \boldsymbol{c},$$

these five entities all being vectors, three of them analyzed as product of scalar and vector. In particular, condition (3) is now automatically satisfied, and

$$\begin{aligned} XP : MD &:: (u \cdot \boldsymbol{c}) : \boldsymbol{c} \\ &:: u, \end{aligned}$$

while

$$\begin{aligned} VX : VM &:: (VO + OX) : (VO + OM) \\ &:: (-\boldsymbol{a} + s \cdot \boldsymbol{a}) : (-\boldsymbol{a} + \lambda \cdot \boldsymbol{a}) \\ &:: \frac{1-s}{1-\lambda} \end{aligned}$$

and

$$XW : MW :: \frac{1+s}{1+\lambda},$$

so that (4) is equivalent to

$$u^2 = \frac{1-s^2}{1-\lambda^2}. \tag{5}$$

We can simplify this by defining $\boldsymbol{b}$ and t so that

$$\frac{1}{\sqrt{1-\lambda^2}} \cdot \boldsymbol{c} = \boldsymbol{b}, \qquad XP = t \cdot \boldsymbol{b}.$$

Then $t/\sqrt{1-\lambda^2} = u$, so that (5) becomes

$$t^2 = 1 - s^2. \tag{6}$$

Also, passing to matrix notation, we have

$$OP = OX + XP = s \cdot \boldsymbol{a} + t \cdot \boldsymbol{b} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}. \tag{7}$$

We now introduce a Euclidean structure to our affine plane. In a rectangular coordinate system whose origin is O, letting

$$P = (x, y), \qquad \boldsymbol{a} = (a_1, a_2), \qquad \boldsymbol{b} = (b_1, b_2),$$

from (7) we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}$$

and therefore

$$\begin{pmatrix} s \\ t \end{pmatrix} = \frac{1}{a_1 b_2 - b_1 a_2} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Plugging this into (6), which we rearrange as $s^2 + t^2 = 1$, and then scaling, we obtain for the ellipse the defining equation

$$(b_2 x - b_1 y)^2 + (a_2 x - a_1 y)^2 = (a_1 b_2 - b_1 a_2)^2. \tag{8}$$

One can then obtain the high-school equation of an ellipse as in (1) by rotating the coordinate system so as to eliminate the xy terms. Alternatively, on can just confirm that this is possible by finding $\boldsymbol{a}'$ and $\boldsymbol{b}'$ such that the equation

$$(b_2 x - b_1 y)^2 + (a_2 x - a_1 y)^2 = (b_2' x - b_1' y)^2 + (a_2' x - a_1' y)^2,$$

which shares its left member with (8), is an identity, but also $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are orthogonal, meaning

$$a_1 b_1 + a_2 b_2 = 0.$$

This is worked out in [11]. Cartesian notation makes it possible. Now we want just to *see* what is happening.

6. Ellipse geometrically

With ruler and compass, we shall *construct* PX parallel to MD as in Figure 4, given the foot X on VW.

Any such X corresponds to two points P on the ellipse, and those two points bound a chord whose midpoint is X. For this reason, VW is a a **diameter** of the ellipse, because it "measures through." Any chord passing through the center of one diameter will be another diameter, in this sense of bisecting each in a family of parallel chords. One can prove this by completing the squares differently in (8), although Apollonius has a geometric method. Again, the details are in [11].

Given V, W, M, D, and X as in Figure 4, we construct P to satisfy (3) and (4) by the following steps.

1. Construct right angle DMB as in Figure 5(a), letting MB have any convenient length for what is to come.

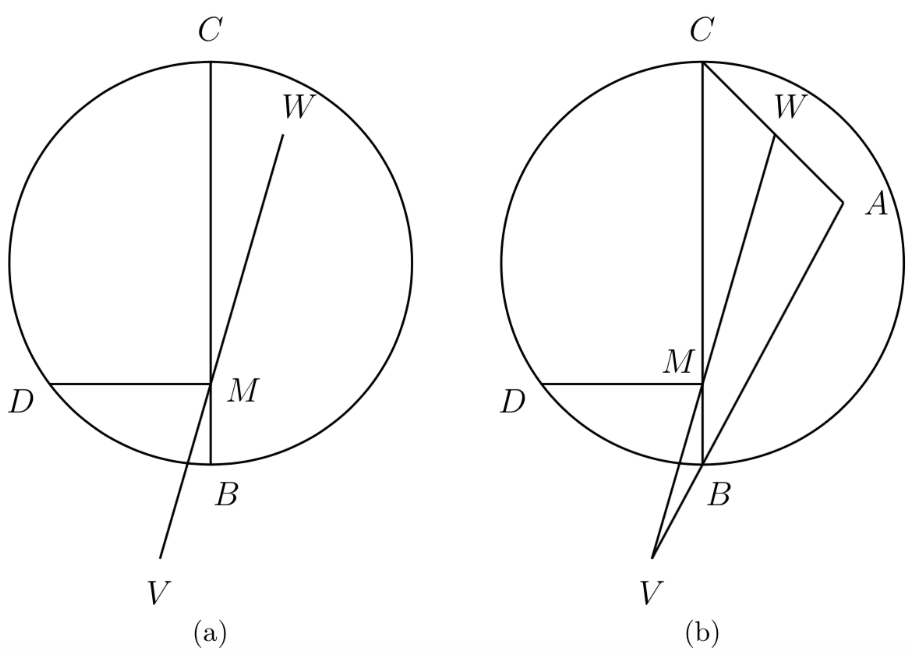

Figure 5: Construction of points of an ellipse

2. Extend BM to C so that angle BDC is right. Then the circle with diameter BC passes through D.
3. We should have chosen B so that VB and WC intersect at a convenient point A, as in Figure 5(b).
4. For an arbitrary point X on VW, as in Figure 6(a), we let
   - AX intersect BC at N,
   - the line through N that is parallel to MD meet the circle with diameter BC at J.
5. Finally, we obtain P on AJ as in Figure 6(b), so that XP is parallel to MD.

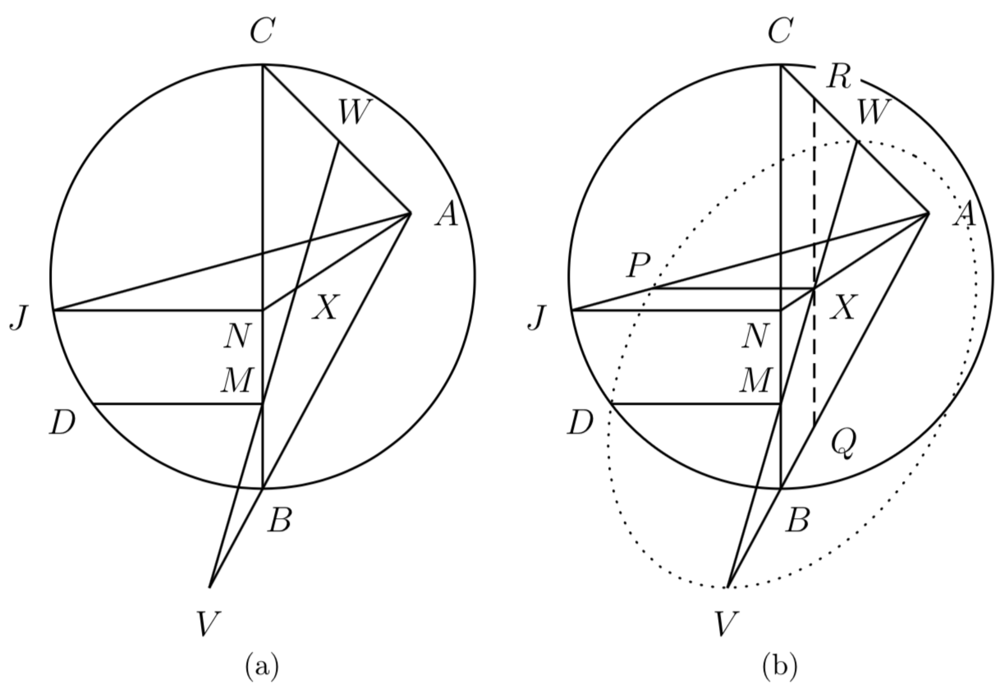

Figure 6: Construction completed

For the *proof* that (4) now holds, through X we draw the line parallel to BC, meeting AB at Q and AC at R. By Thales's Theorem,

$$XP : NJ :: XA : NA,$$

and likewise

$$QX : BN :: XA : NA :: XR : NC.$$

Since we can make the factorization

$$XP : MD :: (XP : NJ)(NJ : MD),$$

from our first two proportions we have

$$\begin{aligned}(XP : MD)^2 &:: (XP : NJ)^2(NJ : MD)^2\\ &:: (XA : NA)^2(NJ : MD)^2\\ &:: (QX : BN)(XR : NC)(NJ : MD)^2.\end{aligned}$$

Since NJ and MD are ordinates of a *circle,* the proportion corresponding to (4) does hold, that is,

$$(NJ : MD)^2 :: (BN : BM)(NC : MC).$$

Plugging this into the previous proportion, reducing, and applying Thales again yields

$$\begin{aligned}(XP : MD)^2 &:: (QX : BM)(XR : MC)\\ &:: (VX : VM)(WX : WM),\end{aligned} \tag{9}$$

which is (4), as desired.

7. Conic sections as such

In the construction of the previous section, M was on the line determined by V and W, but nothing required it to lie *between* those points. If it does not so lie, we get a diagram as in Figure 7, and thus an **hyperbola.**

If instead we push W off to infinity, then condition (4) becomes

$$(XP : MD)^2 :: VX : VM, \tag{10}$$

defining a **parabola.** The construction of P is as before, except that, in step 3, the point A lies on VB so that AC is parallel to VM, as shown in Figure 8. The proof of (10) is as for (4), except that, in the last step, at (9), $XR : MC$ is now a ratio of equals, so it just drops out.

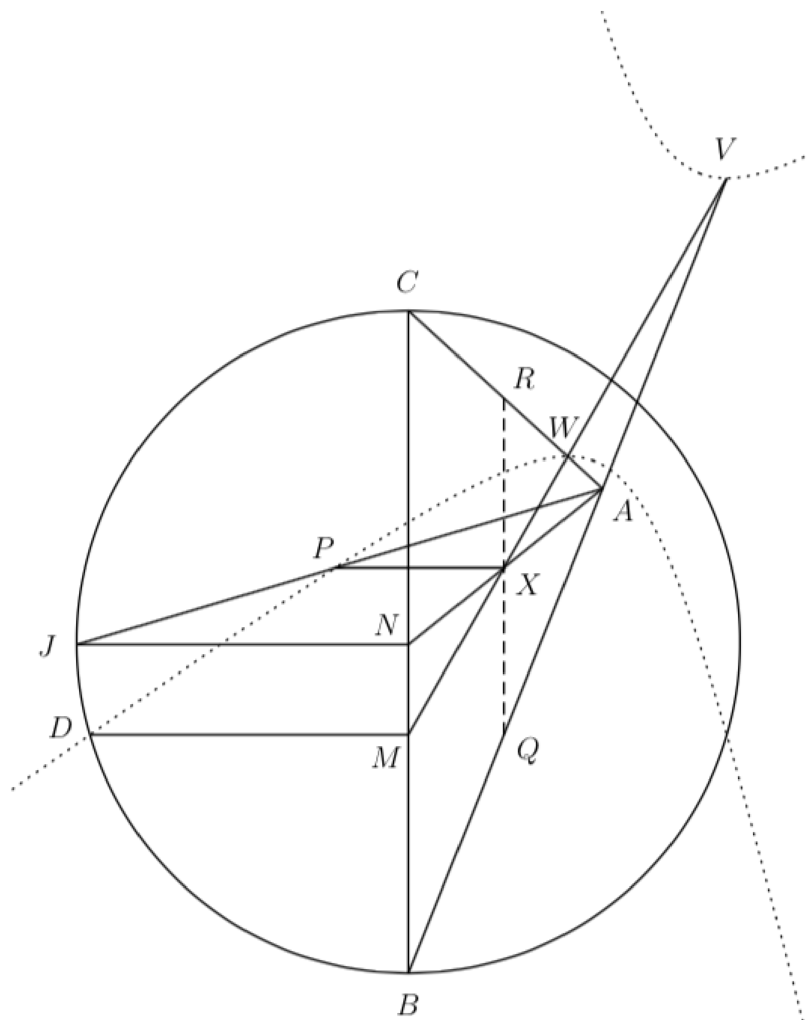

Figure 7: Point of hyperbola

Apollonius is concerned, not just with proportions, but with what we call constants of proportionality. In particular, he cares about the height of the rectangle on the abscissa (or one of the abscissas) that is equal to the square on the ordinate. As the abscissa grows, the height of the rectangle may decrease, grow, or stay the same, and in Greek this is reflected in the terms ellipse, hyperbola, and parabola [10].

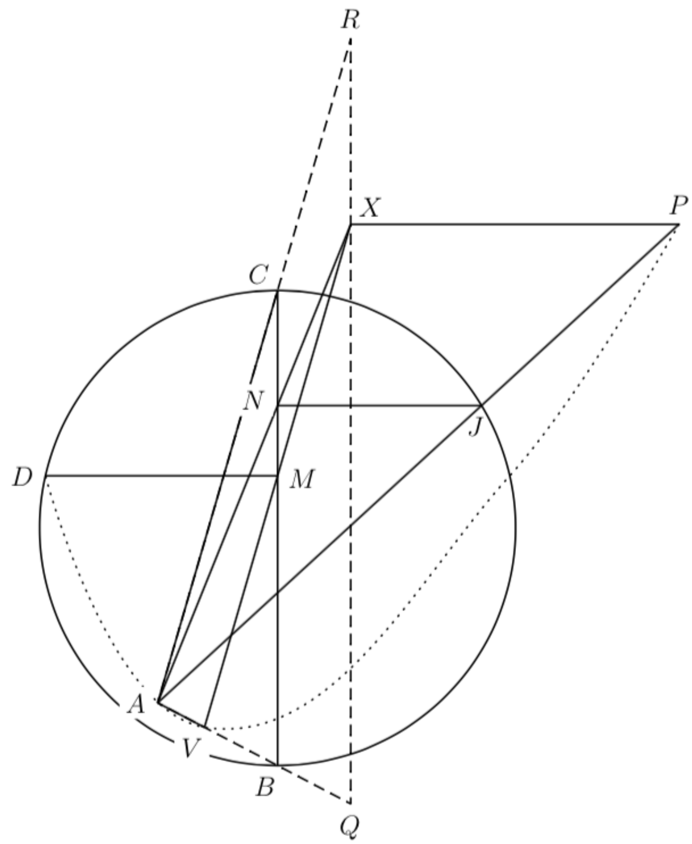

Figure 8: Point of parabola

We have now obtained the three so-called conic sections by means of elementary plane geometry and the algebra of such proportions as arise from Thales's Theorem. Our construction of the point P has not actually required B and C to lie in the plane of V, W, and D. If they do not lie in that plane, then A becomes the apex of a cone whose base is the circle having diameter BC, and the point P lies on a section of the surface of the cone.

If we wish, we can now let our diagrams be parallel projections of the solid situation. Of the circle with diameter BC, the only property that we use is the one that we have now shown to be shared with an arbitrary conic section. Besides D, B, and C, the only point on the circumference of the circle that we need is J. We now know how to construct this point, even if we have replaced the circle with an arbitrary conic section.

Thus, with ruler and compass, we can construct all of the points in a diagram as in Figure 9, where BC is a diameter, and MD an ordinate, of an ellipse in the plane of the diagram.

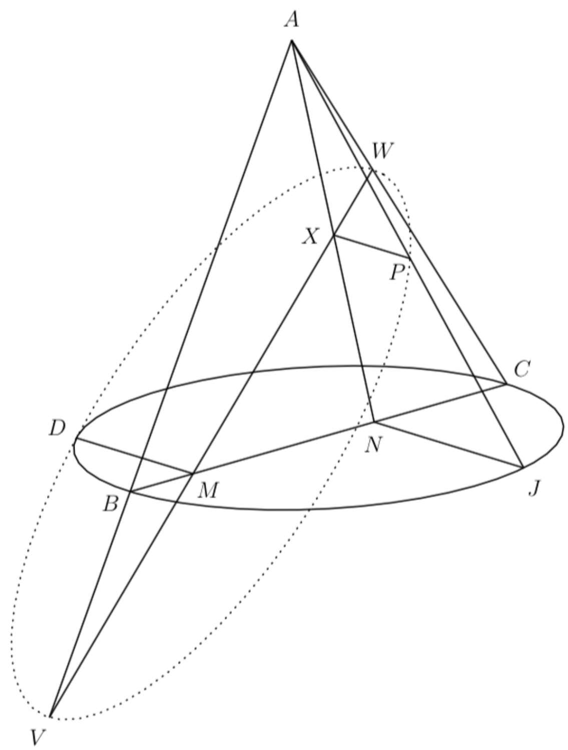

Figure 9: Circular cone in space

We can now read this diagram as an illustration of how to obtain one conic section from another by ruler and compass, or even by two parallel rulers. We may also use three-dimensional intuition to understand the diagram as a demonstration of how to obtain conic sections *as such*; but this intuition is not required.

One also has the option of ignoring one's knowledge and seeing such a diagram (or another one, constructed from one's own chosen proportions) as decoration, if not art.

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