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The Genesis of a Theorem

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Synopsis

We present the story of a theorem's conception and birth. The tale begins with the circumstances in which the idea sprouted; then is the question's origin; next comes the preliminary investigation, which led to the conjecture and the proof; finally, we state the theorem. Our discussion is accessible to anyone who knows mathematical induction. Therefore, this material can be used for instruction in a variety of courses. In particular, this story may be used in undergraduate courses as an example of how mathematicians do research. As a bonus, the proof by induction is not of the simplest kind, because it includes some preliminary work that facilitates the proof; therefore, the theorem can also serve as a nice exercise in induction. Additionally, we use well-known facts from calculus to clarify and enhance what is intrinsically a discrete problem. Making an unexpected but welcome explanatory appearance, the number e is pertinent.

Keywords: mathematical induction, mathematical research, number theory, undergraduate instruction

1. Introduction

At times, when talking with students, I get the impression that some students believe that all of mathematics is already known. It is as if all of mathematics were already available on some tablets that came down the mountain with Moses, and, therefore, since then, there is no need to create any new mathematics. Thus, some students are surprised when I repeat what has been said many times before, namely, that more mathematics has been created since 1900 than had been created before.

It is probably safe to assume that anyone earning an undergraduate degree in mathematics during our time has received instruction in how to read and do mathematical proofs. However, it is also likely that many undergraduates do not receive training in how mathematical theorems actually arise; the students see the theorem already stated, but there is no clue as to how the theorem originated. This is largely understandable because mathematical research generally requires the kind of background that most undergraduates simply do not have. Fortunately, we see a trend toward incorporating mathematical research in the undergraduate mathematics curriculum, so that students can experience the joy of discovering new mathematics, and then writing the proofs that go along with their discoveries.

Finding research problems for undergraduates is not easy; the problems must be easy to understand and amenable to at least a partial solution after a reasonable effort. Undergraduate mathematics students are often accustomed to finishing a homework exercise within at most a few minutes, and so they can easily get discouraged when they cannot solve a problem that requires more time and thought. We should remind students that a number of mathematical problems, such as Fermat's Last Theorem, are passed from generation to generation, hoping that eventually a solution will be found. We should also tell and remind students that the abstract mathematics that may appear to be nothing more than a mental exercise can, in fact, be quite useful; the finite fields that we study in abstract algebra are essential for the applications of coding theory and cryptology that currently permeate and facilitate our daily lives.

In this paper I present a result that can be used for undergraduate instruction in mathematical research. I describe the complete genesis of the result, which is accessible to undergraduates. Students can see where and how the theorem originated, from beginning to end.

2. Genesis

Mathematics research—in fact, all research—begins with a question. So, it is important for students to see examples of how such a question can arise. One example follows.

It used to be that a year of algebra—groups, rings, and fields—was required for an undergraduate mathematics degree. However, for some time now, the requirement is just one semester, typically devoted to mostly groups. When I teach the required one-semester algebra course, I include some material on fields, and I discuss rings as time permits. I also discuss fields in other courses as I can. Many of my students are electrical-engineering majors, who can eventually become interested in coding theory and cryptology, where finite fields are essential. Indeed, at conferences over the years, I have met many electrical engineers who are well versed in algebra, and, especially, in finite fields. So, I always discuss finite fields in my algebra classes and other courses as time allows.

In my classes, I keep my discussion of finite fields as concrete and accessible as possible. Therefore, I usually discuss in detail the fields with 8 and 9 elements. That way, the students can easily and clearly see all the pertinent salient properties, such as the fact that the multiplicative group of nonzero elements in a finite field is a cyclic group.

I did not pay close attention to this the first time I taught the material about fields with $8 = 2^3$ and with $9 = 3^2$ elements, but it so happens that $2^3 + 1 = 3^2$, which I considered a lovely equation. I was then curious: When is it true that $n^{n+1} + 1 = (n+1)^n$ for positive integers? This is it! This is the question that generates the research! It is important for students to realize that the motivating question need not become apparent right away!

3. The Preliminary Investigation

To gain some initial insight, we do what mathematicians generally do, and that is, we examine some particular situations or we do some computing, which, in this case, is natural. Number-theoretic questions naturally lend themselves well to preliminary investigation by computing. A little computing produced the results we show in Table 1. These data suggest that, for natural numbers n, we have $n^{n+1} + 1 = (n+1)^n$ if and only if $n = 1, 2$; moreover, it appears that $n^{n+1} > (n+1)^n$ whenever $n \geq 3$. So, now we have, at last, a conjecture to work on, and we proceed by induction.

Table 1: Values of the sequences $\left(n^{n+1}+1\right)$ *and* $((n+1)^n)$ *for* $n=1,\ldots,10$

n	$n^{n+1}+1$	$(n+1)^n$
1	2	2
2	9	9
3	82	64
4	1,025	625
5	15,626	7,776
6	279,937	117,649
7	5,764,802	2,097,152
8	134,217,729	43,046,721
9	3,486,784,402	1,000,000,000
10	100,000,000,001	25,937,424,601

The first step in the induction proof is already accomplished by examining the data in Table 1, that is, the statement is true when $n=3$. The remainder of a proof by induction directly was not fruitful; roughly, having exponents on both sides of the inequality was a nuisance. So, to reduce the number of exponents, we notice that

$$\begin{aligned} n^{n+1} > (n+1)^n &\iff \frac{n^{n+1}}{n^n} > \frac{(n+1)^n}{n^n} \\ &\iff n > \left(\frac{n+1}{n}\right)^n \\ &\iff n > \left(1+\frac{1}{n}\right)^n, \end{aligned}$$

which is what we proceed to prove.

The induction assumption is that for some natural number $k>3$, we have

$$k > \left(1+\frac{1}{k}\right)^k.$$

We now observe that, for such $k>3$,

$$\begin{aligned} \left(1+\frac{1}{k+1}\right)^{k+1} &< \left(1+\frac{1}{k}\right)^{k+1} \\ &= \left(1+\frac{1}{k}\right)^k \left(1+\frac{1}{k}\right) \end{aligned}$$

$$< k\left(1+\frac{1}{k}\right), \text{ by the induction assumption}$$
$$= k+1,$$

which completes the proof by induction.

Thus, we have proven the following result.

Theorem 1. *If n is a natural number, then $n^{n+1}+1=(n+1)^n$ if and only if $n=1,2$. Moreover, $n^{n+1}>(n+1)^n$ whenever $n\geq 3$.*

4. Insights From Calculus

Mathematical induction is a venerable, widely used tool that is available and present in every mathematician's toolbox. However, after a successful induction application, we are often left partially hungry and craving for knowledge that will help us unravel why the result is actually true; generally, induction does not satisfactorily reveal the reasons why a result is valid.

In this section, let us recollect familiar results from calculus to elucidate Theorem 1; that is, we wish to have a better understanding of why the inequality

$$n > \left(1+\frac{1}{n}\right)^n$$

is true for each natural number $n \geq 3$.

The sequence $\left((1+\frac{1}{n})^n\right)_{n=1}^{\infty}$ excites every mathematician because this sequence is a favorite, traditional example to show that the rational numbers are not complete; that is, it is not true that each convergent sequence of rational numbers converges to a rational number.

For our purposes, we recall the following well-known facts from calculus.

- The sequence $(n)_{n=1}^{\infty}$ is strictly monotonically increasing and not bounded above, so that $\lim_{n\to\infty} n = \infty$.
- The sequence $\left((1+\frac{1}{n})^n\right)_{n=1}^{\infty}$ is strictly monotonically increasing, bounded above by e, and

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

so that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sup\left(\left\{\left(1+\frac{1}{n}\right)^n : n \in \mathbb{N}\right\}\right) = e < 3,$$

where, as usual, $\mathbb{N} := \{1, 2, 3, \ldots\}$ is the set of natural numbers. Consequently,

$$\left(1+\frac{1}{n}\right)^n < 3$$

for all natural numbers $n = 1, 2, 3, \ldots$.

Thus, we now have a better understanding of the fact that

$$n > \left(1+\frac{1}{n}\right)^n$$

for all integers $n \geq 3$.

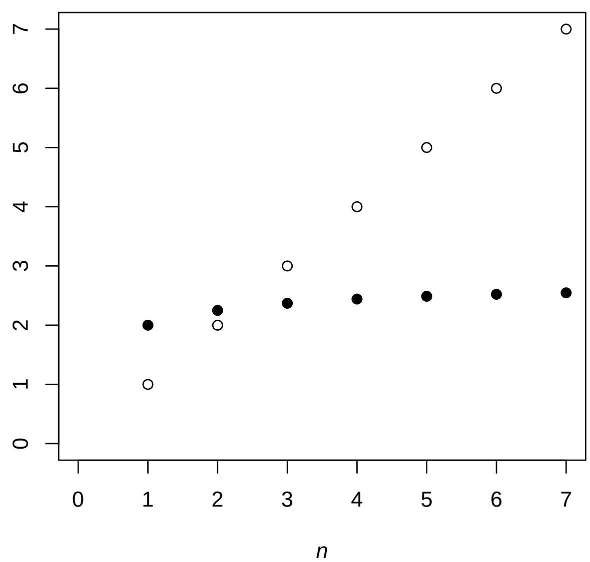

Figure 1: The first seven values of the sequences $(n)_{n=1}^{\infty}$ (symbol: $\circ$) and $((1+1/n)^n)_{n=1}^{\infty}$ (symbol: $\bullet$).

For the first seven values of n, Figure 1 illustrates the relationship between the sequences $(n)_{n=1}^{\infty}$ and $\left((1+\frac{1}{n})^n\right)_{n=1}^{\infty}$; that figure helps to clarify why Theorem 1 is valid.

5. Conclusion

Of course, Theorem 1 can be presented to students without any of the background we have mentioned. In fact, that theorem can be be used as a nice exercise to illustrate a nontrivial application of induction, when some preliminary work facilitates the proof by induction. However, leaving out the motivating background is likely to leave the students hungry for knowing how the result came about, and then some students may perhaps think that this is just another piece of busywork.

In my classes, students invariably prefer knowing about the motivational background, which they appreciate, instead of just being presented the theorem by itself. We invite teachers to present their students the theorem by itself, perhaps as an exercise; and, otherwise, include all the motivating background. Then assess and compare the students's reactions.

Our discussion shows that, for mathematical-induction problems, it is beneficial to further explore the matter at hand. As happens here, such efforts can profitably uncover relationships between topics that initially may appear disparate. We began with a topic that is intrinsically discrete and then we were able to illuminate our theorem with results from calculus. In mathematics everything seems to be related to everything else; this interrelatedness is one more attractive feature that makes mathematics exciting!

For a mathematician, the initial inspirational moment, and then the chase that eventually leads to results, are joys and thrills that are difficult to communicate to others. We can, however, show our students examples of how such exciting moments occur. For some students, this can change mathematics from a boring subject where everything seems to be already known, to a stimulating, very much alive and permanently growing human activity. The example we present in this article is useful for instruction in a variety of courses.