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The Number Systems Tower

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Synopsis

For high school and college instructors and students, this paper connects number systems, field axioms, and polynomials. It also considers other properties such as cardinality, density, subset, and superset relationships. Additional aspects of this paper include gains and losses through sequences of number systems. The paper ends with a great number of activities for classroom use.

1. Pre-Introduction

Zero is the most interesting number—nothing can be divided by it!

This paper considers the intersection of number systems, field axioms, and polynomials with additional dimensions such as cardinality, density, and subset and superset relationships. This material is intended for either high school or college mathematics instructors or their respective students. While some of the material in this paper may be well known by some instructors and college math majors, few universities have in place a course or a series of courses which create the *gestalt* which we intend in this paper. Additionally, even some college math majors may may not recognize some of the number systems mentioned herein, their connections to polynomials, or the transfinite cardinality $2^{\aleph_0}$, among many other ideas in this paper.

The flow of this paper follows the path: Culminating Ideas (§2); Introduction (§3), providing the motivation for the paper; Field Axioms (§4); Polynomials (§5); Cardinality of Sets (§6); Number Systems (§7); Connecting Number Systems through Subsets, Supersets, and Cardinality (§8); Other Interesting Systems (§9); Other Aspects of Gains and Losses (§10); and Activities (§11). Notably, the authors make a heavy use of footnotes as well as little packets of fun facts distributed throughout the paper in small boxes in order to provide the reader some options regarding how to interact with the paper. Some readers may streamline their reading of the paper and only examine the footnotes necessary to complete ideas. This is appropriate for those who do not need the minutia provided in the footnotes. Other readers may carefully examine every footnote to gain deeper understanding of the relative concepts.

The readers are encouraged to look carefully at the extensive list of activities described at the end of this paper in §11 and in the article supplements at http://scholarship.claremont.edu/jhm/vol13/iss2/22. All of these activities have been used through multiple semesters at the authors' university for senior and graduate level mathematics and mathematics education majors. Many of these activities can be used in numerous K-16 classes. These activities bring more meaning to the discussion at hand.

2. Culminating Ideas

$$\boxed{\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4},\ \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5},\ \frac{4\cancel{9}}{\cancel{9}8} = \frac{4}{8},\text{ and } \frac{1\cancel{6},\cancel{6}\cancel{6}\cancel{6}}{\cancel{6}\cancel{6},\cancel{6}\cancel{6}4} = \frac{1}{4}}$$

We have chosen the unusual tack of beginning a paper with its culminating ideas. Figure 1, The Number Systems Maze, depicts subset and superset relationships among various number systems which can be seen in K-16 mathematics and beyond.

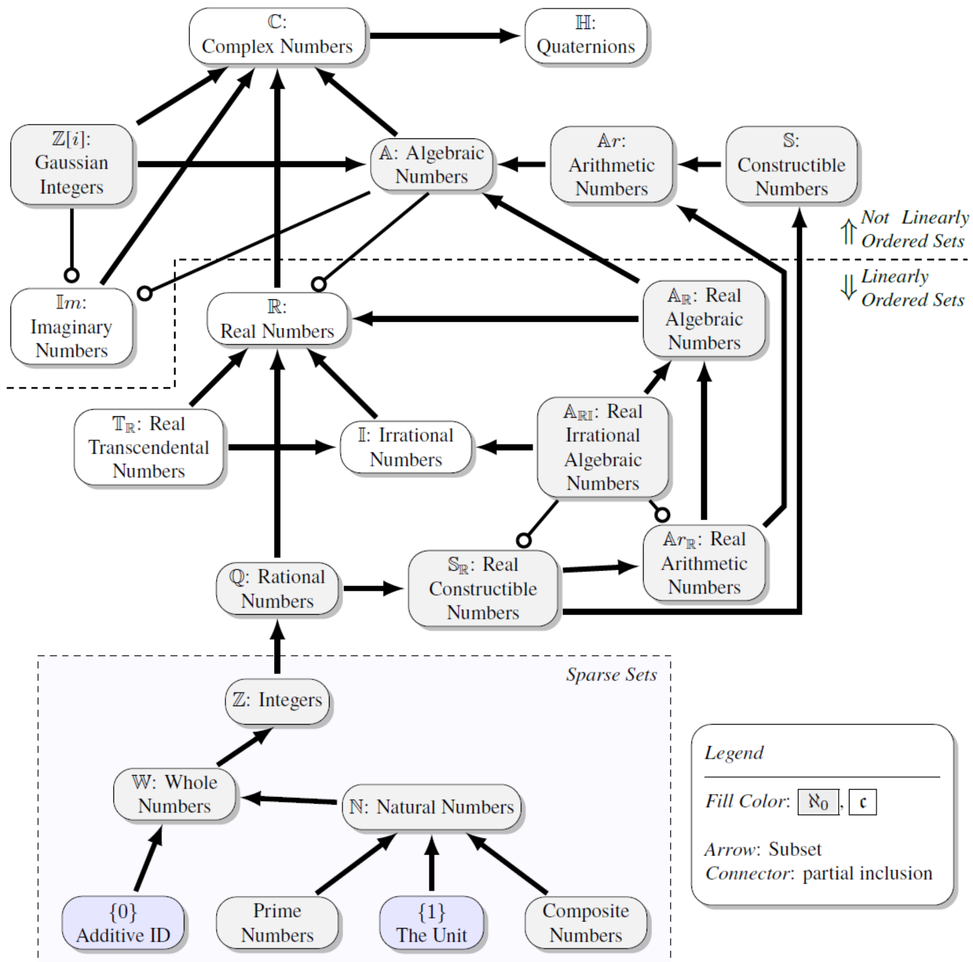

Figure 1: The Number Systems Maze.

Additionally, as seen in the legend for this diagram, this figure also captures whether a particular number system is relatively sparse or dense, ordered or not ordered, and has a cardinality of $\aleph_0$ or $2^{\aleph_0} = \mathfrak{c}$. An accompanying figure is provided as a standalone figure through the link:

http://appstate.edu/∼bossemj/NumberSystems/ImageMap/ImageMap.html

This interactive figure provides mouseover capability, where definitions appear when the mouse runs over the name of the respective number system. and it can be used by instructors and students to further investigate ideas in conjunction with, or independent of, this article.

3. Introduction

$$\boxed{\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \frac{1+3+5+7}{9+11+13+15} = \ldots}$$

Three realms of mathematics are inextricably linked through K-16 mathematics: number systems, axiomatic field theory[1], and solutions to equations. For example, attempt to solve $x + 3 = 1$ in the natural numbers. You are correct; it cannot be accomplished, since negative integers are needed.

To provide a brief motivational platform for this paper, we begin by considering the equation $x^2 - 4 = 0$ and its solutions; see Figure 2 below. Through changes in the constant or the operation involved, we obtain new equations whose solution domains change accordingly. In other words, with each change of constant or operation in the figure, we are thrust into a new number system in which the roots reside. While all the resulting roots are complex numbers, students would most often see ± 2 as integers, $\pm\sqrt{6}$ as irrational, $\pm 2i$ as pure imaginary, and $\pm i\sqrt{6}$, which can be written as $0 \pm i\sqrt{6}$, as complex numbers.

From these two scenarios above ($x + 3 = 1$ and $x^2 - 4 = 0$), we can see that number systems are connected to roots of polynomials. However, the development of number systems is also connected to field axioms. For instance, the set $\mathbb{N}$ of natural numbers is not closed under subtraction or division, and it does not contain an additive identity; inverse elements for addition and multiplication do not always exist in $\mathbb{N}$, and $\mathbb{N}$ has a cardinality of $\aleph_0$ (countable infinity). By contrast, the set $\mathbb{R}$ of real numbers satisfies all of these properties and has a cardinality of $\mathfrak{c}$ (the continuum). Thus, number systems are also connected to axiomatic field theory.

Altogether, Figure 3 depicts our approach in this paper that the three realms of number systems, polynomials, and axiomatic field theory are inextricably

[1]Even though all recognize properties such as the commutative and associative properties, many students may not be familiar with the term field axioms or field properties. When discussed in this investigation, all readers will remember these properties and recognize that they have been used in part since the early elementary grades.

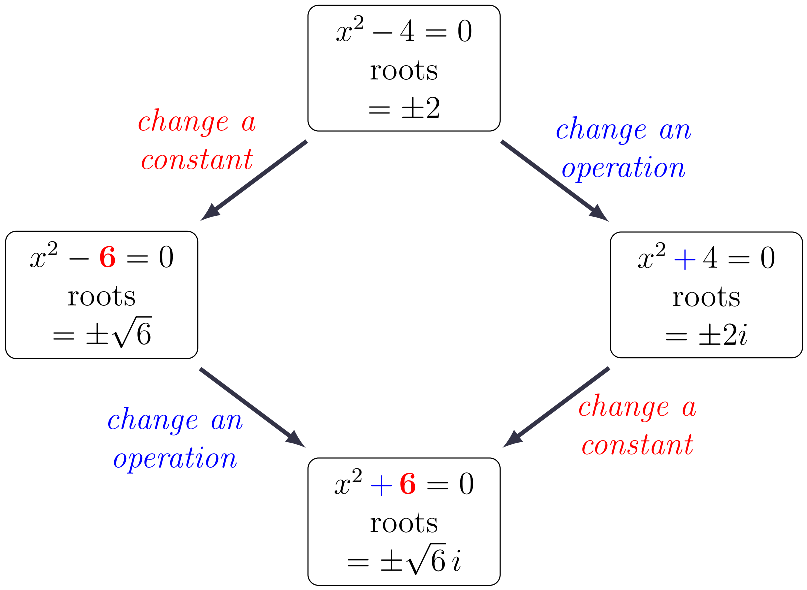

Figure 2: Altering the solution domain.

interconnected. So, how do we attack this intersection of ideas? Indeed, we can focus on any one to develop the other. For instance, it would be as convenient to develop understanding of number systems by considering which number systems are closed with respect to which operations, or equivalently,

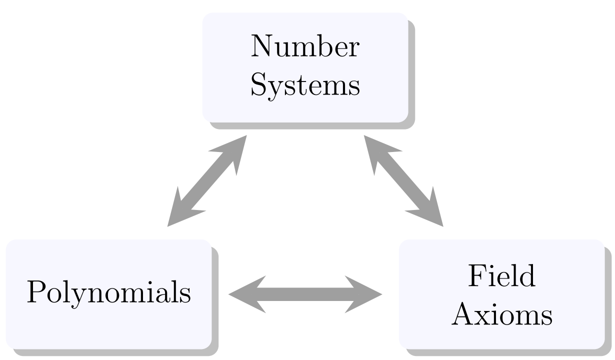

Figure 3: The three realms.

which particular field axioms are satisfied in which number system.[2] Or the discussion could be focused on the order in which number systems and field axiom properties are encountered through K-12 education.

Reviewing all our options, we have decided to provide a development of number systems and connect them to polynomials and field axioms. Notably, the number systems provided in this paper span K-16 mathematics and beyond.[3] To accomplish our goals, we first provide an overview of axiomatic field theory as a reminder to the readers.

4. Field Axioms: A Quick Introduction to Axiomatic Field Theory

In ancient Egyptian mathematics, $\frac{2}{3}$ is the only non-unit fraction not represented as a sum of unit fractions.

Prior to a formal exposition of field axioms, we demonstrate with a few examples that some are considered as early as elementary school. For instance,

- As early as first or second grade, students investigate that $3+5=5+3$ (ithe commutative property).
- Around second grade, students learn that $5+0=5$ and $0+3=3$ (the additive identity).
- This is usually soon followed by exploration of the associate property of addition and possibly multiplication. This may then be followed by the distributive property of multiplication over addition/subtraction.
- Later in elementary school, students encounter negative integers. This allows them to have closure for subtraction and employ the additive inverse to solve very simple (one step) arithmetic equations.

[2]For instance, we could argue that the integers were developed to enforce closure for subtraction for the natural numbers and that the rational numbers were developed to enforce closure for division for the integers, and so forth.

[3]Some of the latter number systems will be quite advanced and usually only investigated in college graduate level mathematics courses.

- This is soon accompanied by the consideration of fractions and decimals, opening the way to the notions of rational numbers, closure for division, and multiplicative inverses.

Thus, while not stated explicitly as field axioms, many of the associated properties of number systems are investigated throughout the K-12 educational system.

The field[4,5,6] axioms are provided in Figure 4 below. This simple table should be a sufficient reminder for most readers.

	Field Properties of $\mathcal{F}$	
	$+ : \forall a,b,c \in \mathcal{F}$	$\times : \forall a,b,c \in \mathcal{F}$
Closed	$a+b \in \mathcal{F}$	$a \times b \in \mathcal{F}$
Commutative	$a+b=b+a$	$a \times b = b \times a$
Associative	$a+(b+c)=(a+b)+c$	$a \times (b \times c) = (a \times b) \times c$
Identity	$\exists\, 0$ s.t. $0+a=a$	$\exists\, 1$ s.t. $1 \times a = a$
Inverse	$\exists$ -a s.t. $a+(\text{-}a)=0$	$\exists\, a^{-1}$ s.t. $a \times a^{-1} = 1$, $a \neq 0$
Distributive Law	$a \times (b+c) = (a \times b) + (a \times c)$	
Non-triviality	$1 \neq 0$	

Figure 4: Field properties.

Prior to considering further how the field axioms intersect with number systems, there are a few additional notions that we think will be important to our investigation. First, as is mathematically common, we define subtraction and division through additive and multiplicative inverse respectively.

[4]Looking into an introductory Modern Algebra textbook for "the" definition of *field* doesn't satisfy our need; nearly all have something like "A *field* F is a commutative ring with $1 \neq 0$ in which every nonzero element a is a unit; that is, there is $a^{-1} \in F$ with $a^{-1}a = 1$." [15, page 230]. or "Division rings that have this property [commutative multiplication] are called *fields*." [8, page 88]. We find much more useful Swokowski's precalculus text, *Fundamentals of Algebra and Trigonometry* (1968), giving the list of field properties collected in Figure 4 on page 8.

[5]An interesting student project is proving addition is commutative from the other properties by investigating the equation $(1+x)\cdot(1+y) = (1+y)\cdot(1+x)$.

[6]Technically, a field is a triple: the underlying set, an "addition," and a "multiplication," that satisfy the properties listed in Figure 4; we usually *abuse the notation* referring to the field by just its underlying set.

Thus, $a - b = a + (-b)$ and $a \div b = a \times \frac{1}{b}$ for $b \neq 0$. With these definitions in place, we do not need to be concerned with closure, identity, or inverses for subtraction and division.

Second, discussions below place some emphasis on the existence of zero and the properties of zero. Zero is an important element in the world of number systems. Zero acts as the additive identity. Also, its properties are instrumental to understanding zero divisors which are later discussed in this investigation. There are two important properties of zero. The first states: $\forall a \in S, a \cdot 0 = 0$. The second states that $\forall a, b \in S,$ if $a \cdot b = 0,$ then $a = 0$ or $b = 0$. Note that "or" in this last statement is the *inclusive or*, meaning one or the other or both.

Our classroom experience has verified that K-16 students can always use more investigations of the field axioms. Although most of the axioms are seen informally in early grades, most student rarely see them as a unified whole. Instructors are encouraged to ask students to define and provide their own examples of each axiom. Also, it is valuable to show them where axioms can be applied to simplify mathematics. For instance, rather than calculating $5(99,999) = 450,000 + 45,000 + 4,500 + 450 + 45 = 499,995$ a student can calculate $5(100,000 - 1) = 500,000 - 5 = 499,995$, employing the distributive property.

5. Polynomials

$\boxed{1 \textit{ and } 8 \textit{ are the only perfect cube Fibonacci numbers.}}$

As a second preparatory discussion prior to considering number systems, we provide a very brief and well known definition for polynomials. We begin with the common notation for a polynomial expression:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots a_2 x^2 + a_1 x^1 + a_0.$$

We define a real polynomial as a polynomial $P(x)$ for which all coefficients and constants a_i are real numbers and all exponents are natural numbers. A rational polynomial would have all coefficients and constants as rational numbers.

Due to their simplicity, many first year college students often do not recognize linear functions as polynomials. However if $2x + 3$ is rewritten as $P(x) =$

$0x^3 + 0x^2 + 2x + 3$, we can see it in the context of a polynomial. Thus, as soon as a student is asked to solve something as simple as $3 + \square = 5$, they are effectively solving a linear polynomial equation, where the x has been replaced by the $\square$ for the variable.

As we will see in later discussions of number systems, the notion of polynomials is paramount. For instance, an algebraic number is defined as a number which is a root of a polynomial function with integer (or rational) coefficients. Transcendental numbers are those which are non-algebraic. Moreover, students generally first encounter irrational, pure imaginary, and complex numbers in the context of solving quadratic equations.

Our experience has revealed that, while students are familiar with polynomials, their understanding is often limited and fragmented. For instance, for

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots a_2 x^2 + a_1 x^1 + a_0,$$

students may not recognize all the information that is readily available (e.g., far-left and -right (limit) behavior, the $y-$intercept, and the number of complex roots and possible real roots). When they see a polynomial in factored form or a quadratic in vertex form, they might value the new information provided through those forms but not truly connect it back to polynomials. In other words, there is much for students to consider with respect to polynomials. Activities #3-5 for $\mathbb{R}$ and $\mathbb{C}$ relate these ideas; see §11.

6. Cardinality of Sets

$$\boxed{9 = 3^2 = 1^3 + 2^3 = 1! + 2! + 3!}$$

Another beautiful notion embedded into the discussion of number systems is their associated cardinality. Most students are familiar with Buzz LightYear's famous quote, "To infinity and beyond." Interestingly, Buzz was correct. There are cardinalities beyond the infinity with which most students may be familiar. Indeed, the infinity recognized by most students is the SMALLEST of the infinities!

Students are quick to recognize that the set $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$ of natural numbers is an infinite set. While the cardinality of this set is indeed infinite, we denote this as $\aleph_0$, said "aleph nought" or "aleph null", depending on the reference source.[7] Interestingly, the respective cardinalities of the following sets are all $\aleph_0$: the natural numbers, integers, rational numbers, and real constructible, arithmetic, and algebraic numbers.

The powerset of a set is the set that includes all the subsets of the set and nothing more.[8] Thus, for any set with a cardinality of n, the cardinality of its power set will be 2^n, and $n < 2^n$. Cantor showed that the powerset of a set always has a larger cardinality. Thus, the cardinality of the powerset of $\mathbb{N}$, denoted $2^{\aleph_0} = \mathfrak{c}$, must be larger than $\aleph_0$: $\aleph_0 < 2^{\aleph_0}$. Taking this further, the powerset of the powerset of $\mathbb{N}$ (i.e., $\mathcal{P}(\mathcal{P}(\mathbb{N}))$) has an ever larger cardinality: $\aleph_0 < 2^{\aleph_0} = \mathfrak{c} < 2^{\mathfrak{c}}$. By continuing to consider powersets of powersets, one can find an infinite number of infinities, each infinitely larger than its predecessor! Buzz Lightyear was right!

Mind-blowingly, the set of real transcendental numbers (of which most people only recognize two, π and e) has a cardinality of $\mathfrak{c}$ (a much larger size of infinity than $\aleph_0$).[9] Since the real transcendental numbers are a subset of the real numbers, the real numbers must have a cardinality at least as large as the real transcendentals. In fact, the cardinality of the reals is also $\mathfrak{c}$. Therefore, the set $\mathbb{R}$ of reals gets its enormous size from the real transcendental numbers! This is cool.

It is fun to consider number systems from the lens of their cardinal sizes. Indeed, all the number systems we will consider in this paper fall under the two cardinalities $\aleph_0$ (countably infinite) and $\mathfrak{c}$ (continuum, uncountably infinite).

Transfinite cardinalities are just plain cool. Student at many levels enjoy investigations of various sizes of infinity. Instructors are encouraged to use notions of infinity to pique the interests of students toward deeper mathematical and philosophical concerns. Zeno's paradoxes and the evolution of the notion of limits can also be fun investigations.

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[7] $\aleph$ (aleph) is the first upper case letter in the Hebrew alphabet.

[8] For instance, for the set $A = \{1, 2, 3\}$, the powerset of A would be $\mathcal{P}(\mathbb{N}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

[9] As stated in [2], "Some authors use $\beth_n$ (Hebrew letter Beth) to represent the n^{th} powerset of $\mathbb{N}$ so $\beth_0 = \aleph_0 = \operatorname{card}\mathbb{N}$, $\beth_1 = 2^{\aleph_0} = \operatorname{card}\mathcal{P}(\mathbb{A})$, $\beth_2 = 2^{\beth_1} = 2^{2^{\aleph}_0}$ etc."

7. Number Systems

15 *is the first product of two distinct odd primes.*

Below is an annotated catalog of number systems. These include some comments on properties, such as whether the particular set is discrete or dense[10], whether it is linearly ordered or not[11], and its cardinaility. Furthermore, we consider the number systems in the context of polynomials when appropriate.

Natural Numbers: $\mathbb{N} = \{1, 2, 3, 4, ...\}$

$\mathbb{N}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity		✓
Inverse		

- $\mathbb{N}$ satisfies properties: closure for $\mathbb{N}^{\mathbb{N}}$[12] and distributive for $\times$ over $+$.[13] $\mathbb{N}$ is discrete, ordered, and has the cardinality $\aleph_0$
- Missing properties include, zero properties and closure for $-$ and $\div$
- $\mathbb{N}$ solves the polynomial equation $x - a = b$ for all $a, b \in \mathbb{N}$. Note how limited this equation is.
- $\mathbb{N}$ is encountered informally (as numbers) as young children, but not formally as a number system.

[10]Sets which are discrete have successive elements such that there are no elements in between (e.g., in $\mathbb{N}$, there are no elements between 2 and 3). Sets which are dense have an infinite number of elements between any two elements — no matter how seemingly close they appear to be. More precisely, a set is dense if given any element x and arbitrarily small distance $\epsilon > 0$, then is some element y within distance ϵ from x (where distance is measured thinking of numbers as points residing in the complex plane). In other words, given an element, there are other elements arbitrarily close to it. On the other hand, a set is discrete if given an element x there is some distance $\epsilon > 0$ such that there are *no elements* (other than x itself within ϵ-distance from x. Consequently, given an element, we can speak of a *closest* neighboring element (e.g., in $\mathbb{N}$ the closest neighbors of 3 are 2 and 4).

[11]Here we speak of a linear ordering (i.e., for any x, y, z our order is transitive: $x < y$ and $y < z$ implies $x < z$ and the trichotomy property holds: exactly one of the following holds: $x < y$, $x = y$, or $x > y$) that respects arithmetic (for any x, y, z we respect addition: $x < y$ implies $x + z < y + z$ and multiplication: $x < y$ and $z > 0$ implies $xz < yz$).

[12]Throughout this paper, $\mathbb{E}^{\mathbb{F}}$ means that elements of $\mathbb{E}$ raised to elements of $\mathbb{F}$ powers result in an element belonging to the number system in question.

Whole Numbers: $\mathbb{W} = \{0, 1, 2, 3, 4, ...\}$

$\mathbb{W}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse		

- $\mathbb{W}$ satisfies the properties: zero[14] and closure for $\mathbb{W}^{\mathbb{W}}$[15]. $\mathbb{W}$ is discrete, ordered, and has the cardinality $\aleph_0$
- Missing properties include closure for $-$ and $\div$. $\mathbb{W}$ still leaves the motivation for a set with closure for $-$.
- $\mathbb{W}$ introduces zero properties and an identity for $+$.
- $\mathbb{W}$ solves the equation $x - a = b$ for all $a, b \in \mathbb{W}$. Note that this is almost the same equation as for $\mathbb{N}$. So, it did not help much. But it certainly leads to the motivation to want a more robust set which solves more equations.
- $\mathbb{W}$ is encountered informally (not as a number system) in grades 1-2. When investigating subtraction, zero is discovered.

Integers: $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

$\mathbb{Z}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	

- $\mathbb{Z}$ satisfies the properties: zero and closure for $-$ and $\mathbb{Z}^{\mathbb{W}}$. $\mathbb{Z}$ is discrete, ordered, and has the cardinailty $\aleph_0$
- Missing properties include closure for $\div$.
- $\mathbb{Z}$ introduces closure for $-$ and inverses for $+$. We may yet want closure for $\div$.
- $\mathbb{Z}$ solves the equation $x \pm a = b$ for all $a, b \in \mathbb{Z}$. This is a great improvement, but it still does not solve an equation even as simple as

[13] Unless otherwise specified or the set does not have closure for $+$ and $\times$, all the sets listed have the distributive property of $\times$ over $+$: $\forall a, b, c \in S, a(b + c) = ab + ac$.

[14] Unless specified otherwise, the Zero Property always includes the two properties: (1) $\forall a \in S, a \cdot 0 = 0$ and (2) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

[15] In all cases of exponentiation, we omit the option of 0^0.

$2x+3=2$. This leads to the motivation to solve linear equations with integer coefficients which are other than 1.

- $\mathbb{Z}$ is encountered semi-formally as the result of investigating subtraction.[16]

Rational Numbers: $\mathbb{Q} = \{p/q : p, q \in Z \text{ and } q \neq 0\}$ or[17]

$\mathbb{Q} = \{x : x$ is a terminating or repeating decimal[18]$\}$

$\mathbb{Q}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{Q}$ satisfies the properties: zero and closure for $\div$[19]. $\mathbb{Q}$ is dense, ordered, and has the cardinality $\aleph_0$.
- $\mathbb{Q}$ introduces closure for for $\div$ and multiplicative inverses and density.
- $\mathbb{Q}$ solves the equation $ax \pm b = c$ for all $a, b, c \in \mathbb{Q}$. Notably, this set still cannot solve equations involving a polynomial function of degree greater than 1.
- $\mathbb{Q}$ is encountered informally in K-1 (as unit fractions), but not formally as a number system. It is then once again encountered, semi-formally, in grades 5-8 as fractions, decimals, and percents.

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[16] We believe that negative integers are encountered far too late in the standard U.S. curriculum (often in fifth grade). They should evolve organically as students perform subtraction as early as a negative value may appear.

[17] It is vitally important to K-16 mathematics that $\mathbb{Q}$ has more than one definition. In essence, the two definitions given indicate that any rational number represented in fraction form can be converted to a decimal and any terminating or repeating decimal can be written as a fraction of integers; this is not always obvious to students.

[18] Students must recognize the distinction between numerals with repeating cycles versus other discernible patterns. For instance, 1.234234234... has a repeating cycle of 234, and 1.1011011101110... has a discernible pattern which is not a cycle of constant length. In fact, playing with these patterned decimals, can lead to an understanding of irrational numbers. The following are all irrational numbers: 1.234567891011121314151617181920...; 1.23434545656767878989891091011...; and 1.5101520253035....

[19] In all discussions involving closure for $\div$, we omit the case of division by zero.

Real Constructible Numbers: $\mathbb{S}_{\mathbb{R}} = \{x : x$ is a real number which can be compass and straightedge constructed[20]$\}$ or $\mathbb{S}_{\mathbb{R}} = \mathbb{R} \cap \mathbb{S}$, where $\mathbb{S}$ will be defined shortly.

$\mathbb{S}_{\mathbb{R}}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{S}_{\mathbb{R}}$ has the properties: zero and closure for $-$ and $\div$, and $\sqrt{x}_{x\geq 0}$, distributive. $\mathbb{S}_{\mathbb{R}}$ is dense, ordered, and has the cardinality $\aleph_0$.
- $\mathbb{S}_{\mathbb{R}}$ introduces closure for square roots and **contains some** irrational numbers. Note that this is not closure for all roots, and only a limited number of irrational numbers are in $\mathbb{S}$, far from the complete set.
- $\mathbb{S}_{\mathbb{R}}$ is rarely encountered before select graduate mathematics courses.

Real Arithmetic Numbers: $\mathbb{A}r_{\mathbb{R}} = \{x : x$ is real number which can be built from natural numbers after a finite sequence of addition, subtraction, multiplication, division, exponentiation, and root taking$\}$ or $\mathbb{A}r_{\mathbb{R}} = \mathbb{R} \cap \mathbb{A}r$, where $\mathbb{A}r$ will be defined shortly.

$\mathbb{A}r_{\mathbb{R}}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{A}r_{\mathbb{R}}$ has the properties: zero, closure for $-$, $\div$, and $\sqrt{x}_{x\geq 0}$, and distributive.
- $\mathbb{A}r_{\mathbb{R}}$ is dense, ordered, and has the cardinality $\aleph_0$.
- $\mathbb{A}r_{\mathbb{R}}$ is rarely encountered outside of graduate studies.

[20]Beginning with two points, $(0,0)$ and $(1,0)$, one is allowed to successively construct lines through two previously constructed points or circles passing through a previously constructed point with previously constructed point as its center. Intersections of such lines and circles give newly constructed points. Coordinates of such points are constructible numbers. Briefly, a constructible number can be constructed from a finite number of uses of a straightedge and compass.

Real Irrational Algebraic Numbers: $\mathbb{A}_{\mathbb{RI}} = \{x : x$ is a real irrational root of a polynomial with integer coefficients$\}$ or $\mathbb{A}_{\mathbb{RI}} = \mathbb{I} \cap \mathbb{A}_{\mathbb{R}}$, where $\mathbb{I}$ and $\mathbb{A}_{\mathbb{R}}$ will be defined shortly.

$\mathbb{A}_{\mathbb{RI}}$	$+$	$\times$
Closed		
Commutative	$\reflectbox{\checkmark}$[22]	$\reflectbox{\checkmark}$
Associative	$\reflectbox{\checkmark}$	$\reflectbox{\checkmark}$
Identity		
Inverse	$\reflectbox{\checkmark}$	$\reflectbox{\checkmark}$

- $\mathbb{A}_{\mathbb{RI}}$ is dense, ordered, and has the cardinality $\aleph_0$.
- Due to lack of closure for $+$ and $\times$[21], we no longer have any of the field axioms.
- $\mathbb{A}_{\mathbb{RI}}$ contains the real irrational solutions of all polynomials with coefficients in $\mathbb{Z}$.
- $\mathbb{A}_{\mathbb{RI}}$ is encountered informally as numbers (but not as a number system) when solving quadratic functions in early high school.

Real Algebraic Numbers: $\mathbb{A}_{\mathbb{R}} = \{x : x$ is a real root of a polynomial with integer[23] coefficients$\}$ or $\mathbb{A}_{\mathbb{R}} = \mathbb{R} \cap \mathbb{A}$, where $\mathbb{A}$ will be defined shortly.

$\mathbb{A}_{\mathbb{R}}$	$+$	$\times$
Closed	$\checkmark$	$\checkmark$
Commutative	$\checkmark$	$\checkmark$
Associative	$\checkmark$	$\checkmark$
Identity	$\checkmark$	$\checkmark$
Inverse	$\checkmark$	$\checkmark$

- $\mathbb{A}_{\mathbb{R}}$ has the properties: zero, closure for $-$, $\div$, and $\sqrt{x}_{x \geq 0}$, and distributive. $\mathbb{A}_{\mathbb{R}}$ is dense, ordered, and has the cardinality $\aleph_0$.
- $\mathbb{A}_{\mathbb{R}}$ contains real solutions of all polynomials with coefficients in $\mathbb{Z}$.
- $\mathbb{A}_{\mathbb{R}}$ is encountered informally in high school when finding roots of polynomials. It is potentially encountered formally in a college modern algebra course.

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[21] Counterexamples are as simple as $\sqrt{2} + \big(-\sqrt{2}\big) = 0$ and $\sqrt{2} \cdot \sqrt{2} = 2$.

[22] The backcheck symbol indicates that, if the operation on two operands produces an element in the set, the operation with those two operands holds the respective property.

[23] Some define $\mathbb{A}_{\mathbb{R}}$ as the set $\{x : x$ is a root of a polynomial with **rational** coefficients and $x \in \mathbb{R}\}$. These two definitions are essentially the same since any polynomial with rational coefficients can be multiplied by some constant to make all coefficients integral, and both polynomials would have the same roots.

Real Transcendental Numbers: $\mathbb{T}_{\mathbb{R}} = \{x : x \text{ is a real number but not algebraic (i.e., } x \notin \mathbb{A}_{\mathbb{R}})\}$ or $\mathbb{T}_{\mathbb{R}} = \mathbb{R} \cap \mathbb{T}$, where $\mathbb{T}$ will be defined shortly. Also $\mathbb{T}_{\mathbb{R}} = \mathbb{R} - \mathbb{A}_{\mathbb{R}}$[24]

$\mathbb{T}_{\mathbb{R}}$	$+$	$\times$
Closed		
Commutative	✓	✓
Associative	✓	✓
Identity		
Inverse	✓	✓

- $\mathbb{T}_{\mathbb{R}}$ is dense and ordered and introduces the cardinality $2^{\aleph_0} = \mathfrak{c}$.
- Due to lack of closure for $+$ and $\times$[25], we no longer have any of the field axioms.
- $\mathbb{T}_{\mathbb{R}}$ is encountered informally in approximately third grade as students begin to work with calculations of areas of circles with $\pi \approx 3$. It is also considered semi-formally (but not as a number system) in high school precalculus courses and above generally in the form of π and e. It is potentially considered formally in a college modern algebra course.

Irrational Numbers: $\mathbb{I} = \{x : x \text{ is a non-terminating and non-repeating decimal}\}$ or $\mathbb{I} = \mathbb{R} - \mathbb{Q}$ or $\mathbb{I} = \mathbb{A}_{\mathbb{R}\mathbb{I}} \cup \mathbb{T}_{\mathbb{R}}$

$\mathbb{I}$	$+$	$\times$
Closed		
Commutative	✓	✓
Associative	✓	✓
Identity		
Inverse	✓	✓

- $\mathbb{I}$ is dense and ordered, and since $\mathbb{T}_{\mathbb{R}} \subset \mathbb{I}$, $\mathbb{I}$ has the cardinality $2^{\aleph_0} = \mathfrak{c}$.
- Due to lack of closure for $+$ and $\times$, we no longer have any of the field axioms.
- $\mathbb{I}$ is encountered informally in early high school when solving quadratic equations. It is later considered semi-formally (but not as a number system) in high school precalculus courses and above. $\mathbb{I}$ is potentially considered formally as a set in a college modern algebra course.

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[24] $A - B = \{x \in A : x \notin B\}$ is the complement of the set B in the set A (also called a set difference).

[25] Counterexamples are as simple as $\pi + (-\pi) = 0$ and $\pi \cdot \frac{1}{\pi} = 1$.

Real Numbers: $\mathbb{R} = \{x : x \text{ is a decimal number}\}$
or $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ or $\mathbb{R} = \mathbb{A}_{\mathbb{R}} \cup \mathbb{T}_{\mathbb{R}}$

$\mathbb{R}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{R}$ has the properties: zero, closure for $-$, $\div$, and ${\mathbb{R}_{\geq 0}}^{\mathbb{R}}$, and distributive. $\mathbb{R}$ is dense, ordered, and has the cardinality $\mathfrak{c}$.
- Given a polynomial $P(x)$ with real coefficients and $a \in \mathbb{R}^+$, there are real solutions of $P(x)^a = 0$.
- $\mathbb{R}$ introduces closure for $(\mathbb{R})^{\mathbb{R}}_{\geq 0}$
- $\mathbb{R}$ is encountered informally in middle grades in the investigation of decimal numbers and the number line. It is also considered semi-formally throughout high school and above. $\mathbb{R}$ is potentially considered formally as a set in a college modern algebra course.

Imaginary Numbers: $\mathbb{I}m = \{ai : a \in \mathbb{R}, i^2 = -1\}$

$\mathbb{I}m$	$+$	$\times$
Closed		
Commutative	✓	✓
Associative	✓	✓
Identity		
Inverse	✓	✓

- $\mathbb{I}m$ is dense, but not ordered,[26] and has the cardinality $\mathfrak{c}$.
- Due to lack of closure for $+$ and $\times$, we no longer have any of the field axioms.
- $\mathbb{I}m$ is encountered informally when finding roots of quadratic functions. It is also considered semi-formally (but not as a number system) in high school precalculus courses and above. $\mathbb{I}m$ is potentially considered formally as a set in a college modern algebra course.

Gaussian Integers: $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z} \text{ and } i^2 = -1\}$

$\mathbb{G}_{\mathbb{Z}}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	

- $\mathbb{Z}[i]$ has the properties: zero, closure for $-$ and distributive. $\mathbb{Z}[i]$ is discrete, not ordered, and has the cardinality $\aleph_0$.
- We have lost closure for $\div$.

[26] We can order $ai < bi$ for real numbers $a < b$. Such an ordering respects addition but not multiplication: since $0 = 0i < 1i = i$ but $i^3 = -i = -1i < 0i = 0$ (a "positive" cubed isn't positive).

- $\mathbb{Z}[i]$ introduces a set which is not ordered.[27,28]
- $\mathbb{Z}[i]$ is encountered informally when solving quadratic equations. It is also considered semi-formally (but not as a number system) in high school precalculus courses and above. It is potentially considered formally as a set in a college modern algebra course.

Constructible Numbers: $\mathbb{S} = \{x + yi$: the point (x, y) can be constructed with finitely many uses of compass and straightedge and $i^2 = -1\}$

$\mathbb{S}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{S}$ has the properties: zero, closure for $-$ and $\times$, and $\div$, and distribution. $\mathbb{S}$ is dense, not ordered, and has the cardinality $\aleph_0$.
- $\mathbb{S}$ is rarely encountered before graduate mathematics courses.

Arithmetic Numbers: $\mathbb{A}r = \{x : x$ can be built from natural numbers after a finite sequence of addition, subtraction, multiplication, division, exponentiation, and taking roots$\}$

$\mathbb{A}r$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{A}r$ has the properties: zero, closure for $-$ and $\times$, and $\div$, and distribution. $\mathbb{A}r$ is dense, not ordered, and has the cardinality $\aleph_0$.
- $\mathbb{A}r$ is rarely encountered before graduate mathematics courses.

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[27] As soon as the elements of a set possess both non-zero real and imaginary parts, it is no longer ordered. This stems from the fact that we can no longer respect multiplication: non-zero squares must be positive but $i^2 = -1 < 0$.

[28] The Eisenstein integers are complex numbers of the form $x + \omega y$ where x and y are integers and ω is a nontrivial cube root of 1. This set has hexagonal symmetry.

Algebraic Numbers: $\mathbb{A} = \{x : x \text{ is a root of a non-zero polynomial with integer (or rational) coefficients}\}$[29]

$\mathbb{A}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{A}$ has the properties: zero, closure for $-$ and $\div$, and distributive. $\mathbb{A}$ is dense, not ordered, and has the cardinality $\aleph_0$.
- $\mathbb{A}$ is encountered informally in high school when finding roots of polynomials. It is potentially encountered formally in a college modern algebra course.

Complex Numbers: $\mathbb{C} = \{a+bi : a, b \in \mathbb{R} \text{ and } i^2 - 1\}$ or $\mathbb{C} = \mathbb{R} \cup \mathbb{I}m$ or $\mathbb{C} = \mathbb{A} \cup \mathbb{T}$

$\mathbb{C}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	✓
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{C}$ has the properties: zero, closure for $-$, $\div$, and $\mathbb{C}^{\mathbb{C}}$, and distributive. $\mathbb{C}$ is dense, not ordered, and has the cardinality $\mathfrak{c}$.
- $\mathbb{C}$ introduces closure for $\mathbb{C}^{\mathbb{C}}$.
- Given a polynomial $P(x)$ with complex coefficients and $a \in \mathbb{C}$, some elements of this set are solutions of $P(x)^a = 0$.
- $\mathbb{C}$ is encountered informally when solving quadratic equations. It is also considered semi-formally (but not as a number system) in high school precalculus courses and above. It is potentially considered formally as a set in a college modern algebra course.

Quaternions: $\mathbb{H} = \{a+bi+cj+dk : a, b, c, d \in \mathbb{R} \text{ and } i^2 = j^2 = k^2 = ijk = -1\}$[30]

$\mathbb{H}$	$+$	$\times$
Closed	✓	✓
Commutative	✓	
Associative	✓	✓
Identity	✓	✓
Inverse	✓	✓

- $\mathbb{H}$ contains the properties: zero, closure for $-$, $\div$, and $\mathbb{H}^{\mathbb{H}}$, and distributive. $\mathbb{H}$ is dense, not ordered, and has the cardinality $\mathfrak{c}$.

[29] This set is sometimes denoted by $\overline{\mathbb{Q}}$ because it is the algebraic closure of the rational numbers.

[30] On Monday, the sixteenth of October, 1843, Hamilton had the epiphany that the relation $i^2 = j^2 = k^2 = ijk = -1$ defined the quaternions; he carved this formula into the stone of the Brougham Bridge in Dublin.

- Since $\mathbb{H}$ is noncommutative for multiplication, $\mathbb{H}$ is a skew-field (also called a division ring), not a field.
- $\mathbb{H}$ is rarely encountered before graduate mathematics courses.

8. Connecting Number Systems through Subsets, Supersets, and Cardinality

$\boxed{\textit{For } n = 22, 23, 24 \textit{ (only), } n \textit{ is the number of digits in } n!}$

The annotated list of number systems of §7 can be considered through another interesting and entertaining light. Some sets have subset and superset relationships. This also leads to considerations of sets with cardinality $\aleph_0$ or $2^{\aleph_0}$. The following expressions demonstrate some of these relationships.

$$\underbrace{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{S} \subseteq \mathbb{A}\mathrm{r}_{\mathbb{R}}}_{\aleph_0} \subset \underbrace{\mathbb{R} \subset \mathbb{C}}_{\mathfrak{c}},$$

$$\underbrace{\mathbb{A}_{\mathbb{R}\mathbb{I}}}_{\aleph_0} \cup \underbrace{\mathbb{T}_{\mathbb{R}}}_{\mathfrak{c}} = \underbrace{\mathbb{I} \subset \mathbb{R}}_{\mathfrak{c}}, \; \underbrace{\mathbb{I}m \subset \mathbb{C}}_{\mathfrak{c}},$$

$$\underbrace{\mathbb{Z}[i]}_{\aleph_0} \subset \underbrace{\mathbb{C}}_{\mathfrak{c}}$$

Considering these set relationships introduces another interesting notion: as elements from a proper subset are inherited into a superset, the superset also inherits all the field axioms associated with the subset. For instance, since $\mathbb{Z} \subset \mathbb{Q}$, all the field axioms inherent in $\mathbb{Z}$ are also in $\mathbb{Q}$. Notably, the superset may bring in additional field axioms not held in the subset. However, and maybe unexpectedly, the superset may also lose some properties too. For instance, while $\mathbb{R} \subset \mathbb{C}$, $\mathbb{C}$ loses order, and while $\mathbb{C} \subset \mathbb{H}$, $\mathbb{H}$ loses multiplicative commutativity.

Instructors are encouraged to lead students through investigations of various number systems through the lens of subset and superset relations (see activity #7 under $\mathbb{R}$ and $\mathbb{C}$ in §11). Through investigating these superset and subset structures, the notion of inheriting field axioms among subsequent number systems naturally evolves. These can be informative and engaging investigations.

9. Interesting *Systems* Other than Number Systems

In English, 77 is the smallest number requiring five syllables to name.

There are four other *systems* which we wish to present for the sake of comparison with previously presented number systems. These include the integers $\mathbb{Z}_p$ under a prime modulus, integers $\mathbb{Z}_c$ under a composite modulus, matrices, and field extensions. We discuss these less formally than the number systems of §7.

- **Integers Mod p where p is prime:** $p = \{2, 3, 5, 7, 11 \ldots\}$. $\mathbb{Z}_p$ is a field with all the properties including zero, closure $(+, -, \times,$ and $\div)$, and distributive. Notably the cardinality of $\mathbb{Z}_p$ is p and $\mathbb{Z}_p$ is discrete.
- **Integers Mod c where c is composite:** $c = \{4, 6, 8, 9, 10, \ldots\}$ $\mathbb{Z}_c$ is not a field. $\mathbb{Z}_c$ contains *zero divisors*[31]. While zero divisors do not affect closure, they affect other properties such as the identity and inverse properties.
- **Matrices**. Most notably, matrix multiplication is noncommutative. Since matrices are investigated as early as high school, they are usually the first system which students investigate which is noncomutative for multiplication.
- **Field Extensions**. We can also build towers of fields. For example, look at $\mathbb{Q}[\sqrt{2}] = \{p + q\sqrt{2} : p, q \in \mathbb{Q}\}$ or $\mathbb{Q}[\sqrt[3]{2}] = \{p + q\sqrt[3]{2} + r\sqrt[3]{2^2} : p, q, r \in \mathbb{Q}\}$. This concept is studied in many college-level algebra courses.
- **Computable numbers**. Alan Turing defined a computable number as a number that can be approximated to an arbitrarily chosen precision given a finite amount of time. Rational numbers and algebraic numbers are computable. While transcendental numbers like π or e which have convergent series expansions are computable, most real numbers are not computable. For a nice exposition, see [11].

[31] We previously stated that if $a \cdot b = 0$, then $a = 0$ or $b = 0$. However, for instance, under $\mathbb{Z}_6, 2 \cdot 6 = 0$, but neither 2 nor 3 is zero. Thus, 2 and 3 are *zero divisors* and we say that the system includes zero divisors.

There are countless additional systems which have been recognized by others or can be constructed. The study of any and all of these help us to understand others. Instructors can encourage students to investigate number systems to build their enthusiasm regarding mathematical pursuits.

10. Other Aspects of Gains and Losses

> 113 *is the smallest three-digit number for which all arrangements of its digits are also prime.*

There are other ways to envision the movement among number systems. This can be seen through the lens of gains and losses. Some of these are depicted in Figure 5.

- Progressing from $\mathbb{N}$ to $\mathbb{Z}$, we gain closure for subtraction but lose the representation of a unique absolute value. For instance, $|-1| = |1|$ and $|-3| = |3|$.
- Progressing from $\mathbb{Z}$ to $\mathbb{Q}$, we gain closure for division and lose unique representation. For instance $1/2 = 2/4 - 100/(-200) = 0.04/0.08 = (1.\overline{234}/2.\overline{468} = 0.5 = 0.4\overline{9}$.
- Progressing from $\mathbb{Q}$ to $\mathbb{R}$, we gain Dedekind completeness[32] and lose countability[33] and simple representation[34].
- Progressing from $\mathbb{R}$ to $\mathbb{C}$, we gain algebraic completeness[35] and lose order[36].

[32] Dedekind completeness is encountered in a typical introductory real analysis course. It guarantees that subsets of real numbers with upper and lower bounds in fact have greatest upper bounds and least upper bounds. Alternatively, this kind of completeness guarantees that Cauchy sequences (that is, sequences in which as one goes further out in the sequence, the terms gets closer and closer together) must converge to some real number.

[33] The cardinality moves from $\aleph_0$ to $2^{\aleph_0} = \mathfrak{c}$.

[34] There is no simple way to represent $r = 1.01\,002\,0003\,00004\,000005\,0000006\,\ldots$.

[35] Every nth degree polynomial has exactly n roots (counting multiplicity).

[36] We can no longer say that for any two numbers $a, b \in \mathbb{C}$, that $a < b$ or $a = b$ or $a > b$. Let us consider why one complex number cannot be greater than another. Suppose $i > 0$ and multiply by i. Then $i > 0 \Longrightarrow i^2 > i \cdot 0 = 0 \Longrightarrow -1 > 0$, which is somewhat unsettling.

- Progressing from $\mathbb{C}$ to $\mathbb{H}$ gains an ability to depict 3D geometry and loses commutivity for multiplication.

Students can be encouraged to investigate other series of number systems for gains and losses in other dimensions. For example, 'what properties are gained or lost when moving from irrational numbers to real numbers?' This is a relatively unique mechanism through which to view, compare, and contrast various number systems. This could provide much interesting fodder for university mathematics learners.

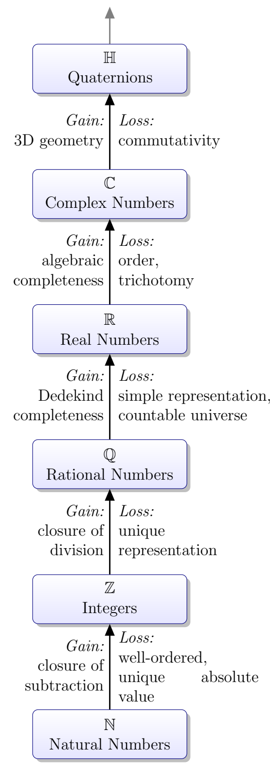

Figure 5: The Tower of Number Systems

11. Activities

101 *is a palindromic prime.*

We conclude this paper with some activities designed to help make the *gestalt* among the topics and subtopics in this paper, leading to deepening students' understanding of the topics. These activities are not grade-related; they are aligned by number system. Thus, they can be used in multiple ways in different classes. In the following we provide a brief description of each activity. A booklet containing full descriptions of all activities can be found as an article supplement on the website for this article (http://scholarship.claremont.edu/jhm/vol13/iss2/22) in two formats: a PDF file for effective printing and a MS Word doc, so that it can be easily edited and modified by instructors to give students the learning experiences they wish them to have.

Now suppose $i < 0$. Then multiply by i. Then $i < 0 \Longrightarrow i^2 > i \cdot 0 = 0 \Longrightarrow -1 > 0$, which still is discomforting. So i can be neither greater than nor less than 0 — there is no 'less than' for the complex numbers.

All of these activities have been field tested in middle school, high school, and college classrooms and revised and improved accordingly. The reader should not be deceived by the seeming simplicity or complexity of some of the activities; at times college math majors are surprised by what they learn in the simple activities and high school students are surprised how far they can get with the more advanced activities.

Activities for $\mathbb{N}$*,* $\mathbb{W}$*, and* $\mathbb{Z}$

1. Blank Modulo Tables: This activity asks readers to complete some blank addition and multiplication tables in various modular arithmetic systems with leading questions at the end. It is an introduction to zero divisors. This activity compares systems with prime versus composite moduli. This activity can be completed by students as young as in middle grades. Understanding zero divisors leads to better understanding of the far more familiar zero properties. This activity leads students to some fun and unanticipated findings. In another dimension regarding number systems, this investigation deals with sets of cardinality $n < \aleph_0$.

3. The Magic of 1: This activity investigates some fun facts about the values 0 and 1. The reader must complete proofs about some commonly encountered ideas often seen as early as in middle grades. These proofs should be immediately at hand for any teacher of grade 6 and above to readily answer student questions. The values 0 and 1 are very important in mathematics in general and in respect to the consideration of number systems and field axioms.

4. Order of Operations: This activity uses modulo operation tables in order to investigate order of operations. Too many students think they understand the order of operations. This activity both challenges them and solidifies these notions. In respect to number systems, a firm grasp of operations is needed to understand field axioms.

5. Rules for Divisibility: Students are often expected to use divisibility rules. However, few understand why these work. This worksheet asks the reader to prove various rules. Students at many levels are expected to apply these rules without an understanding as to why they work. Admittedly, this worksheet is not needed in the context of an investigation of number systems. But students often enjoy proving these ideas, particularly for divisibility by 3 and 9. Younger students can empirically experiment with the rules to see if they believe that are true.

6. Theorem Investigation: This activity is fun based on the cryptic form through which the Fundamental Theorem of Arithmetic is presented. This activity introduces many students to the operation $\prod$ rather than only the $\sum$ operation. This is important regarding the Fundamental Theorem of Arithmetic and later applications of such. While students often find this activity fun in its cryptic nature, the activity subtlety emphasizes the notion of uniqueness – a concept later lost in more advanced number systems.

7. Other Bases: Understanding numbers and operations in other bases helps us understand base 10. This activity forces students out of their comfort zone regarding base 10 and helps them to better conceptually understand the arithmetic they do daily. Although beyond the scope of this activity, students are surprised to find that number systems in other bases hold the same properties as does base 10 numbers, including prime and composite values.

Activities for $\mathbb{Q}$

1. Algebraic Structures: This chart can be used as either a reference tool or an instructional/learning tool in respect to the notion of a field and the field axioms. This is much more of a very helpful resource than an activity. Many abstract algebra students struggle with the nomenclature of groups, rings, and fields. This chart collects many of these ideas in one graphic.

2. Playing with Fractions: This activity looks more deeply at relationships with fractions. Some students find these activities both interesting and challenging, and are quite surprised at recognizing the gaps in their knowledge and what they learn. In respect to number systems, this activity helps students better understand aspects of $\mathbb{Q}$ which might have previously escaped them.

3. Terminating and Repeating Decimals: Students can investigate fun ideas regarding terminating and repeating decimals, including delayed cycles. Many students are quite surprised what they learn through this exercise, and how they can use these ideas to understand delayed terminating and repeating decimals. In respect to number systems, this activity helps students better understand aspects of $\mathbb{Q}$ which may have previously escaped them.

4. Mapping Between Number Systems: This activity considers mappings between $\mathbb{N} \to \mathbb{Z}$ and $\mathbb{N} \to \mathbb{Q}$. It also considers other mappings and the notion of cardinality. Students generally like this exercise as the see many sets with cardinality $\aleph_0$ and are introduced to the cardinality $2^{\aleph_0}$.

Activities for $\mathbb{I}$

1. Geoboard Numbers: This activity introduces the notion of constructible numbers. Many students will not have seen these numbers before and may be surprised at what number can be constructed and which cannot.

2. Rational Approximations of Irrational Numbers: This activity expands on student understanding of rational and irrational numbers. It helps students to see that a rational number can be made to approximate any irrational number to a particular decimal place. This activity also helps with the notion that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.

3. Continued Fractions: This activity expands on student understanding of rational and irrational numbers. It invites students to consider the nature of continued fractions, a concept only infrequently investigated in high school and college classes.

4. Infinite Irrational Decimals: This activity considers a historic proof of the infinity of irrational numbers from a very cryptic and incomplete presentation. As in a precious exercise, students appreciate the mathematically cryptic form of the exercise. It allows them to fill in missing ideas, and learn thereby. It also helps them to understand that the cardinality of $\mathbb{I}$ must be greater than $\aleph_0$.

5. Fibonacci Sequences: An investigation of the Fibonacci sequence leads to some nice connections to irrational numbers. Particularly, this activity helps students understand one number which is often misunderstood as irrational: ϕ.

Activities for $\mathbb{R}$ *and* $\mathbb{C}$

1. Functions with Characteristics: This activity considers various functions with zeros and asymptotes from various number systems. This activity deepens many students understanding of polynomial functions. Many students enjoy solving these problems and inventing their own.

2. Finding Domains: This activity considers how the domains of functions intersects with number systems. Students are often challenged by these exercises.

3. Taylor Polynomials and Euler's Formula: Some mathematicians and educators consider Euler's Formula to be the culmination of all mathematics that should be learned in high school. This activity makes some nice connections among trigonometric functions and complex numbers. Students often enjoy recognizing the connection of trigonometry with Euler's Formula.

4. Some Complex Ideas: This is an extension of investigating Euler's Formula. Even students who are quite familiar with complex numbers only occasionally consider complex numbers in this manner.

5. Readings on Complex Roots of Polynomial and Rational Functions: These provide interesting, fun, and unexpected results. Most interesting is that, through the graph alone of a polynomial or rational function, students can approximately locate the position of complex roots.

- **5.1** *Quadratic through Quartic:* https://scholarworks.umt.edu/cgi/viewcontent.cgi?article=1440&context=tme
- **5.2** *Circle Constructions:* https://php.radford.edu/~ejmt/deliveryBoy..php?paper=eJMT_v11n2n1
- **5.3** *Quintics:* https://scholarworks.umt.edu/tme/vol15/iss3/12/
- **5.4** *Rational Functions:* https://php.radford.edu/~ejmt/deliveryBoy.php?paper=eJMT_v12n2n1

6. Number Systems Table: This activity connects number systems with field axioms, cardinaliy, and other dimensions. Its simple form allows for students to recognize gaps in their own knowledge and the instructors to efficiently assess student understanding.

7. Number Systems Diagram: This assignment challenges the student to better understand subset and superset relationships. Additionally, this activity allows them to see how many poor representations are provided on the internet.

8. System-Oscillating Functions: This is a fun exercise which considers functions hopping between number systems. More advanced students generally greatly enjoy this activity, and often develop very creative functions to complete the tasks.

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