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Ana J. Lemes
University of Lille

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I am grateful to the trainees of the course Historia de la Matematica: Abordajes para la Ensenanza Superior, Diploma ANEP-UdelaR, Uruguay, who allowed me to use their survey responses.

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Gödel’s Theorem in the Continuing Education of Mathematics Teachers

Ana Jimena Lemes

Université de Lille, Lille, Hauts-de-France, FRANCE
jimenalemes@gmail.com

Abstract

The notion of dépaysement épistémologique (epistemological disorientation) aims to capture the sense of disorientation when a learner is led to question their prior assumptions and understandings, generating uncertainty in a context in which they thought they had certain knowledge. This article describes an activity used with a group of practicing mathematics teachers in Uruguay that integrates elements of the history of mathematics related to Gödel’s incompleteness theorem, with the aim of provoking in the participants the experience of dépaysement épistémologique. Results show that several of the teachers participating in the activity felt dépaysement épistémologique, and this feeling triggered empathy towards their own students. The article ends with a discussion of the real possibilities of integrating the history of mathematics in secondary mathematics courses and in the training of teachers.

Keywords: didactic of mathematics, history of mathematics, training of mathematics teachers.

1. Introduction

This article analyzes the implementation of an activity designed from elements of the history of mathematics, in the context of the degree in mathematics organized by the National Administration of Public Education (ANEP) and the University of the Republic, in Uruguay. The main objective of this diploma, which consists of four semesters, is to train practicing secondary school mathematics teachers, as trainers of future mathematics teachers.
In this context, I work as the instructor in charge of a second-semester seminar called “History of Mathematics: Approaches to Higher Education.” The contents of this seminar are organized around the reflection on the potentialities of the history of mathematics in mathematics classes, particularly in teacher training.

The discussion about the integration of the history of mathematics in teaching is not new. Some arguments in favor discussed in the international literature are that the history of mathematics motivates and awakens the curiosity of students, humanizes mathematics by involving historical characters in their context, and allows us to reflect on the construction and evolution of mathematical notions [5, 11, 12, 13, 14, 15, 16, 18, 19].

One of the objectives of my doctoral research was to try to understand the influence of the history of mathematics on the education of mathematics teachers. I was interested in identifying some conceptions of this audience regarding the nature of the discipline, and in how, through explicitly engaging with the history of the discipline, it is possible to enhance the learning of mathematics by making its human, cultural, and social dimensions explicit. In this article, I focus on one of the most widespread conceptions regarding mathematics that I could find among Uruguayan teachers: “In mathematics it is always possible to decide whether a sentence is true or false, and in order to justify this result, it is necessary to present a demonstration.” After confirming the presence of this conception among the seminar attendees, I designed an activity based on Gödel’s incompleteness theorem, with the objective that teachers can question such a conception.

Much research shows that teachers learn models and teaching practices during their own experience as students, throughout their schooling [1, 7, 17, 4]. Thus, it is possible to affirm that for future teachers to integrate the history of mathematics in their classes, it is convenient for them to experience it during their experiences as students. Given that integrating the history of mathematics in the training of mathematics teachers is a key objective of my History of Mathematics seminar, I designed this activity revolving around Gödel’s incompleteness theorem to enable the trainees to experience the history of mathematics from a student’s perspective.

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1 Before the seminar began, I shared a questionnaire with the trainees, from which different conceptions could be identified.
In addition to this broad objective within the framework of the seminar, the specific objective of this activity is to provoke the questioning of the aforementioned conception. This experience lived by the students, which provokes the questioning of a supposed truth, and which generates uncertainty in a field in which one thought to have certainties, is defined by Barbin [2] as dépaysement épistémologique— the French phrase is usually translated into English as “epistemological disorientation”. Questioning knowledge, notions, or concepts that we think we have mastered, but that we do not recognize because of the characteristics of a historical source or an unknown context, dépayse us, it disorients us. Such disorientation can be valuable for various reasons. For example, Guillemette states that teachers experiencing dépaysement épistémologique can develop greater empathy towards their students once they have been in this place of uncertainty [11, 12].

2. Participants, objectives, and design of the activity

2.1. Participants
The activity described in this article was implemented in the eight-week seminar “History of Mathematics: Approaches for Higher Education”, within the framework of the diploma in mathematics offered by the National Administration of Public Education and the University of the Republic. The trainees of this postgraduate course are graduates of teacher training in the mathematics specialty, and most of them have several years of experience as high school teachers. The high school students in Uruguay are between 15 and 18 years old.

The seminar was taught in virtual mode in the second semester of 2020. It included one compulsory activity per week for eight weeks, with an effective participation of 14 trainees out of 19 enrolled. Each activity consisted of a reflection exercise based on recommended readings and participation in a discussion forum. Some of the topics we covered were: the history of mathematics and the history of mathematics education in teaching contexts [3], the history of mathematics in teacher training [19], analysis of a proposal for the integration of the history of mathematics in a high school class [14], and the present activity, based on Gödel’s incompleteness theorem.

2.2. Objectives
In developing this activity I was seeking to achieve at least three objectives.
First, I wanted to bring a historical episode of the twentieth century closer to an audience of practicing mathematics teachers, taking into account the context in which it takes place and the historical figures involved. Knowing this context can deepen the teachers’ understanding of the historical dimension of mathematics, identifying the links between the discipline and its context of construction.

Secondly, I wanted to make explicit the evolution of the mathematical thought of David Hilbert (1862-1943), where the search for the justification of mathematical foundations can be appreciated. Explaining Hilbert’s thought allows the trainees to naturally see his way of considering mathematics as a tool that allows them to position themselves before any type of statement with a proof that justifies it.

The third objective I had in mind while developing this activity was to help these teachers understand the statement of the incompleteness theorem of Kurt Gödel (1906-1978), which I expected would provoke a dépaysement épistémologique with respect to the conception mentioned above.

These three objectives can all be summarized in my one central goal as a teacher: to present a dimension of mathematics that is more human, fallible, collectively constructed, and with limitations. Thinking of mathematics in this way as educators enables our students to question the discipline.

2.3. Design

The activity was designed to have three stages. The first two objectives were to be achieved in the first stage and the third in the second stage.

**Stage 1.** In this first stage, participants were asked to read chapter eight of the popular book *The Music of the Primes* by du Sautoy [6]. The aim of this reading was for the trainees to be introduced to the historical episode, and at the same time, to be able to appreciate two relevant aspects that took place in that period: David Hilbert’s position with respect to mathematical foundations, and the appearance of Kurt Gödel’s incompleteness theorem. I provide a brief review of this historical episode, following [6], in Appendix A.

**Stage 2.** In the second phase, an excerpt from Gödel’s “On Formally Undecidable Propositions of Principia Mathematica and Related Systems I” was presented to the participants, offering the original 1931 text written in German (Figure 1) and its translation into Spanish by Mosterin [10].
Here is an excerpt from the first paragraph, translated by Meltzer into English in 1992, in which Gödel makes it explicit that it is not always possible to decide on all the mathematical questions that can be formulated:

The development of mathematics in the direction of greater exactness has—as is well known—led to large tracts of it becoming formalized, so that proofs can be carried out according to a few mechanical rules. The most comprehensive formal systems yet set up are, on the one hand, the system of *Principia Mathematica* (PM) and, on the other, the axiom system for set theory of Zermelo-Fraenkel (later extended by J. v. Neumann). These two systems are so extensive that all methods of proof used in mathematics today have been formalized in them, i.e. reduced to a few axioms and rules of inference. It may therefore be surmised that these axioms and rules of inference are also sufficient to decide all mathematical questions which can in any way at all be expressed formally in the systems concerned. It is shown below that this is not the case, and that in both the systems mentioned there are in fact relatively simple problems in the theory of ordinary whole numbers which cannot be decided from the axioms. [9, pages 37-38]

With this second reading, the aim was to bring the trainees closer to the statement of Gödel’s incompleteness theorem, trying to provoke the experience of the *dépaysement épistémologique*. In this way, the trainees could come to question their own conceptions about the possibility of non-demonstration of true sentences. Precisely stating or proving the theorem was beyond the scope of this activity.
3. Results

Below I have transcribed and translated some comments made by the trainees in the forum of this activity. In this selection of comments it is possible to identify the conceptualization of the theorem and indications of the objectives proposed.

Here is what some trainees wrote regarding their conceptualization of the theorem:

[...] what the article, I believe, is proposing is a demonstration that formal mathematical systems are not “complete”. That is to say, there are statements for which their truth value cannot be decided in those same systems. Gödel’s demonstrations, as he himself explains, are constructive, finding statements that can neither be proved nor disproved within a system of axioms. (Trainee 11)

[...] I think what Gödel showed is that in every axiomatic system there are propositions [for] which it cannot be proved whether they are true or false. Axiomatic systems work like a kind of machine which, starting from its initial propositions and following a certain finite number of steps, can incorporate or discard them. A problem occurs when such a “machine” can neither accept nor reject a proposition. Without being mathematically rigorous, one could interpret Gödel’s idea as follows: [if the machine] chooses the proposition P: “This proposition is not true”. In this way, according to the axiomatic system, if the proposition is accepted by the machine, it would be true (leading to a contradiction), while if it is not accepted, there would be a true proposition that the machine does not prove, rendering the axiomatic system inconsistent. (Trainee 2)

That the first objective (“to bring the historical episode closer to the trainees”) was achieved can be seen in comments such as the following:

[...] the debate about the solidity of the foundations of knowledge shows that mathematical knowledge is indeed a social and cultural construction, in which the most abstract concepts arise from concrete problems and that the questioning of such knowledge has as a consequence its deepening and the development of creativity in the search for valid explanations. On the other hand,
when we refer to the history of mathematics, we think of events that are various centuries old, and this case is very close in time [...]. It makes me think of mathematics as a “living” discipline. (Trainee 1)

In a context where everything is change and uncertainty, where people have stopped believing in what they once had faith in, theories emerge that reflect this change in thinking. From the theory of relativity to other mathematical theories that arise from questioning the very functioning of the foundations of the discipline. It was thought that mathematics worked like a perfect machine and that it was capable of proving everything, until problems arose that seemed to be unanswerable. This generated discomfort among many mathematicians who thought they were working in a solid building, until various arguments cast doubt on the foundations of this building. (Trainee 2)

[...] this made me think about the relevance of the history of mathematics. While I was aware of some of these ideas, either through an article or a video, I had never read an account of this kind that shows the evolution of mathematical ideas over time and how they influence different mathematicians, according to the concerns of the time. This historical perspective clearly shows how prejudices, culture, and the technology of the time influence the construction of mathematics. (Trainee 3)

In these comments it can be seen that through exposure to a context previously unknown to them, the trainees were able to perceive other aspects of the construction of the discipline.

Here are some comments related to the second objective (“to make the evolution of Hilbert’s thought explicit”):

[...] it made me think about that conception that students, and to a great extent we as well, have of mathematical knowledge as something completed and unquestionable. (Trainee 1)

[...] many times we perceive the idea in our students that mathematics is a science that provides exact and completed knowledge (ideas of testability and efficacy, seen as formulas or principles). (Trainee 4)
I think the way of thinking about mathematics in teacher education is still Hilbert’s way of thinking about consistency and non-contradiction. (Trainee 5)

His [Gödel’s] proposal shakes the “mathematical building” because what up to that moment was a consolidated science, consistent and without flaws, here it is demonstrated to be not so. I find it fabulous and strange at the same time, that a theory free of contradictions can be obtained but it cannot be demonstrated that there are no contradictions within that theory. The truth is that I was unaware of this whole proposal, and it made me curious to continue reading to see how this story continued, what other findings it generated, besides Turing’s work. (Trainee 6)

In this last comment in particular, trainee 6 shows his surprise at Gödel’s result by referring to the instability caused in the foundations of the discipline. At the same time, this bewilderment leads us to identify indications that the third objective pursued (“to provoke a dépaysement épistémologique”) has been achieved, thanks to Gödel’s incompleteness theorem:

[...] I had always maintained that it is important to present mathematical proofs to justify the validity and give meaning to propositions and now I don’t know what to think... I will have to start studying logic. (Trainee 7)

My head exploded; it was so hard! I’m still not sure I understood it at all [...]. If it is so revolutionary and controversial what he did, that in our own education we continue to study mathematics as the building that rests on firm foundations, and we continue to transmit the idea (or at least we do not deny it) that mathematics explains everything and that everything can be demonstrated from a good axiomatic system. (Trainee 8)

The famous idea of the solid building falls to the point of questioning the existence of sequences of deductions that can simultaneously prove the veracity and falsity of the same result, although this may never happen. Moreover, the possibility that there are true statements according to the theory that are not provable in that theory is even more disconcerting [...]. I believe that Gödel’s work gives us a new conception of what mathematics is, or at least of what it is not. (Trainee 4)
From these comments it can be inferred, in the first place, that there is a questioning of the conception of a mathematics that can prove every true statement. It can thus be confirmed that several of the trainees experienced the *dépaysement épistémologique*, and through that experience, have appreciated more the human dimensions of the discipline:

Personally, I am comforted by the idea that the human dimension of mathematics comes alive in the fact that mathematics is not necessarily perfect, or at least as perfect as it was once believed to be. Moreover, it makes it even more real and interesting. (Trainee 5)

[...] in a way I am “reassured” by this human dimension, beyond what I still must learn, it is reassuring to assume that [mathematics] it is not necessarily perfect. (Trainee 9)

Secondly, the trainees express their point of view as teachers, concerned with planning time, but their empathy for their students is also activated:

[...] reading these texts I thought of several things to bring to the classroom but probably due to lack of time I did not manage to do it [...]. Reflect on the importance of demonstrations and their role in the classroom. To present open problems. [...] make students feel “uncertainty” about mathematics, which I think is what many of us feel when reading these texts. (Trainee 7)

[...] when I read the articles, I felt totally ignorant. [...] reading these articles I imagined many things that could be addressed with the students [...]. Probably very little of this I can carry out. But I think the important thing is to change my mind and think about other types of activities with the students. Just as it has made us reflect, I can imagine what high school students could reflect if we present some of this to them. (Trainee 10)

Changing your mind and being aware that there are other ways of doing things is necessary to be able to change some of the classroom practices. In addition, it is also important to have good examples and ideas to plan our classes. [...] it is important that for the time we devote to it we have good examples, and we can generate challenging and stimulating practices for the students. (Trainee 7)
Thanks to the reading and discussion in the seminar forum, it is possible to affirm that the trainees experienced the *dépaysement épistémologique*. This experience helped them to become aware that mathematics is constructed, and that it is not infallible. At the same time, as they lived through this experience, they demonstrated empathy for their students; the experience helped them think more clearly about the goals they might pursue in activities that incorporate the history of mathematics. To this end, during the following weeks of the seminar, they were guided in the exercise of analyzing the stages of this same activity, allowing them to appropriate a design possibility.

The time that can be dedicated to integrating the history of mathematics into the mathematics class is a current issue given the large amount of content to cover in the courses. However, the trainees’ comments show an openness to the possibility of this integration once they have concrete examples at their disposal.

4. Conclusion

At the end of the seminar, the trainees were asked to make an overall evaluation, considering aspects such as: the quality of the recommended readings, the topics of discussion proposed in the forums and the dynamics of the interaction, the feedback during the evaluations, the proposal of activities, etc. From this overall view of the seminar, I could state that the activity presented in this article was the one that mobilized them the most.

In addition to the three specific objectives that we described in Section 2.2 in relation to the proposed activity on Gödel’s incompleteness result, there were indications that three more global conclusions could also be reached associated with the seminar.

The first of these global conclusions is that indeed, having identified a given conception, it is possible to design an activity that triggers an experience of *dépaysement épistémologique*. This experience can provoke surprise, confusion, instability, and awareness of the lack of mathematical and historical knowledge.

The second overall result, considering that the targeted audience consists of practicing teachers, is the activation of empathy. Experiencing the desired *dépaysement épistémologique* led several teachers to think spontaneously of situations in which they could involve their students with different purposes.
The third result is related to the idea that the role of the history of mathematics is all too often reduced to telling an anecdote or introducing a topic. Thanks to the design of this activity, the trainees experienced the possibility of using history as a tool for learning mathematics by highlighting its human dimension. This provides the trainees with a positive outlook on the possibility of integrating history into their courses.

At this point, I must say that this experience does not allow me to affirm that the use of the history of mathematics produces measurable changes in the professional skills of mathematics teachers, nor in student learning. Analysis that could lead to such conclusions is beyond the scope of this article. However, I have indeed observed teachers experience a process of questioning about the nature of mathematics, and I have seen that the instability caused by this questioning induced an empathetic reflection towards their own students. In this way and following Guillemette [11, 12], it is possible to affirm that one of the consequences of dépaysement épistémologique on teachers is that it can activate empathy.

More generally, how best and most effectively to use the history of mathematics in a given teaching context is not often obvious, and achieving a dépaysement épistémologique in one’s students takes time of study and planning. However, the proposal to plan one activity of this kind per year could be a modest objective—in terms of time to invest—and the result can contribute to the students’ awareness of the human dimension of mathematics.

References


A. Hilbert and Gödel’s Incompleteness Theorem

David Hilbert (1862-1943) was undoubtedly the most influential mathematician of the first half of the twentieth century. He published in 1899 the famous *Grundlagen der Geometrie* (Foundations of Geometry), which had an immediate impact on contemporary mathematics. A year later, in 1900, Hilbert gave one of the central lectures at the Second International Congress of Mathematics held in Paris, in which he presented ten of the twenty-three open problems in mathematics from his well-known list “Hilbert’s twenty-three problems”. This mathematician’s studies of non-Euclidean geometries raised a disturbing question: are we sure that we can never prove that a statement is both true and false? Hilbert was convinced that it would be possible to use mathematical logic to show that the discipline did not contain contradictions of this kind, and the statement of his second problem challenged the mathematical community to prove it.

Several years later, on September 7, 1930, Hilbert was named favorite son of Königsberg, his hometown. That year he had also left the chair in Göttingen, and he ended his thank-you speech with an appeal to all mathematicians: *Wir müssen wissen. Wir werden wissen.* (We must know. We will know). However, during a lecture delivered a short distance away, at the University of Königsberg, Kurt Gödel (1906-1978), a twenty-five-year-old Austrian logician, showed that it was impossible to use the axioms of mathematics to prove that those same axioms would not lead to contradictions. Gödel had shown that you can get a theory free of contradictions, but you cannot show within that theory that there are no contradictions.
In 1900, Hilbert had declared that in mathematics there is nothing that is impossible to know, and thirty years later, Gödel proved that ignorance is an integral part of mathematics. Hilbert learned of Gödel’s theorem a few months after his speech in Königsberg.

Gödel’s doctoral thesis contained a second statement: if the axioms of mathematics are consistent, then there will always be true statements that cannot be formally proved from those axioms. No matter how many new axioms are added to the foundations of mathematics, there will always remain some true statement impossible to prove. This result took the name of Gödel’s incompleteness theorem: any system consisting of axioms is necessarily incomplete, i.e., there will always be true statements that cannot be deduced from the axioms. Gödel was able to show that for any choice of axioms there will always be true statements that cannot be proved.

Gödel’s incompleteness theorem modified the way of reasoning of some mathematicians: if there are problems so difficult to solve, perhaps it is because they are simply unprovable with the logical instruments and with the axioms that apply.

Gödel did not question the truth of what was already demonstrated. His theorem demonstrates that mathematical reality is not reduced to the deduction of theorems from axioms. It is necessary that the incessant work of construction of the mathematical edifice be accompanied by a continuous evolution of the foundations on which the mathematical edifice is based. Unlike the formal nature of the rules for the construction of the edifice, the evolution of the foundations must be based on the intuitions of mathematicians about the choice of axioms that, in their opinion, can provide a better description of the world of mathematics.

Many mathematicians felt satisfaction in interpreting in Gödel’s theorem a confirmation of the superiority of the mind over the mechanistic spirit brought about by the industrial revolution.

In this brief review it is possible to identify some remarkable historical elements, such as: the international congress of mathematics, David Hilbert’s formalist thinking on mathematical foundations and Kurt Gödel’s results. A group discussion could emphasize some of these elements, depending on the facilitator’s wishes.