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Ramanujan Cab Numbers: A Recreational Mathematics Activity

Cover Page Footnote

I thank Ma (Kalyani Banerjee) for teaching me mathematics and Baba (Tarakeswar Banerjee) for instilling in me the love of mathematics.

Ramanujan Cab Numbers: A Recreational Mathematics Activity

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Synopsis

In this paper, I introduce teaching activities about Ramanujan's cab numbers and related software that can inspire people and help them enjoy these beautiful mathematical creations.

Srinivasa Ramanujan's story is inspiring and gives us hope. He rose from the poorest of backgrounds and rose to rarefied heights in mathematics, all within a very short life.

One inspiring story about him concerns the Ramanujan taxicab number 1729 and the sequence of numbers n defined by the equation:

$$n = x^3 + y^3 = p^3 + q^3$$

where x, y, p, and q are distinct positive integers. 1729 is the smallest number that can be expressed as the sum of two cubes in two different ways:

$$10^3 + 9^3 = 12^3 + 1^3 = 1729.$$

The story of Ramanujan and all the people who helped him, suggest to us how the pursuit of mathematics can bring out redeeming qualities in humans.

In this paper, I introduce teaching activities and software that can inspire people and help them enjoy these beautiful mathematical creations. I hope these resources and activities can inspire minorities and students in developing nations all over the world to better appreciate the beauty in mathematics. Ramanujan would certainly have wanted all of us to appreciate the inherent beauty in these numbers.

1. Introduction

Mathematics gave Srinivasa Ramanujan hope, in a life otherwise filled with poverty and grief. His life and his mathematics can give all of us hope and inspiration.

Born in an India which was ruled ruthlessly by the British, Ramanuijan's genius was recognized by an Englishman, Professor G.H. Hardy. Ramanujan had little formal education. He was working as a clerk in the city of Chennai in India, when his mathematical talents were brought to the attention of his supervisor S. Narayana Aiyar. S. Narayana Aiyar was a patron of mathematics and was a member of the Indian Mathematical Society. He encouraged him and in turn communicated with Francis Spring and Alfred Gibbs Bourne. They encouraged Ramanujan to get in touch with prominent mathematicians in England. Of the many mathematicians he corresponded with, only Professor Godfrey Harold Hardy (at the University of Cambridge) responded. Professor Hardy along with his colleague Professor Littlewood, recognized the genius in Ramanujan. They arranged to bring Ramanujan to Cambridge, where they would start an immensely productive collaboration.

This story suggests to us the redeeming qualities in humans. Littlewood and Hardy were very receptive and without their kindness and persistence, the world would not have come to know of Ramanujan.

Ramanujan is well known for his approach to mathematics which put mathematical form and beauty centre stage. He also had a great fondness for numbers. A famous incident in his life involved a taxicab. Briefly, Ramanujan lay ill in a hospital far from home in London. G.H. Hardy visited him one day and remarked that he had just been in a taxicab numbered 1729 and that this was a very unremarkable number. Upon hearing this, Ramanujan remarked that actually 1729 is quite a remarkable and beautiful number. Indeed, 1729 is the smallest number that can be expressed as the sum of two cubes in two different ways:

$$10^3 + 9^3 = 12^3 + 1^3 = 1729.$$

Inspired by this story, the **Ramanujan taxicab numbers** are defined as those numbers n that can be written as:

$$n = x^3 + y^3 = p^3 + q^3, \tag{1}$$

where x, y, p and q are distinct positive integers.

The next Ramanujan taxicab number is:

$$4104 = 9^3 + 15^3 = 2^3 + 16^3.$$

You can read more on this story and on taxicab numbers online at the relevant `Wikipedia` entry (https://en.wikipedia.org/wiki/Taxicab_number), on Bill Butler's website *Durango Bill's Ramanujan Numbers and The Taxicab Problem* (http://www.durangobill.com/Ramanujan.html) and Christian Boyer's website *New Upper Bounds for Taxicab and Cabtaxi Numbers* (http://www.christianboyer.com/taxicab/), all last accessed on July 24, 2022.

2. Activities

In this section, I outline some recreational activities for the reader.

2.1. Reading and viewing for background

As a starter activity, students or readers at any level of education may like to watch the short video by Brady Haran, titled *1729 and Taxi Cabs - Numberphile*, available at https://youtu.be/LzjaDKVC4iY (last accessed on July 24, 2022). You can then watch the 2015 movie *The Man Who Knew Infinity* directed by Matthew Brown, which discusses the life of Ramanujan in Cambridge. If you are a student, ask your teacher to check out a DVD from the local library. (See the `Wikipedia` article on the movie.)

Another useful activity is to read the book *The Man Who Knew Infinity: A Life of the Genius Ramanujan* by Robert Kanigel [3].

For more ideas on what other movies students can watch, instructors can check out the list of movies about mathematics provided by Oliver Knill at https://people.math.harvard.edu/~knill/index.html, last accessed on July 24, 2022. Movies with some math in them can encourage students to develop a love for mathematics, and think about areas in mathematics they would like to work on.

2.2. Programming activities

In this section, I outline some programming activities. All code and instructions for generating Ramanujan taxicab numbers on a computer are available at the following repositories:

https://github.com/neelsoumya/ramanujan_number_generator

https://bitbucket.org/neelsoumya/ramanujan_number_generator/

There is a Python program which can be used to generate and plot Ramanujan numbers. A plot of the Ramanujan numbers on the number line is shown in Figure 1. The reader has to install the Python and R programming languages on the computer and download or clone the repository. Detailed installation instructions are available in the repositories mentioned above. Briefly, the student requires a modern computer/laptop with an internet connection and needs to install R, which can be obtained at https://www.r-project.org/, and Python, which can be obtained at https://www.python.org/downloads/, both last accessed on July 24, 2022.

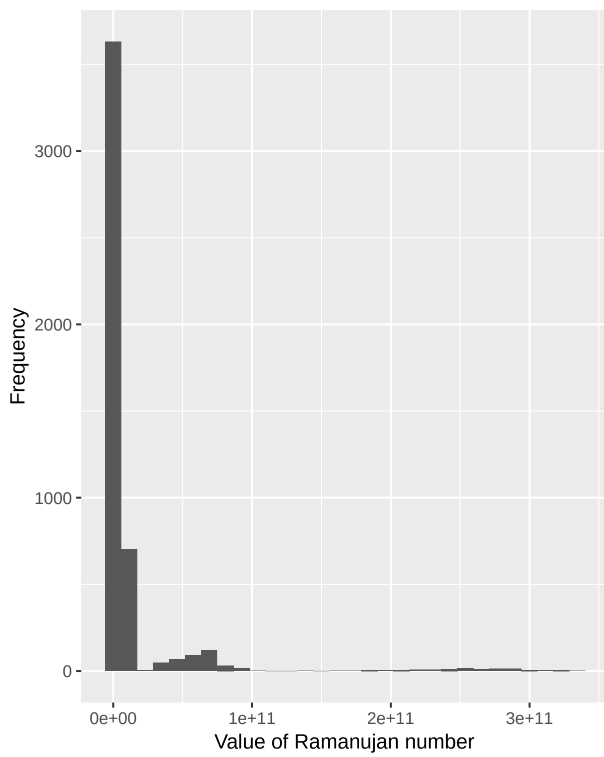

Figure 1: Histogram of Ramanujan taxicab numbers generated using the Python and R programs.

The following command can be used to generate the Ramanujan taxicab numbers:

`python3 ramanujan_test_v1.py`

The program will save the Ramanujan taxicab numbers as a tab separated file. The histogram of the numbers can then be generated by running the following command from the terminal:

`R - -no-save < analysis.R`

You can try to find the next three Ramanujan taxicab numbers. Can you try to find the number of Ramanujan taxicab numbers which are less than 10^5? As an advanced activity (for undergraduate students), I invite the reader to make the software more efficient and generate more Ramanujan taxicab numbers.

2.3. Activities using a computer

Activity 1. Let us go back to the first Ramanujan number:

$$10^3 + 9^3 = 12^3 + 1^3 = 1729.$$

Can you think of solutions for the following equation for real values of x?

$$10^x + 9^x = 12^x + 1^x.$$

We can rearrange the equation to have the following form:

$$12^x + 1^x - 10^x - 9^x = 0.$$

I invite the reader to solve this using WolframAlpha. A link to the solution is available here: https://www.wolframalpha.com/input/?i=12%5Ex+%2B+1%5Ex+-+10%5Ex+-++9%5Ex

The expression above has two real roots, $x = 0$ and $x = 3$.

A plot of this expression is shown in Figure 2 on the next page.

The expression has a minimum of -60.350 achieved at the value of $x \approx 2.5794$.

I propose calling this number (2.5794), that minimizes the expression, the *Komalatammal-Janakiammal number*, in memory of Ramanujan's mother and wife, respectively.

Finally, the reader can take note of the following asymptotic relationship:

$$\lim_{x\to-\infty} (12^x + 1^x - 10^x - 9^x) = 1.$$

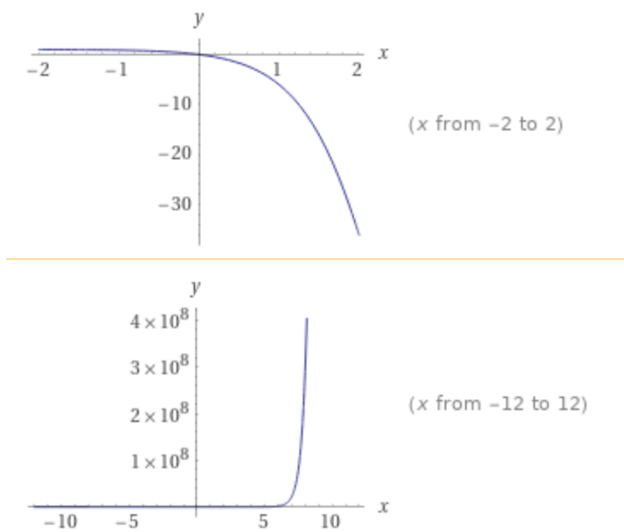

Figure 2: Plot of Ramanujan expression $12^x + 1^x - 10^x - 9^x$ as the values of x are changed. The expression has two real roots, $x = 0$ and $x = 3$.

Activity 2. The second Ramanujan number is:

$$15^3 + 9^3 = 16^3 + 2^3 = 4104.$$

We can generalize this equation to have the following form:

$$15^x + 9^x = 16^x + 2^x.$$

Rearranging we have

$$15^x + 9^x - 16^x - 2^x = 0$$

for all real values of x.

We can again solve this equation using WolframAlpha: https://www.wolframalpha.com/input/?i=16%5Ex+%2B+2%5Ex+-+15%5Ex+-++9%5Ex

This equation also has two real roots, $x = 0$ and $x = 3$. Can you find the minimum value attained by this expression? Can you find a series expansion for it?

Activity 3. We defined the Ramanujan taxicab numbers by the following equation:

$$a^3 + b^3 = c^3 + d^3,$$

where a, b, c and d are distinct positive integers.

We can generalize this notion of Ramanjuan number to the following equation:

$$a^x + b^x = c^x + d^x,$$

where x is a real number (and as before a, b, c and d are distinct positive integers).

This equation has at least two roots: $x = 0$ and $x = 3$. Can you find if this equation can have any other solutions?

The equation can be further generalized to the following form:

$$a^p + b^q = c^r + d^s,$$

where a, b, c, d, p, q, r and s are real numbers. As hopefully the reader will see, the possibilities are limitless and like a kaleidoscope, Ramanujan's simple number can lead into almost infinite possibilities.

2.4. Activity for extending to inequalities

Here I introduce some activities for high school students. Let us again go back to the definition of the Ramanujan taxicab numbers:

$$x^3 + y^3 = p^3 + q^3, \tag{1}$$

where x, y, p and q are distinct positive integers. Can you think of some inequalities that can be derived from this?

To make a start, let us recall the following relationship:

$$\frac{a+b}{2} \geq \sqrt{ab}. \tag{2}$$

This is commonly phrased as "the arithmetic mean is greater than equal to the geometric mean". We will briefly prove this. Let us square both sides:

$$(\frac{a+b}{2})^2 \geq ab.$$

Simplifying we get

$$\begin{aligned} (a+b)^2 &\geq 4ab. \\ a^2 + b^2 + 2ab &\geq 4ab. \end{aligned}$$

Simplifying even further we get

$$a^2 + b^2 - 2ab \geq 0.$$

Completing the square we get

$$(a - b)^2 \geq 0,$$

which is true since the square of a real number must be greater than 0. This proves the inequality:

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

We will rewrite this as:

$$a + b \geq 2\sqrt{ab}.$$

Let us now compare it to the equation (1) for Ramanujan taxicab numbers. Can you now think of how to turn this equation into an inequality? Compare it to the equation before it. Think of a as x^3 and b as y^3. Then we can rewrite the left hand side of the equation $x^3 + y^3$ as

$$x^3 + y^3 \geq 2\sqrt{x^3 y^3}$$

We have derived our first inequality from the equation for Ramanujan taxicab numbers. Can you extend this even further? Let us look at the left hand side of the above equation. Can we substitute it for something else? How about using the relationship $x^3 + y^3 = p^3 + q^3$?

We can now rewrite the left hand side of the inequality

$$x^3 + y^3 \geq 2\sqrt{x^3 y^3}$$

as

$$p^3 + q^3 \geq 2\sqrt{x^3 y^3}.$$

Can you think of some more inequalities that can be derived from these relationships? These exercises can be used as fun activities for school students.

Let us think of another inequality. Let us rewrite Ramanujan's equation (1) as the following:

$$p^3 + q^3 = x^3 + y^3.$$

Now can you make another inequality out of this? Let us look at the inequality (2) again and substitute $a = p^3$ and $b = q^3$. This gives us:

$$p^3 + q^3 \geq 2\sqrt{p^3 q^3}.$$

Let us again compare this to the rewritten Ramanujan equation $p^3 + q^3 = x^3 + y^3$. In particular look at the left hand side of the inequality and the left hand side of the Ramanujan equation: both are $p^3 + q^3$. Let us substitute that with $x^3 + y^3$.

This means we can rewrite the inequality:

$$p^3 + q^3 \geq 2\sqrt{p^3 q^3}$$

as

$$x^3 + y^3 \geq 2\sqrt{p^3 q^3}.$$

Hence out of Ramanujan's equation (1), we have derived two inequalities:

$$\begin{aligned} x^3 + y^3 &\geq 2\sqrt{p^3 q^3} \\ p^3 + q^3 &\geq 2\sqrt{x^3 y^3} \end{aligned}$$

I propose calling these two inequalities the *Ramanujan taxicab inequalities*.

Concretely, what do these inequalities mean? Let us substitute the values of the first Ramanujan taxicab numbers into these equations. Remember this means that we are using $x = 10$, $y = 9$, $p = 12$ and $q = 1$.

The first inequality then becomes

$$10^3 + 9^3 \geq 2\sqrt{12^3 1^3}.$$

Simplifying we get:

$$1729 \geq 83.14.$$

As you can see, the possibilities are endless. As a fun activity, can you try to find other hidden inequalities?

2.5. Activities to detect patterns in taxicab numbers

These are activities for advanced students of mathematics, for those who are in high school or studying for an undergraduate degree. We will try to find some simple patterns in the Ramanujan taxicab numbers.

Let us look at the first taxicab number: 1729. Can you try to factorize this number? i.e. try to find two numbers, which, if multiplied together, will give 1729.

Try a few numbers starting from 2, 3, 4 onwards and you can use a calculator on your computer. After a few trials, you will come across the number 19.

This is a factor, since $19 \times 91 = 1729$. Can you try to spot a pattern in the above equation? Stare at the equation for a few minutes ....

And soon you will notice that 19 and 91 are related! When the digits of 91 are reversed, you get 19.

Can you find if there are other Ramanujan numbers which can be written as the product of two numbers which are reverses of one another? Start with 4104 and work your way up to other Ramanujan numbers. Use WolframAlpha as shown in the examples below.

Let us look at the equation $19 \times 91 = 1729$ again and rewrite it as the following:

$$(10 \times 9 + 1)(9 + 10) = 1729.$$

We can generalize this equation to:

$$(10 \times a + 1)(a + 10) = 1729,$$

and simplify:

$$10a^2 + 101a = 1719.$$

Can you find out how many solutions this equation has? Let us ask WolframAlpha to find solutions for this equation: https://www.wolframalpha.com/input/?i=10+*+a%5E2+%2B+101+*a+-+1719

There are two solutions: $a = 9$ and $a = 19.1$.

Let us look at the equation $19 \times 91 = 1729$ again and try to break each factor as two sums:

$$(9 \times 10 + 1)(9 + 10) = 1729.$$

Can you observe a pattern in the above equation? The numbers 9 and 10 recur in the equation. Hence we can write it as the following general equation:

$$(p \cdot q + 1)(p + q) = 1729,$$

where $p = 9$ and $q = 10$. Can you find solutions of the above equation? Let us use WolframAlpha for this: https://www.wolframalpha.com/input/?i=%28p*q++%2B1%29%28p++%2Bq%29+%3D+1729

It turns that there are many solutions for this equation, including $p = 1729$ and $q = 0$. 1729 itself is one of the roots!

That was fun! Team up with your best friend and try to find more such patterns.

Let us look at the equation $19 \times 91 = 1729$ again and rewrite it one more time as the following:

$$(10 \times 9 + 1)(9 + 10) = 1729$$

We can write this more generally as:

$$(10 \times a + b)(10 \times b + a) = 1729.$$

Can you find how many solutions the above equation has? Once again you can use WolframAlpha for this. This link will give you the answer: https://www.wolframalpha.com/input/?i=%2810a+%2B+b%29%2810b+%2B+a%29+%3D+1729

2.6. Activities for generalizing Ramanujan taxicab numbers

This is an advanced activity for students studying mathematics in undergraduate or postgraduate degrees. The student can think about other more complex relationships like:

$$x^3 + y^3 + z^3 = p^3 + q^3 + r^3,$$

where x, y, z, p, q and r are distinct positive integers.

What would happen if you relaxed some of the restrictions and also allowed negative numbers (or complex numbers)?

Let us take the case of what would happen if you allowed the numbers to be negative in the equation (1) for taxicab numbers. Allowing negative numbers gives us one possible solution:

$$91 = 4^3 + 3^3 = 6^3 + (-5)^3.$$

(This came up in the NumberPhile video mentioned in Section 2.1.) Here is another solution:

$$152 = 3^3 + 5^3 = 6^3 + (-4)^3.$$

Can you find other solutions? Can you modify the Python computer program (see Section 2.2) to automatically search for these solutions?

Let us relax the restrictions a bit more. Experiment with these equations and let your imagination go wild!

For example, you could come up with an equation like the following:

$$n = x^7 + y^7 = a^3 + b^3 + c^3,$$

where x, y, z, a, b and c are integers (negative numbers are allowed) and need not be distinct. Can you find a solution for this equation? One solution is:

$$127 = 2^7 + (-1)^7 = 5^3 + 1^3 + 1^3.$$

As a fun activity, experiment with these and come up with more equations. Here are some relations to get you started:

$$343 = 6^3 + 2^7 + (-1)^3 = 7^3.$$

$$216 = 4^3 + 5^3 + 3^3 = 6^3.$$

Let us now look at another equation:

$$3^x + 4^x + 5^x = 6^x.$$

Can you try to find a solution such that x is a positive integer? Do you think $x = 2$ is a solution? What about $x = 3$?

You can also generalize the concept even further to have relationships of the following form:

$$x^3 + y^3 = p^3 + q^3 = r^3 + s^3,$$

or include higher powers:

$$x^5 + y^5 = p^5 + q^5.$$

As of 2022, we have not yet found distinct positive integer solutions for the equation shown above. (See the `Wikipedia` article on generalized taxicab numbers for more details: https://en.wikipedia.org/wiki/Generalized_taxicab_number, last accessed on July 24, 2022.) Can you accept the challenge and try to find a solution?

2.7. Reflection activity

This activity is for advanced students studying for an undergraduate degree. They can reflect on the role of modern computing tools and computers in the pursuit of mathematics.

Activities include reading about the Ramanujan machine [6], an automated program to find beautiful mathematical expressions in the style of Ramanujan; you can learn more about it at http://www.ramanujanmachine.com/, last accessed on July 24, 2022.

Students can also reflect on what would be the role of computers in mathematics in the future. Can intelligent machines and humans co-create mathematics? What does this mean for the myth of the lone genius mathematician working in complete isolation with only pen and paper?

Finally, students can read Ramanujan's "lost" notebooks, available in digitized format at:

http://www.math.tifr.res.in/%7Epubl/nsrBook1.pdf

http://www.math.tifr.res.in/%7Epubl/nsrBook2.pdf

http://www.math.tifr.res.in/%7Epubl/nsrBook3.pdf

all last accessed on July 24, 2022.

3. Discussion

As a person who was always afraid of mathematics and was a minority, when I moved to USA to pursue my PhD, I took solace and inspiration from Ramanujan's life. I hope that doing the activities outlined in this work will allow the readers, especially minorities, to feel a kinship with Ramanujan, who overcame enormous obstacles in his short life.

My first exposure to Ramanujan's genius was when I was in middle school. Since then I have always been fascinated by Ramanujan's work. In this work, I introduced some recreational mathematical activities based on Ramanujan's number.

These recreational activities can also help tackle imposter syndrome. Many people may feel that this is not "real" mathematics. However it is the opinion of the author that if you can enjoy it on a lazy Sunday afternoon, it is mathematics and is worth pursuing. This may run counter to advice given by G.H. Hardy in his book *A Mathematician's Apology* [2], but I believe that mathematics, if enjoyed (e.g. recreational mathematics), can also help in healing people who have suffered from trauma in the past; see for example [8].

The computer program that accompanies this manuscript currently performs an exhaustive brute force search. At the time of writing this, the largest numbers that can be generated is still being searched for by the program. I hope the readers will continue the search for ever larger Ramanujan numbers.

This work may help build richer instructional resources to teach basic mathematics in schools, like the *Math-U-See* program, available at https://mathusee.com, last accessed on July 24, 2022.

Our search for Ramanujan taxicab numbers is not complete and will never be complete. This incompleteness has a beauty, similar to the Japanese philosophy of *wabi-sabi*, which has remained under-appreciated in mathematics [4, 5]. *Wabi-sabi* emphasizes the beauty in creations that are imperfect, impermanent, and incomplete.

Almost a century ago, Jacques Hadamard in his book *The Mathematician's Mind: The Psychology of Invention in the Mathematical Field* [1], remarked that mathematicians frequently pursue beauty in their quest to find solutions to mathematical problems. Aesthetic considerations remain foremost in the minds of many of the greatest mathematicians [7], including Ramanujan. We may never know what Ramanujan saw in these equations. However all of us can try to find in our own way, some beauty in these equations.

Can you implement a faster approach? One that looks for beauty in numbers? What wonders would some of these numbers hold? Ramanujan certainly would want to know.

The story of Ramanujan and all the people who helped him, suggest to us how the pursuit of mathematics can bring out redeeming qualities in humans.

I hope these resources and activities can inspire minorities and students in developing nations all over the world to better appreciate the beauty in mathematics. Ramanujan would certainly have wanted all of us to appreciate the inherent beauty in these numbers.

Acknowledgements. I thank Ma (Kalyani Banerjee) for teaching me mathematics and Baba (Tarakeswar Banerjee) for instilling in me the love of mathematics. I thank Mrs. Joyeeta Ghose and Dr. Clarice Sonali Ghose for helpful discussions. This work is dedicated to the memory of Ramanujan's mother Komalatammal and his wife Janakiammal.

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