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One Theorem, Two Ways: A Case Study in Geometric Techniques

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One Theorem, Two Ways: A Case Study in Geometric Techniques

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Synopsis

If the three sides of a triangle $\mathrm{AB\Gamma}$ in the Euclidean plane are cut by points H on AB, Θ on $\mathrm{B\Gamma}$, and K on $\Gamma\mathrm{A}$ cutting those sides in the same ratio:

$$\mathrm{AH} : \mathrm{HB} = \mathrm{B}\Theta : \Theta\Gamma = \Gamma\mathrm{K} : \mathrm{KA},$$

then Pappus of Alexandria proved that the triangles $\mathrm{AB\Gamma}$ and $\mathrm{H\Theta K}$ have the same centroid (center of mass). We present two proofs of this result: an English translation of Pappus's original synthetic proof and a modern algebraic proof making use of Cartesian coordinates and vector concepts. Comparing the two methods, we can see that while the algebraic proof gets to the heart of the matter more efficiently, the synthetic proof does a better job of revealing hidden aspects of the geometric configuration. Moreover, as Pappus presents it, the synthetic proof provides a real element of surprise and a sense of discovering unexpected connections. We conclude with some general observations about synthetic versus algebraic techniques in geometry and in the teaching and learning of mathematics.

1. Introduction

Book VIII of the *Mathematical Collection* (*Synagogē*) of Pappus of Alexandria (ca. 290–ca. 350 CE) is devoted to a geometric treatment of various problems from mechanics. In the third numbered proposition from that book, after recalling ideas about centers of mass drawn from Archimedes' *Equilibria of Planes I*, Pappus proves the following interesting statement about centroids (centers of mass) of triangles to begin his exposition.

Theorem 1. *Let* $\mathrm{AB\Gamma}$ *be a triangle in the Euclidean plane. If the sides are cut in the same proportion by points* $\mathrm{H}, \Theta, \mathrm{K}$ *so that*

$$\mathrm{AH} : \mathrm{HB} = \mathrm{B\Theta} : \Theta\Gamma = \Gamma\mathrm{K} : \mathrm{KA},$$

then the triangles $\mathrm{AB\Gamma}$ *and* $\mathrm{H\Theta K}$ *have the same centroid (center of mass).*

A particular case of the situation described in the theorem is given in Figure 1. In physical terms, the triangle $\mathrm{AB\Gamma}$ may be visualized as a thin plate made of material with constant mass density. The centroid is then the same as the center of mass of the plate. In preparing the figure, we have made

$$\mathrm{AH} : \mathrm{HB} = \mathrm{B\Theta} : \Theta\Gamma = \Gamma\mathrm{K} : \mathrm{KA} = 1 : 3,$$

so the points $\mathrm{H}, \Theta, \mathrm{K}$ satisfy Pappus's hypotheses. The point Z is the centroid of both triangles.

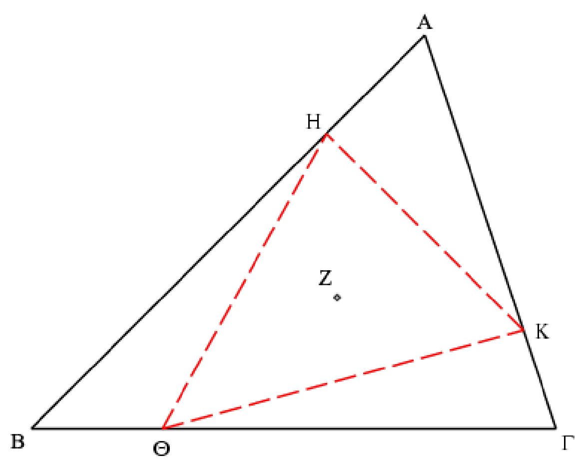

Figure 1: The situation of Theorem 1.

Pappus does not give an attribution for this result. However, it is possible that he is making use of a section of the *Mechanica* of Heron of Alexandria (ca. 10–ca. 70 CE) that is now lost. The surviving text was preserved only in an Arabic translation made in the ninth century by Qusta Ibn Luqa and it is not known whether the Arabic version represents the complete text. In fact, the first part of Pappus's proof is essentially the proof Heron gives for the location of the centroid of a triangle in Proposition 35 of Book II of the *Mechanica* (see for instance [1, page 111]).

The Greek text of Pappus's proof and a more or less literal Latin translation of the Greek are available in Friedrich Hultsch's edition of the *Mathematical Collection*, [4, pages 1034–1041]. Since neither version is very accessible for many current readers, and no translation of Book VIII as a whole into English has been published as of this writing, we give our own translation of Pappus's proof in §2.

To make the argument clearer for modern readers, we use the modern fraction notation that Hultsch used for ratios in parts of his Latin version. In other words, this is not exactly a literal translation of Pappus's text, which uses the usual verbal formulas found in Greek mathematical texts to express ratios and proportions. There is, of course, some anachronism involved in writing Pappus's ratios as fractions. However, in this particular case, we claim that the anachronism is essentially a matter of notation and not a significant difference in the mode of reasoning. Every operation with the fractions is an instance, in fact, of one of the transformations of ratios and proportions codified in propositions from Book V of Euclid's *Elements* (expressed in words with the terms *alternando, componendo, convertendo, invertendo* — we explain these in footnotes when they occur). In other words, the fractions are not being treated as ways to represent numbers.

Following this, in §3, we give a second, modern proof of the same result using Cartesian coordinates. One obvious difference here is the relative simplicity of this second proof compared with Pappus's proof. As is often true, the use of algebra makes the result "fall out" with almost no effort. By comparing the two proofs, in §4, we make some observations about what is gained or lost with each of these techniques of geometric proof.

2. A translation of Pappus's proof

In this section, we give our version of Pappus's proof, following the notation of Hultsch's Latin version in [4]. This is a rather close translation, but we have sometimes rearranged sentences for greater clarity and added a few comments (shown in square brackets) and footnotes giving reasons for claims. We have used colors and different line styles in the accompanying figures for greater legibility, and we have also written the names of the points using capital Greek letters rather the lower case letters that Hultsch used in his Latin translation of the proof.

Pappus's proof of Theorem 1 refers to the diagram reproduced in Figure 2 below. We pick up after the statement of the theorem in [4].

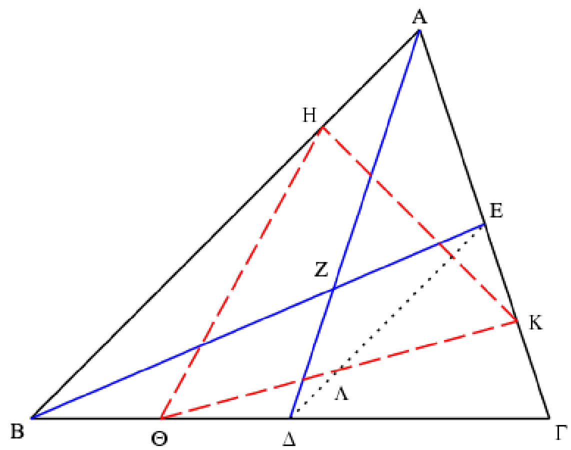

Figure 2: Pappus's main figure for Theorem 1.

Proof: Let BΓ and ΓA be bisected by the points Δ and E and let AΔ and BE be joined. We claim that the centroid of the triangle ABΓ is the point Z [i.e. the intersection of AΔ and BE]. For if the triangle is placed on a perpendicular plane containing the line AΔ, then it will not tend toward either side,[1] since the triangle ABΔ is equal [in area] to the triangle AΓΔ. But similarly if the triangle is placed on a perpendicular plane containing the line BE, then it will not tend toward either side, since the triangles ABE and ΓBE are equal. So if the triangle is placed along either of the lines AΔ or BE, then equilibrium will be maintained, and the common point Z of those two lines will be the centroid. Moreover, it is clear that $\mathrm{AZ} = 2\mathrm{Z}\Delta$ and $\mathrm{BZ} = 2\mathrm{ZE}$.[2] Also,

$$\frac{\Gamma\mathrm{A}}{\mathrm{AE}} = \frac{\mathrm{AB}}{\Delta\mathrm{E}} = \frac{\mathrm{BZ}}{\mathrm{ZE}} = \frac{\mathrm{AZ}}{\mathrm{Z}\Delta},$$

[1]That is, it will "balance" along that line.

[2]The standard fact that the centroid cuts the medians in the ratio 2 : 1 is proved in Proposition 35 in Book II of Heron's *Mechanica*. Since Archimedes also used this fact several times in other works, it is often taken to be a result of his from a hypothetical original version of the *Equilibria of Planes*. Later in the argument, Pappus will also use the fact that the point cutting any one of the medians of a triangle in the ratio 2 : 1 (with the larger part toward the vertex and the shorter part toward the opposite side) must be the centroid of that triangle to conclude his main argument.

since the triangles ΔZE and AZB are similar, as are the triangles E$\Delta\Gamma$ and ABΓ.[3]

Let the line ΔE intersect the line ΘK in the point Λ. By composite ratios

$$\frac{B\Theta}{\Theta\Gamma} = \frac{\Theta B}{\Delta\Theta} \cdot \frac{\Delta\Theta}{\Theta\Gamma},$$

and by hypothesis $\frac{B\Theta}{\Theta\Gamma} = \frac{\Gamma K}{KA}$.[4] From this it follows, *componendo*, that $\frac{B\Gamma}{\Gamma\Theta} = \frac{\Gamma A}{AK}$.[5] Hence, $\frac{\Delta\Gamma}{\Gamma\Theta} = \frac{EA}{AK}$ by taking halves, and *convertendo*, $\frac{\Delta\Gamma}{\Delta\Theta} = \frac{EA}{EK}$.[6] Moreover, $\Delta\Gamma = B\Delta$ and $EA = \Gamma E$,[7] so $\frac{B\Delta}{\Delta\Theta} = \frac{\Gamma E}{EK}$. Hence, *componendo*, $\frac{B\Theta}{\Delta\Theta} = \frac{\Gamma K}{EK}$. By composite ratios,

$$\frac{B\Theta}{\Theta\Gamma} = \frac{B\Theta}{\Delta\Theta} \cdot \frac{\Delta\Theta}{\Theta\Gamma} = \frac{\Gamma K}{EK} \cdot \frac{\Delta\Theta}{\Theta\Gamma},$$

or, since by hypothesis $\frac{B\Theta}{\Theta\Gamma} = \frac{AH}{HB}$,

$$\frac{AH}{HB} = \frac{\Gamma K}{EK} \cdot \frac{\Delta\Theta}{\Theta\Gamma}.$$

But, as will be shown in the following Lemma 1, it is also true that

$$\frac{\Delta\Lambda}{\Lambda E} = \frac{\Gamma K}{EK} \cdot \frac{\Delta\Theta}{\Theta\Gamma}.$$

Therefore, $\frac{AH}{HB} = \frac{\Delta\Lambda}{\Lambda E}$. In addition, AB and ΔE are parallel [as noted above] and the lines AΔ and BE intersect in the point Z. Therefore, H, Z, Λ are collinear (as will be shown later in Lemma 2).

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[3]Both of these claims follow because the line ΔE is parallel to the side AB, a fact that follows from Proposition 2 in Book IV of the *Elements* of Euclid. Apparently, Pappus expects his readers to know this so no explicit reference is needed.

[4]This was added in the Latin version by Hultsch; it does not appear explicitly in the Greek.

[5]In classical terms, if $A : B :: C : D$, *componendo*, the equivalent proportion $A + B : B :: C + D : D$ is produced.

[6]Again, in classical terms, if $A : B :: C : D$, assuming $A > B$ and $C > D$, *convertendo*, the equivalent proportion $A : A - B :: C : C - D$ is produced.

[7]This is by construction since Δ and E are the midpoints of the sides they lie on.

Because of the parallels,[8] it is true that $\frac{\mathrm{BZ}}{\mathrm{ZE}} = \frac{\mathrm{HZ}}{\mathrm{Z\Lambda}}$. Since, as we showed above, $\mathrm{BZ} = 2\mathrm{ZE}$,[9] it is also true that $\mathrm{HZ} = 2\mathrm{Z\Lambda}$. Lemma 1 also shows that $\mathrm{K\Theta}$ is bisected by Λ. Therefore Z is also the centroid of the triangle $\mathrm{H\Theta K}$.[10]
□

We now show the claims whose proofs were deferred [in the argument above].

Lemma 1. *Let* $\frac{\Gamma\Delta}{\Delta\Theta} = \frac{\Gamma\mathrm{E}}{\mathrm{EK}}$ *and join* $\Delta\mathrm{E}$ *and* $\mathrm{K\Theta}$*, meeting in* Λ*. Then* $\Theta\Lambda = \Lambda\mathrm{K}$ *and*

$$\frac{\Delta\Lambda}{\Lambda\mathrm{E}} = \frac{\Delta\Theta}{\Theta\Gamma} \cdot \frac{\Gamma\mathrm{K}}{\mathrm{KE}}.$$

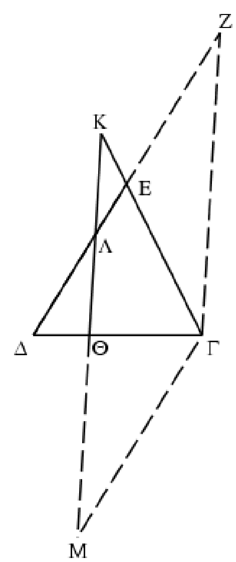

Figure 3: Figure for Lemma 1.

Proof: [Referring to Figure 3], through Γ, construct the line $\Gamma\mathrm{Z}$ parallel to $\Theta\mathrm{K}$, and let Z be the point of intersection with the line $\Delta\mathrm{E}$, produced. Because the lines $\Delta\Lambda$ and $\Lambda\mathrm{E}$ are given, and the line $\mathrm{Z\Lambda}$ is obtained by construction, by the formula for composite ratios,

[8] Note that the triangles BHZ and $\mathrm{E\Lambda Z}$ are similar because HB is parallel to $\mathrm{E\Lambda}$.

[9] Actually this was just stated.

[10] Hultsch adds his own footnote here that this can be shown by the same sort of argument used before to identify Z as the centroid of the triangle $\mathrm{AB\Gamma}$.

$$\frac{\Delta\Lambda}{\Lambda\mathrm{E}} = \frac{\Delta\Lambda}{\Lambda\mathrm{Z}} \cdot \frac{\Lambda\mathrm{Z}}{\Lambda\mathrm{E}}.$$

ΓZ and KΘ are parallel, so $\frac{\Delta\Lambda}{\Lambda\mathrm{Z}} = \frac{\Delta\Theta}{\Theta\Gamma}$. By the similarity of the triangles ΓEZ and KEΛ, and *componendo*, it follows that $\frac{\mathrm{Z}\Lambda}{\Lambda\mathrm{E}} = \frac{\Gamma\mathrm{K}}{\mathrm{KE}}$. Therefore,

$$\frac{\Delta\Lambda}{\Lambda\mathrm{E}} = \frac{\Delta\Theta}{\Theta\Gamma} \cdot \frac{\Gamma\mathrm{K}}{\mathrm{KE}}.$$

By the same reasoning, it can also be shown that

$$\frac{\mathrm{K}\Lambda}{\Lambda\Theta} = \frac{\mathrm{KE}}{\mathrm{E}\Gamma} \cdot \frac{\Gamma\Delta}{\Delta\Theta},$$

when the line ΓM is drawn through Γ parallel to EΔ, meeting the line KΘ, produced, in M. Again [by composite ratios],

$$\frac{\mathrm{K}\Lambda}{\Lambda\Theta} = \frac{\mathrm{K}\Lambda}{\Lambda\mathrm{M}} \cdot \frac{\Lambda\mathrm{M}}{\Lambda\Theta}.$$

And again because EΛ and ΓM are parallel, $\frac{\mathrm{K}\Lambda}{\Lambda\mathrm{M}} = \frac{\mathrm{KE}}{\mathrm{E}\Gamma}$. Moreover, by the similarity of the triangles ΔΘΛ and ΓΘM and *componendo*, it follows that $\frac{\Lambda\mathrm{M}}{\Lambda\Theta} = \frac{\Gamma\Delta}{\Delta\Theta}$. Therefore

$$\frac{\mathrm{K}\Lambda}{\Lambda\Theta} = \frac{\mathrm{KE}}{\mathrm{E}\Gamma} \cdot \frac{\Gamma\Delta}{\Delta\Theta}.$$

From the hypothesis, $\frac{\mathrm{KE}}{\mathrm{E}\Gamma} = \frac{\Theta\Delta}{\Delta\Gamma}$,[11] and hence

$$\frac{\mathrm{K}\Lambda}{\Lambda\Theta} = \frac{\Delta\Theta}{\Delta\Gamma} \cdot \frac{\Gamma\Delta}{\Delta\Theta},$$

which is the ratio of a magnitude to an equal magnitude. Therefore KΛ = ΛΘ. □

We now show the other claim that was deferred [in the proof of Theorem 1].

[11] Pappus is also using the fact that if $A : B :: C : D$, then $B : A :: D : C$. This transformation of the proportion was known as *invertendo*.

Lemma 2. *Let* AB *and* $\Gamma\Delta$ *be parallel. Let* Z *be a point on* AB *and let* Θ *be a point on* $\Gamma\Delta$ *such that* $\frac{\mathrm{AZ}}{\mathrm{ZB}} = \frac{\Gamma\Theta}{\Theta\Delta}$. *Let* $\mathrm{A}\Gamma$ *and* $\mathrm{B}\Delta$ *be joined and let the intersection point of those lines be* E. *Then* $\mathrm{Z}, \mathrm{E}, \Theta$ *are collinear.*

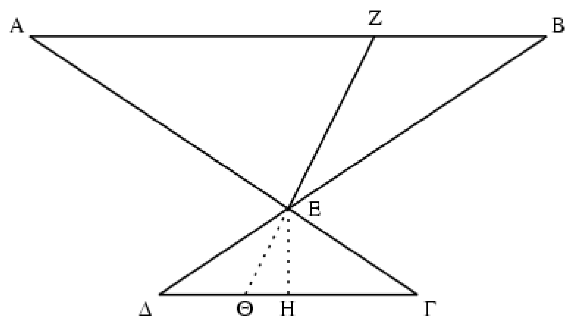

Figure 4: Figure for Lemma 2.

Proof: Suppose not and suppose that the straight line through Z and E intersects $\Delta\Gamma$ in H [a point different from Θ]. Because AB and $\Delta\Gamma$ are parallel,

$$\frac{\mathrm{AZ}}{\Gamma\mathrm{H}} = \frac{\mathrm{ZE}}{\mathrm{EH}} = \frac{\mathrm{ZB}}{\mathrm{H}\Delta}.$$

and hence $\frac{\mathrm{AZ}}{\mathrm{ZB}} = \frac{\Gamma\mathrm{H}}{\mathrm{H}\Delta}$.[12] It follows that

$$\frac{\Gamma\Theta}{\Theta\Delta} = \frac{\Gamma\mathrm{H}}{\mathrm{H}\Delta},$$

since by hypothesis $\frac{\mathrm{AZ}}{\mathrm{ZB}} = \frac{\Gamma\Theta}{\Theta\Delta}$. But this is not possible [if H and Θ are different], hence the straight line through Z and E must pass through Θ. □

3. Cartesian coordinates and a second proof

In this section, we present an alternative proof of Theorem 1 using Cartesian coordinates. The fact that coordinate proofs of geometric statements, using algebra, are often shorter and easier than synthetic, Euclidean-style proofs is something that every student of mathematics exposed to both geometric

[12]Pappus is using the fact that $A : B :: C : D$ implies $A : C :: B : D$, applied to the outside terms here. This transformation is sometimes called *alternando*.

techniques has surely noticed. Indeed, we can see from Descartes' own writings that it was partly the dissatisfaction that he felt with proofs from works like Pappus's *Mathematical Collection* that led him to begin the process of marrying algebra with geometry through the use of coordinates. In the first book of his groundbreaking work *La Géométrie*, for instance, (in connection with a different theorem of Pappus) Descartes states that

> In passing, I ask you to remark that the reluctance [i.e. scruples] the ancients had about using arithmetic terms in geometry, which could only have come from the fact that they did not see the connection very clearly, caused much obscurity and difficulty in the ways they expressed themselves.[13]

For the alternative proof of Pappus's Theorem 1, we only need to know that if the vertices of a triangle $\mathrm{AB\Gamma}$ in the plane are given in coordinates by

$$\mathrm{A} = (x_1, y_1), \quad \mathrm{B} = (x_2, y_2), \quad \Gamma = (x_3, y_3),$$

then the coordinates of the centroid Z are given by

$$\mathrm{Z} = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right),$$

or in vector form by

$$\mathrm{Z} = \frac{1}{3}(\mathrm{A} + \mathrm{B} + \Gamma). \tag{1}$$

This is discussed in many geometry textbooks and is presented in many online videos and websites. Also note the analogy with the formula for the midpoint of a line segment.

Proof: Let the vertices of the triangle $\mathrm{AB\Gamma}$ be as above. We will use the same letters to represent the points and their coordinate vectors. Suppose the points $\mathrm{H}, \Theta, \mathrm{K}$ are given as before, but now assume:

$$\frac{\mathrm{AH}}{\mathrm{AB}} = \frac{\mathrm{B\Theta}}{\mathrm{B\Gamma}} = \frac{\mathrm{\Gamma K}}{\mathrm{\Gamma A}} = \rho,$$

[13]This is my English translation rather than the one given in [2], which seems a bit garbled. The original French text is: *Ou ie vous prie de remarquer en passant, que le scrupule, que faisoient les anciens d'vser des termes de l'Arithmetique en la Geometrie, qui ne pouuoit proceder, que de ce qu'ils ne voyoient pas assés clairement leur rapport, causoit beaucoup de d'obscurité, & d'embarras, en la façon dont ils s'exploient.* [2, pages 19-20].

for some real number $0 < \rho < 1$.[14] Then the coordinates of these points can be computed easily from the vector formulation:

$$\begin{aligned} \mathrm{H} &= \mathrm{A} + \rho(\mathrm{B} - \mathrm{A}), \\ \Theta &= \mathrm{B} + \rho(\Gamma - \mathrm{B}), \\ \mathrm{K} &= \Gamma + \rho(\mathrm{A} - \Gamma). \end{aligned} \tag{2}$$

Hence, when we apply the formula (1) above to compute the centroid of the triangle HΘK, we have

$$\begin{aligned} \frac{1}{3}(\mathrm{H} + \Theta + \mathrm{K}) &= \frac{1}{3}\left(\mathrm{A} + \rho(\mathrm{B} - \mathrm{A}) + \mathrm{B} + \rho(\Gamma - \mathrm{B}) + \Gamma + \rho(\mathrm{A} - \Gamma)\right) \\ &= \frac{1}{3}(\mathrm{A} + \mathrm{B} + \Gamma) \\ &= \mathrm{Z}, \end{aligned}$$

since the other terms cancel in pairs because of the alternating signs. Hence the triangles ABΓ and HΘK have the same centroid. □

To be clear, I am not claiming any originality for this proof. For instance, it is certainly a special case of the results on systems of particles with a common centroid from [3]. I am sure that it has been found many other times, too, because it is simply what comes out when one understands a suitable way to set up the problem, and then "turns the crank" of the algebra machine.

4. Observations and Conclusions

Why do we do mathematics? What is it for? I hope that no one would argue with the idea that (at least) one reason that humans do mathematics is *to gain understanding* — to explain patterns and to find reasons why things that seem to be true are true (or not). Proofs have a key role to play in this enterprise. Working through a proof or, even better, finding a proof, should help us gain understanding, and then perhaps take further steps in exploring the mathematical world.

But as Hermann Weyl said in his famous address "Topology and abstract algebra as two roads of mathematical comprehension,"

[14]Note that to say these are all equal, we have applied another transformation of proportions to the hypotheses as Pappus states them. These are the ratios of the lengths of the segments to the whole corresponding sides.

> We are not very pleased when we are forced to accept a mathematical truth by virtue of a complicated chain of formal conclusions and computations, which we traverse blindly, link by link, feeling our way by touch. We want first an overview of the aim and of the road; we want to understand the idea of the proof, the deeper context. ([5, page 453])

Because of the length of the chain of deductive steps, really coming to grips with and understanding Pappus's proof of Theorem 1 from §2 requires much more time and persistence than understanding the steps presented in the algebraic proof from §3. Part of the issue is that Pappus has not been very good about explaining the key idea of the proof before plunging into the technical details. If he had said from the beginning that the idea was to show that $\mathrm{H}, \mathrm{Z}, \Lambda$ are collinear, $\Theta\Lambda = \Lambda\mathrm{K}$, and $\mathrm{HZ} = 2\mathrm{Z}\Lambda$, that would have been a real help in sorting out why Z also has to be the centroid of the triangle $\mathrm{H}\Theta\mathrm{K}$. But Pappus clearly does not want to give away the punchline too early. Greek mathematical writing seemingly hardly ever takes the reader's needs into account in the way Weyl wanted. We will see a possible reason for this shortly.

On the other hand, the algebraic proof seems almost too easy. As we said before, that proof is (just) a matter of seeing a good way to set up the problem and "turning the crank."

To make a fairer comparison, we should point out that a substantial chunk of the beginning of Pappus's proof is devoted to the argument showing that the centroid is the intersection of the two medians of the triangle. To even things out, we probably should have included a derivation of the formula (1) in the second proof. We did not do that, though, to make the point that modern, algebraic proofs in coordinate geometry are often simpler *precisely because* they are also often farther from the geometric foundations. This means that, without the derivation of (1), most of the physical intuition providing the meaning of the centroid is lost in the algebraic proof.

Algebraic proofs often make full use of sophisticated algebraic reformulations of the problem at hand, such as the formulas from (2) in the last section. Those equations essentially rely on parametrizing the line segments making up the edges of the triangle and interpreting the hypothesis that $\mathrm{H}, \Theta, \mathrm{K}$ cut the edges in the same ratios in those terms. "Seeing a good way to set up the problem" requires understanding that thinking that way would be useful, and then knowing how to do that. And of course, that requires a good understanding of Cartesian coordinates as vectors, the algebra of vectors, parametrizations of lines, and so forth. The passage from Weyl's address quoted above continues,

> A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details. ([5, page 453])

In a different direction, to foster understanding, a "good" proof may try to catch the attention and interest of the reader. One way this can be done is through a *surprising*, or *unexpected* deduction. In my opinion, on this score, Pappus's proof passes with flying colors! What reason do we have to expect before it is demonstrated that the three points $\mathrm{H}, \mathrm{Z}, \Lambda$ in Figure 2 are actually collinear or that $\Theta\Lambda = \Lambda\mathrm{K}$? This is the possible reason for not giving away the punchline too early that I alluded to above and, in my opinion, it really does make the eventual conclusion more surprising. The algebraic cancellations in the proof from §3 seem rather humdrum by comparison.

A reader might be forgiven at this point if they have formed the impression that I am about to advocate for the abolition of algebraic coordinate proofs in geometry. But of course that would be quixotic and surely counterproductive for mathematics. My only real issue with algebraic proofs in geometry is that, especially for some students learning a geometric subject, "turning the crank" to get quick results without any real understanding can unfortunately become the perceived *raison d'être* for the whole exercise.[15] Even if it takes longer and is more frustrating, I would argue that successfully finding a complete and solid synthetic proof of a geometric theorem can provide more real understanding and be a more meaningful learning experience than manipulating symbols to establish an algebraic identity. And for that reason, it would be a real shame if synthetic geometry were to disappear entirely from our secondary and undergraduate curricula.

[15] This statement may seem unexpected from an algebraic geometer whose work features applying techniques from symbolic computation to geometric questions. But it reflects the experience gained in a career of undergraduate teaching.

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