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Bootstraps and Scaffolds: What a Cognitive-Historical Analysis of the Complex Number System Reveals About Numerical Cognition

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Synopsis
The following investigation is a cognitive-historical analysis of the conceptual development of complex numbers. The history we explore spans nearly two millennia, from the earliest appearance of the square root of a negative quantity in the calculations of Heron of Alexandria (first century CE) to the full flowering of complex numbers in the first half of the 19th century. The approach we use is Nersessian’s, including her formulations of model-based reasoning and mental models. Additional aspects feature the prominent roles played by process representations, including object-process complementarities, and by core numerical systems. Our analysis provides support for mental rotation as a fourth core system and for bootstrapping scenarios in both the senses of Nersessian and of Carey. We use the results of our analysis and discussion to update previous conceptual metaphor mapping approaches to complex numbers by Fauconnier and Turner and by Lakoff and Núñez. Finally, we review recent empirical studies of complex number learning and find them to be consistent with the present work.

1. Introduction
The universe is imaginary! Complex numbers are deeply real—without them quantum physics falls apart, and a deep understanding of reality is lost [121].
What follows is an account of humanity’s two-thousand-year journey of discovery, acceptance and utilization of the complex number system. Like the subject itself, the story told is complex—one that winds through the history and philosophy of mathematics as it engages process philosophy and cognitive linguistics, developmental psychology and the neuro- and cognitive sciences of numeration. Ultimately our inquiry arrives at a more complete understanding of four core systems of numerical cognition and a detailed account of the cognitive scaffolding and bootstrapping processes that are utilized in the construction of the complex number system.

1.1. Concerning the Subject and Method of this Analysis
For more than four decades, research in numerical cognition has played across behavioral, developmental, and neural sub-disciplines of cognitive science and psychology, as well as mathematical education research. Studies have ranged from the cognitive capacities of newborn infants to the problem-solving strategies of expert mathematicians, from animal studies of angel fish, parrots, chicks, and chimpanzees, to the learning of mathematics by pre-school pupils and onward to university students. In so doing, these initiatives have drawn on neuroimaging modalities such as fMRI, EEG and tES, on a large assortment of qualitative and quantitative experimental design strategies, and on a cluster of elaborate statistical analytical methods.

Amid this panoply of scientific subjects, methods and technologies, there resides another investigative approach—cognitive-historical analysis—although it is not widely known and only infrequently utilized. The following work is predicated upon this approach, with the historical development of complex numbers as the particular subject to be closely scrutinized. As remote as this approach and as obscure as this topic might seem to be, this inquiry will be seen to produce an abundance of results that are central to many of the concerns of contemporary research in numerical cognition, with implications that extend even to the realm of quantum cognition.

1.2. Nersessian’s Analytical Method and Model
Although historical analyses undertaken for the investigative purposes of cognitive science pre-date the seminal work of Nersessian, the analysis that follows will be carried out in the manner characterized by her as a cognitive-historical analysis [101, 102, 103, 105, 107]. According to Nersessian’s conception, a cognitive-historical analysis is an integrative and reflexive enterprise. Primary source material is first extracted from the historical records.
of the conceptual development that is the subject of interest, and it is then combined with observations and analysis from relevant areas of philosophy, along with pertinent concepts and constructs from various branches of cognitive science, in order to illustrate and provoke a deeper understanding of the conceptual development under scrutiny. Insights from these results are then reflected back to cognitive science where their implications promote revision and discovery in their respective cognitive domains.

In addition to carrying out a cognitive-historical case study of the conceptual development of complex numbers, the following analysis will draw upon Nersessian’s accompanying formulation of model-based reasoning with its attendant notion of a mental model to explore in greater detail the specific modeling processes that characterize the conceptual bootstrapping of complex numbers [102, 104, 105, 106, 107, 108, 109, 110]. Model-based reasoning consists primarily of abductive inferential reasoning processes that are claimed by Nersessian to constitute a distinct category of reasoning, with mental models and their constraints comprising the core of the reasoning processes. Nersessian defines a mental model as “… a structural, behavioral or functional analog representation of a real-world or imaginary situation, event or process,” [107, page 93]. The representational modes of model-based reasoning processes include imagistic representation, simulation (thought experiment), and particularly analogy. Imagistic representation involves the production of internal imagistic mental models that frequently become a coupled system with external representations such as diagrams and schematic drawings. Simulation draws upon the dynamic nature of mental models to subject them to visualizable changes that anticipate the states and behaviors projected for the model. Analogy is the mainstay of model-based reasoning, so to carry out her analyses, Nersessian draws upon Gentner’s structure-mapping theory [52, 53, 56] with its procedures for mapping from a base (or source) domain to a target domain, subject to the requirements of structural consistency, structural focus, and systematicity as criteria for satisfactory mapping of the structure of an analogy. Nersessian also draws upon Holyoak and Thagard’s multiconstraint theory [75, 76] to implement semantic and pragmatic constraints in analogy formation that Genter’s syntactic approach does not incorporate.

Model-based reasoning begins with the construction of the model by using processes of abstraction that include idealization, approximation, limiting case, and generic abstraction. The model is then subjected to simula-
tion, evaluation, and adaptation, where constraints imposed by the target, source and model itself are incorporated. These stages comprise a cycle of model construction that is iteratively and incrementally repeated, producing a succession of hybrid, intermediate models that ultimately results in a final model that fully satisfies the analogical constraints. Nersessian regards this constraint-satisfaction process as a type of bootstrapping, where,

“... constraints from target and sources each provide one strap, the intermediary hybrid models are strap crossings, and each crossing supports or contributes to further model building and enhanced target understanding by the entire group. The problem solution does not arise from a single well-articulated model, but is an incremental process where a constructed model leads to partial target insights and new constraints, which in turn lead to further construction, until an adequate conceptualization is achieved.” [110, page 186]

In sum, model-based reasoning is portrayed to be a self-standing category of reasoning not ancillary to propositional logic. It is most closely aligned with abductive reasoning processes that are ampliative and at times regarded to be genuinely creative. Nersessian considers model-based reasoning to span a continuum from mundane problem-solving tasks to the self-critical elaborations of scientific discovery. As well as characterizing the processes of conceptual discovery and development at the level of the individual, it also pertains to the uptake of conceptual change in society as a whole.

1.3. A Vital Role for Representation of Process

Nersessian’s conceptualization of a mental model as an analog representation capable of depicting an event or process calls up the first of two major themes developed and utilized in this present analysis—specifically that there is a need to attend to the representation of processes, as well as of objects, that characterize the phenomena of interest, and further, to recognize the extent to which a complementarity between object and process conceptualizations may be present. In fact, the dynamical process aspect of Nersessian’s mental model conceptualization is one of several distinct but intertwined strands that range from contemporary studies of the historical development of scientific theories by their originators to those of the initial learning of scientific and mathematical subjects by students—each strand reinforcing the recognition of the vital role played by descriptions of process.
1.3.1. Views of Process from the Linguistics and Philosophy of Science

Over the course of their development from their early beginnings in ancient Greece, the languages of science and mathematics have evolved to be predominantly object-based, with the processes that affect them retreating into relative obscurity. Halliday [66] documented the onset and progression of this linguistic transformation, which he calls *nominalization*, as he carried out extensive examinations of the role of grammatical metaphor in texts that range from Chaucer’s *A Treatise on the Astrolabe*, Newton’s *Opticks*, Priestley’s *History and Present State of Electricity*, Darwin’s *On the Origin of Species*, to contemporary articles in *Scientific American*. A single instance of nominalization occurs when, for example, a simple clause such as ‘planets move’ undergoes a grammatical metaphorical change such that the verb ‘move’ is transformed to a noun ‘motion’ and the clause itself becomes nominalized as ‘planetary motion’. When this process is carried out literally thousands of times, a technical language emerges that is overwhelmingly based on nouns. As such, it reshapes the experience of the reality that it describes. The world becomes noun-like, thus stable in time and amenable to observation, experimentation, measurement and reasoning. However, in doing so, the roles previously attributed to processes are diminished.

For the most part, science progresses through ‘normal’ episodes of conceptual modifications that are able to be accommodated by the object-based technical language at hand. However, from time to time a radical conceptual change is required, one that engenders a transformative revolution in science, and the agency of such change may be the result of the re-assertion of a role for processes in the subject at hand.

Starting initially from a Kuhnian perspective of conceptual change in scientific theories as being analyzable in terms of taxonomic structural shifts, Chen ultimately arrived at a perspective on scientific revolutions that suggests that transition across seemingly incommensurable conceptualizations might be the result of the transformation from an object concept to a process concept [22, 25]. Specifically, Chen has offered the Copernican/Keplerian Revolution in 16th/17th century astronomy, the Optical Revolution in 19th century optics, and the Darwinian Revolution in 19th century biology as three examples of scientific revolutions that resulted from an object-to-process conceptual transformation [23, 26]. In the first case above, the Aristotelian/Ptolemaic conceptualization of a material object, *orb*, was ultimately replaced by the dynamical process concept of *orbit*. In the second, the classical Newtonian
concept of a particle of light was supplanted by that of a wave, while correspondingly, light polarization underwent conceptual realignment from an object concept, side, to a process concept, phase difference. Finally, in the case of the theory of evolution, the concept of species was transformed from one with a fixed, object-like taxonomic definition to one based on the processes of evolutionary change.

Chen would further argue for the existence of an object bias that is generally part of human cognition and one that is specifically evident in scientific theory construction [24]. He advanced a provisional account of this bias as one that results from the discrepancy of the time of onset of core object knowledge systems available to infants early in their first year compared to their later-arriving ability to distinguish object from process concepts as they approach the age of seven [25]. He also reviewed cognitive science research resulting from lesion studies, PET and fMRI imaging, and EEG studies that indicate that process concepts are neurologically independent of object concepts, with processing occurring in distinct areas of the brain. Based on this cognitive science research, Chen advocated for a shift of focus in the study of scientific revolutions that has implications for all sciences and for cognitive science in particular: “...we need to shake off the object bias—the tendency to view all episodes of scientific revolutions as changes between object concepts, and to re-examine many of the assumptions, methods, and practices that we have taken for granted in our studies of scientific revolutions. In many ways, this transformation in the study of scientific revolutions itself is a revolutionary change” [25, page 190].

A prominent role for process also emerges in Gooding’s work on the use of visual representation and reasoning in the construction of scientific theory [58, 59, 60]. His approach is converse to Nersessian’s postulation of internal imagistic representations that become coupled to external visual representations, but it is one that nevertheless arrives at a compatible viewpoint to hers. From his analysis of scientific discoveries as diverse as the rendering of Burgess Shale fossil data, Teutsch’s modeling of the action of liver enzymes, magnetometer data implicating sea floor spreading with continental drift, and Faraday’s visual conceptualization of electromagnetic rotation, Gooding was led to posit a Pattern-Structure-Process (PSP) Schema that could be identified to be at work in each case. The image-based reasoning process begins with the initial rendering of empirical data typically as a two-dimensional pattern, followed by the application of innate pattern matching and mental rotation
cognitive functions that eventually lead to a three-dimensional *structure* representation, then finally by the incremental variation of structure representations results in the attainment of a four-dimensional *process* representation. Each step in the progression from pattern to structure to process is considered by Gooding to be an instance of visual abductive inference, and when a progression in the reverse sense—from process to structure to pattern—is undertaken, each step is understood as an instance of visual deductive inference. For Gooding, “The extra dimensionality of visual process representations helps explain the source phenomenology and can predict new phenomena. Moving from interpreted sources to structural models and on to process models generates visual theories that satisfy the explanatory aims of science” [60, page 25].

While the above writings of Chen and Gooding share a concern with the role played by processes in the development, revision and replacement of scientific theories through the closely reasoned and scrutinized analyses of experts, Chi’s work is concerned with the role of processes in the initial learning of science, specifically physics, by novice learners. In her early work [27], she proposed the existence of three ontological categories—*matter, events, and abstractions*. Later these categories were revised to be *entities, processes, and mental states* [28], although throughout her investigations, her primary focus has been only with the first two categories. She held that while conceptual revision within a particular ontological category may be relatively straightforward though not necessarily easy, there are no operations or mechanisms that provide similar ease of revision across ontological categories. Change across ontological categories requires *radical* conceptual change, and radical conceptual change entails *incommensurability* of concepts. Nevertheless, learning certain science topics in domains such as physics requires conceptual change across ontological categories, and this causes considerable difficulty for learners—a circumstance that she would later characterize in the formulation of her *Incompatibility Hypothesis* [29]. Specifically, for students to properly understand physical concepts related to concepts of force, light, heat, and current, they need to change their conception of these entities as objects or *material substances* and conceive of them as occurrences of a type of process, i.e., as *constraint-based events*. Chi regarded the ontological conceptual shift undergone by the individual learner as analogous to the radical theory change underlying a paradigm shift in a scientific theory, although paradigm shifts at a global level also entail sociological and political processes.
1.3.2. Views of Process and Object-Process Complementarity in Mathematics

In the realm of mathematics learning, Dubinsky and Sfard have individually explored the roles played by process conceptualizations, and they have also advanced the discussion from conceptually separate object and process aspects to the circumstance of their joint involvement as a pair of complementary concepts. When object and process aspects of a given conceptualization are simultaneously mutually exclusive and jointly completing aspects, they may be regarded as complementary. Such complementarities are consistent with the concept of complementarity, introduced first by Niels Bohr to characterize quantum phenomena but soon generalized to other areas, including biology, psychology, and philosophy [13, 82, 116]. It has been suggested that Bohr may have first encountered the notion of complementarity in the psychological writings of William James [74], and without a doubt it is re-emerging in contemporary cognitive science through recent developments in quantum cognition research [10, 117, 144].

Issuing from Piaget’s development of genetic epistemology, particularly from his notions of equilibration and reflective abstraction, APOS Theory as articulated by Dubinsky [36, 37] engaged the appearance of object-process complementarity in the development of mathematical concepts. APOS is the acronym composed from the specified concepts of action, process, object and schema. The notion of process is understood by Dubinsky as “interiorized action”, where interiorization is the conscious reflection on an action. The schema is understood as “a circular feedback system” comprised of the objects and processes considered as a dynamical whole.

Initially Dubinsky applied APOS analysis to the genetic decomposition of the concepts of mathematical induction and compactness and to the consideration of predicate calculus as a schema constructed from the schemas of propositional calculus and the function concept. In the intervening years to the present, APOS analysis has been applied to a variety of mathematical topics, including studies of limits, the chain rule, graphing via derivatives, related-rates problems, polar coordinates, vector spaces, etc. Today, the defining concepts and methods of APOS theory have been refined and expanded, and the theory now plays a role as a methodology for mathematics education research and pedagogy [2].

The work of Sfard is a second stream of research originating from a Piagetian standpoint with the intent to explore object-process complementarity.
Specifically, Sfard put forward an analysis of the conceptual development of mathematical entities, including functions, number systems, and algebras, that draws upon Bohr’s generalized notion of complementarity [127, 128, 129, 130]. She explored operational-structural complementarities where the operational aspect includes processes, algorithms and actions and the structural aspect includes objects regarded as static or ‘timeless’ entities. She posited a progression of conceptual development from process aspect to object aspect, where the object aspect that resides at a higher conceptual level is arrived at by a three-stage transition involving cognitive processes of **interiorization**, **condensation** and **reification**, commencing at the process aspect of the lower level. For Sfard, **interiorization** is the stage at which a learner becomes initially acquainted with the processes that will eventually give rise to a new concept. **Condensation** is the next stage whereby lengthy sequences of operations are “squeezed” into more manageable units. Finally, **reification** is the appearance of the conceptualization as a fully-fledged object. Sfard held that reification is inherently difficult to achieve, but when it is accomplished, it has the appearance of a sudden leap. The difficulty of reification is the result of an ontological shift from the lower-level-process aspect to the upper-level-object aspect. It is these two aspects that Sfard conjoined as a complementary duality. This developmental process is claimed to lead to the formation of conceptual hierarchies such as the hierarchical development of number systems from natural to complex numbers.

**1.4. Essential Roles for Core Numerical Cognition Systems and Quinean Bootstrapping**

The second major theme to be developed and utilized in the following analysis are the essential roles played by the action of core systems of numerical cognition and by Quinean bootstrapping. Different usages of the term **core** abound, and the meaning ascribed to it here is derived from the six defining criteria set out by Carey [18]: core representational systems (i) are associated with dedicated input analyzers; (ii) are innate; (iii) have a long evolutionary history; (iv) utilize an iconic format; (v) are constant through the lifespan, and (vi) are domain-specific learning mechanisms. In the portion of her research that is oriented toward identifying the core cognitive processes underlying the conceptualization of natural number and of learning to count, Carey has distinguished three innate representation systems concerned with numerical content: (i) the analogue magnitude system (AMS) [alternatively characterized as the approximate number system (ANS)],
an evolutionary ancient system readily observed in the magnitude comparison behaviours of human adults and infants, as well as numerous non-human animals; (ii) the parallel individuation system (PI) [alternatively characterized as the object tracking system (OTS)], a system associated with the rapid distinction and tracking of sets of one to four objects, and (iii) set-based quantification (SBQ), the representation system underlying the prelinguistic capacities by which are formed natural language quantifiers—e.g., ‘a’, ‘some’, ‘all’, ‘each’—that are part of the human innate endowment for language. Specifically, set-based quantification is regarded as the underlying prelinguistic bracketing process supporting the distinction of individuals, sets, chunks and ensembles [44, 45, 64, 65, 123, 151]. Carey asserted that these three systems are the only innate systems with numerical content for which there is evidence. The research on these systems is extensive and rapidly growing, but beyond Carey’s references herein, good entry points can be found in [8, 35, 51].

Furthermore, in the course of her research on the developmental origins of natural number conceptualization and counting, Carey was led to postulate the existence of a conceptualization process that she referred to as Quinean bootstrapping [15, 16, 17, 18, 19, 20, 21]. In so doing, she drew upon Quine’s use of several colorful metaphors to illustrate the bootstrapping predicament, including his Chimney metaphor, in which “The child scrambles up an intellectual chimney, supporting himself against each side by the pressure of the others” [120], Neurath’s Boat, in which, “… we are in the position of a mariner who must rebuild his ship plank by plank while continuing to stay afloat on the open sea” [119], and Wittgenstein’s Ladder, whose originator explained that, “[m]y propositions serve as elucidations in the following way: anyone who understands me eventually recognizes them as nonsensical, when he has used them—as steps—to climb beyond them. (He must, so to speak, throw away the ladder after he has climbed it)” [146]. The common notion here is that of a pre-existing supportive structure, a ladder or scaffold, that facilitates the construction of a new higher-level conceptual structure but that itself is not integral to the finished structure. In Carey’s formulation, this supportive conceptual structure is identified as a placeholder. A placeholder is a system of symbols obtained from the initial conceptual system but nevertheless one that is only partially interpreted with respect to that antecedent system.
The existence of a placeholder is essential for Quinean bootstrapping to proceed, and an equally essential accompaniment to the placeholder is the collection of processes by which the placeholder is interpreted and acted upon so that the bootstrap is formed. In her characterisation of these processes, Carey associates them with the modeling processes specified by Nersessian—i.e., analogical mapping, inductive and abductive inferences, imagistic reasoning, simulations, thought experiments, and limiting-case analyses. As a consequence of these modeling processes, the conceptualization of the higher-level cognitive system is ultimately attained—an attainment unreachable via the attributes of the lower-level cognitive system alone.

Carey developed her Quinean bootstrapping theory while exploring the cognitive processes entailed with learning to count with natural numbers, but she went on to incorporate it generally into analyses where conceptual discontinuities were thought to exist—e.g., in the conceptualization of rational numbers, of weight and density relations, and further, following Nersessian, in the realms of the scientific conceptual revolutions resulting from the work of Kepler, of Maxwell, and of Darwin.

1.5. Proceeding with the Analysis

With the aforementioned analytical tools assembled—namely (i) the model-based reasoning scenario with its associated conception of mental model that together stipulate a central role for analogical, imagistic and abductive reasoning processes, (ii) process representations in addition to or complementary to object representations, and (iii) core numerical representation systems complemented by Quinean bootstrapping—we now begin our cognitive-historical analysis of complex numbers. In the course of this analysis, the bootstrapping approaches of both Nersessian and Carey will be seen to resonate with and to supplement pre-existing analyses of complex numbers drawn from conceptual metaphor and blending theory in cognitive linguistics, in particular from the works of Fauconnier and Turner and Lakoff and Núñez.

Although complex numbers might be regarded as an obscure, conceptually remote abstraction with no apparent relevance to issues of human cognition, a detailed examination of their historical development will reveal instances of incommensurable concepts and the cognitive means by which the associated discontinuities are surmounted, examples of hybrid, intermediate mental models and the bootstrapping roles that they play, the scaffolding provided
by core numerical representation systems for the construction of number systems ranging from the natural to the complex numbers, and the identification of a fourth core numerical representation system, mental rotation. In sum, the prospects for a cognitive-historical analysis as articulated by Nersessian will be shown to be a mutually enriching, multidisciplinary enterprise with benefits both for numerous branches of cognitive science and for the history and philosophy of mathematics, well justifying the effort to carry out a fine-grained analysis of a database rich in historical detail with no immediately apparent connection to the central concerns of cognitive science.

2. Case Study: The Historical Development of the Complex Number Concept

2.1. First Encounters

The birth of the complex number concept followed a long and difficult labor; nearly two millennia passed from the first historical encounter with the square root of a negative quantity to the full emergence of a generally accepted conceptual understanding of a new category of number, eventually formally recognized and christened by Gauss in 1832 as the “complex” number [94]. In his Stereometria, Heron of Alexandria (c.10-c.70) presented the calculation of the volume of the frustum of a pyramid that if it were properly carried out, should have produced the “square root of the quantity (81−144)”, clearly the square root of a negative quantity, as a solution. In fact, Heron’s manuscript displayed the answer as the “square root of the quantity (144−81)”, with no remark indicating what the reasoning might have been behind this ‘correction’, or even whether it had been made by Heron or a scribe [62]. Two centuries later, Diophantus (c.201/215-c.285/299), in his Arithmetica, was led to dismiss the possibility of obtaining solutions to a particular quadratic equation based on the fact that it would produce square roots of negative numbers, but he did not pursue the implications of this circumstance further.

With the passage of nearly six centuries, overt reference to the possibility of a square root of a negative number emerged, only to be swiftly denied. In the 9th century, the mathematician Mahâvîra (c.850) briefly remarked that, “as in the nature of things a negative (quantity) is not a square (quantity), it has therefore no square root.” In the same era Abd al-Hamîd ibn Turk (c.830) used a geometric demonstration of the sort preferred by Islamic mathematicians of that time (e.g. Muhammad al-Khwârizmî, who in al-Jabr
also regarded the square root of a negative quantity as impossible) to entertain the prospect of a geometrically impossible rectangle with negative area. This result was analogous in algebraic terms to the square root of a negative number, and it provoked ibn Turk to acknowledge, “the logical necessity of impossibility” [83]. Nearly three centuries later, the mathematician and astronomer Bhāskara II (c.1114-c.1185) would affirm that, “[t]here is no square root of a negative quantity; for it is not a square.” At nearly the same time, Abraham bar Hiyya (c.1120), working with quadratic equations, offered the somewhat less strident observation that “trouble could arise if the square root had to be that of a negative quantity.”

2.2. The Usefulness of the Impossible

In the centuries that followed, mathematicians were drawn to explore this “trouble” as they pursued solutions to quadratic, cubic and higher-order equations, which led to increasingly frequent encounters with the square root of negative numbers. In his work, Summa di arithmetica geometrica, published in 1494, Luca Pacioli (c.1445-c.1500) restricted the solution of a quadratic equation to real roots, indicating that he was aware of the possibility for an “impossible” solution involving square roots of negative quantities. Nicolas Chuquet (c.1445- c.1500) would also refer to “impossible solutions” in his unpublished manuscript of 1484.

With the work of Girolamo Cardano (1501-1576), particularly his Ars magna of 1545, the awareness of the need to consider and to work with the “impossible” began to rapidly develop [85]. Cardano, as with many of his contemporaries, struggled to accept the usage of zero in algebraic equations, and he regarded negative numbers also as impossible. Nevertheless, these impossible numbers continued to assert themselves in the equations that he examined. Building upon Cardano’s work, Rafael Bombelli (c.1526-1572) truly began to grapple with these impossibilities. In fact, he established the four rules for complex numbers (as they were later to be called) much as they are currently expressed, obtained conjugate pairs of complex numbers, and improved the notation used to express them. In 1629 Albert Girard (1595-1632) introduced the now-familiar symbol \(\sqrt{-1}\), and by 1637 René Descartes (1596-1650) had established in his La Géométrie, the terms, “racines réelles et imaginaires” (real and imaginary roots), referring to the two separable constituents of complex numbers—terms that have become part of the contemporary lexicon of complex numbers [3].
Descartes’ greatest contribution to the later development of the complex number concept was not due to a direct interest in the topic; he disavowed both imaginary numbers and even negative numbers as suitable solutions to algebraic equations. What Descartes did contribute was his synthesis of algebraic and geometric methods into a single field of study, analytic geometry. In setting the foundations of analytic geometry, Descartes was not the sole architect. Pierre de Fermat (1601-1665) had developed a synthesis of his own at roughly the same time, but Fermat’s algebra was less modern and his notation more obscure, whereas Descartes’ presentation remains largely accessible to the present day.

Descartes did not develop analytic geometry further after the publication of *La Géométrie* in 1637. Its eventual wide dissemination was the result of the efforts of Frans van Schooten (the younger) (1615-1660), who first created for Descartes the illustrations that appeared in *La Géométrie*. In 1649, he published the first Latin version of *La Géométrie*, making it accessible to scholars throughout Europe. In the period from 1659-1661, he published a second edition that included comments, explanations, and clarifications of Descartes’ ideas, transforming them into a systematic theory, as well as appendices written by three mathematicians who had studied with van Schooten. Among the many scholars who came in contact with van Schooten’s work was John Wallis (1616-1703), English mathematician and theologian, and his younger contemporary, Isaac Newton (1642-1727).

2.3. *The Analogical Reasoning of John Wallis*

John Wallis was largely self-taught in mathematics, but he taught himself well enough to attain the Savilian Chair of geometry at Oxford in 1649, holding it for more than fifty years [118]. In his *Treatise on Algebra* of 1685 (1693, Latin version), Wallis devoted a section of the work (Chapters LXVI-LXIX) to “. . . Negative Squares and their Imaginary Roots in Algebra” [142]. In less than nine pages, Wallis managed to characterize the status of imaginary numbers of his day and to advance their consideration, obtaining results that would help to develop a graphical representation of complex numbers.

Wallis began his discussion by characterizing the prevailing opinion of the time with regard to imaginary numbers:

These Imaginary Quantities (as they are commonly called) arising from the Supposed Root of a Negative Square (when they
happen,) are reputed to imply that the Case proposed is Impossible.

And so indeed it is, as to the first and strict notion of what is proposed.

Wallis showed why this is so, given the mathematical properties of negative and affirmative (positive) numbers. He then began to construct an analogy with regard to ‘impossibility’ between imaginary numbers and negative numbers, which were also regarded as impossible:

But it is also Impossible, that any Quantity (though not a Supposed Square) can be Negative. Since that it is not possible that any Magnitude can be Less than Nothing, or any Number Fewer than None.

Wallis continued by arguing for a different understanding of what negative quantities might represent:

Yet is not that Supposition (of Negative Quantities) either Unuseful or Absurd; when rightly understood. And though, as to the bare Algebraick Notation, it import a Quantity less than nothing; Yet, when it comes to a Physical Application, it denotes as Real a Quantity as if the Sign were +1 but to be interpreted in a contrary sense.

To explain what he meant by physical application, Wallis introduced positive and negative displacements that could lead to the circumstance that a man might be found to have,

\[ \ldots \text{advanced 3 Yards less than nothing.} \]

Which in propriety of Speech, cannot be (since there cannot be less than nothing.) And therefore \ldots the case is Impossible.

Wallis argued that the apparent impossibility is resolved if the displacements can be characterized as “forward” or “backward”, “advanced” or “retreated”, along a straight line. With this argument he arrived at a representation of a number line characterized by points associated with both positive and negative integers—seemingly the first instance of the number line conceptualization required for the development of analytic geometry. Earlier anticipations of the analytic geometrical number line may be found, such as the number line representations of Napier (Nepair, or Neper, 1550-1617) in his development of logarithms, but his number lines involved only positive integers [100, 113].
Wallis next asserted that,

Now what is admitted in Lines, must on the same Reason, be allowed in Plains also.

With this assertion he reasoned that a negative square should be considered possible, for example in the case of the area of land gained or lost due to the flooding of the sea, and he asked what the side of a supposed negative square might be. He then employed the long-established result that the *mean proportional* (say $x$) of two numbers $b$ and $c$ represents the square root of the product of those numbers. That is,

$$b : x :: x : c \text{ giving } x^2 = bc \text{ and } x = \pm \sqrt{bc}.$$  

By allowing either $b$ or $c$ to be negative, he gained valuable insight into the algebraic notion of an imaginary number as a mean proportional between a positive and a negative number.

The algebraic concept of *mean proportional* is analogous to the geometric concept of *geometric mean* that has been studied at least from the time of Euclid. In Chapter VI, Proposition 13 of his *Elements* [38], Euclid constructed the geometric mean of two line segments—e.g., $AB$ with length $b$ and $BC$ with length $c$—by first constructing a semicircle whose diameter is composed of these two line segments and then projecting a perpendicular line segment $BP$ (the sine) upward from the point $B$ on the diameter where the segments join to point $P$ on the circumference (see Figure 1). The constructed line segment is the geometric mean whose length is the mean proportional of the lengths $b$ and $c$.

![Figure 1: Geometric mean, from Wallis, *A Treatise...* (1685) [142].](image-url)
From the perspective afforded by analytic geometry, Wallis was able to make progress towards the geometrization of imaginary numbers by carrying out the construction of the geometric mean (in this case the tangent resulting from projection from the segment $AB$ to $P$ on the circumference of the semicircle) whose mean proportional corresponded to an imaginary quantity.

With additional geometrical argument, Wallis was able to show that for quadratic equations that produced real roots—whether positive or negative—such roots would correspond to points that lay upon a straight line determined in a plane, whereas quadratic equations that produced imaginary roots would have those roots located not on the straight line but somewhere in the plane containing the line. Wallis would summarize his argument thus:

So that, whereas in the case of Negative Roots, we are to say, the Point $B$ cannot be found, so as is supposed in $AC$ Forward, but Backward from $A$ it may be in the same Line: We must here say, in case of a Negative Square, the Point $B$ cannot be found so as supposed, in the Line $AC$; but Above that Line it may in the same Plain.

*This I have more largely insisted on, because the Notion (I think) is new; and this, the plainest Declaration that at present I can think of, to explicate what we commonly call the Imaginary Roots of Quadratick Equations. For such are these.* (Italics added)

With the foregoing argument, Wallis made use of the emerging analytic geometry of his day to progress towards the geometrization of imaginary numbers. His argument suggested that if real roots could be located as points on a straight line imbedded in a plane, then impossible imaginary numbers could be conceived as lying in the plane at some distance from the line. Perhaps if Wallis had at hand a mature version of analytic geometry, then he might well have achieved a full geometrization of complex numbers. However, such means were not yet available to him; analytic geometry would require nearly another century to develop to that point, an achievement for which he also would play a significant role.

### 2.4. The Development of the Cartesian Coordinate System

The Cartesian coordinate system familiar from present-day analytic geometry did not originate with Descartes or Fermat. Descartes did not use coordinates, even to the extent already available from the work of Nicole Oresme
He did not entertain the possibility of negative coordinates, and neither he nor Fermat used the term “coordinate system” or the notion of two axes. The plotting of curves in the now customary manner was not a part of Cartesian analytic geometry [9].

Wallis appears to have been the first to utilize both positive and negative ordinates, as well as abscissae, a practice not generally adopted by his contemporaries [99]. On the Continent, Wallis was widely held in disrepute because he had accused Descartes of plagiarizing Thomas Harriot (1560-1621)—an accusation apparently without sufficient foundation—and as a consequence, his work was given less attention than was van Schooten’s. For his part, van Schooten did not appear to have been aware of Wallis’s work on negative coordinates when he published his *Exertationis mathematicae* of 1656-57.

By 1679, Philippe de La Hire (1640-1718) had begun to systematically apply terminology to the various characteristics of the emerging coordinate system. In his correspondence of 1694, Gottfried Wilhelm Leibniz (1646-1716) first used the term “coordinate” in the modern sense, and he recognized that the two coordinates had the same status. By the end of the century, the terms “origin,” “axis,” “coordinates,” “abscissa,” “ordinate”, (or “appliquée”) had been adopted with their now familiar meanings. In 1704, Newton published as an appendix to his *Opticks*, the *Enumeratio linearum tertii ordinis*, in which he constructed a coordinate system of two positive and negative axes and four quadrants, though the origin was fixed for the abscissa only, and the axes were assumed to be oblique. With this work, which he had withheld from publication for decades, analytic geometry had come into its own as the means by which the development of a general theory of curves would be established in the 18th century. In Europe, Gabriel Cramer’s (1704-1782) *Introduction à l’analyse des lignes courbes algebraiques* (1750) showed the influences of Newton’s *Enumeratio* and of James Stirling’s (1692-1770) *Mathematical Commentaries*, and it resembled works by Jean-Paul de Gua de Malves (1713-1785) and Leonhard Euler (1707-1783) that were published at nearly the same time. Cramer was among the first to make formal use of two axes for which the two sets of coordinates were simultaneously and systematically defined. By the middle of the 18th century, long, sustained resistances to the use of negative numbers had begun to dissipate, and the two-dimensional rectangular Cartesian coordinate system had become increasingly standardized and extended to three dimensions.
With a mature analytic geometry at hand in the latter half of the 18th century, the conceptual structure was in place for constructing a complete geometrical representation of complex numbers. Within a span of less than three decades, this was accomplished not by a single individual but by at least three individuals working independently—Wessel, Argand, and Buée—and there are good reasons to believe that there may have been at least three more persons who separately had arrived at the same result.

2.5. Wessel’s Analytical Representation of Direction

Caspar Wessel (1745-1818) was born in the village of Vestby, Norway, the sixth of fourteen children with whom his mother and father, a curate, were blessed [6, 7]. Wessel began his study of law at the University of Copenhagen in 1763, but as a result of limited financial means, after a year he began to work with his older brother, Ole Christopher, as a surveyor for the Royal Danish Academy of Sciences in order to finance his education. His surveying and map-making skills were highly prized and heavily utilized, to the extent that he was not able to complete his law degree until 1778. Even with this attainment, Wessel would continue to make his livelihood from surveying and cartography, and he persisted with it even after his formal retirement.

Wessel was not a professionally trained mathematician, but the calculations required by surveying involved a substantial knowledge of geometry, algebra, and trigonometry. In the course of his work, he discovered the means by which complex numbers could be given a geometric interpretation, and he presented these results in 1787 in a report that he wrote for his employer, the Royal Danish Academy, about his development of more sophisticated mathematical methods for surveying. However, a full disclosure of his methods did not occur until March 10, 1797, at a meeting of the Royal Danish Academy. Because Wessel was not a member of the Academy, his work was presented on his behalf by Johannes Nikolaus Tetens (1736-1807), a professor of mathematics and philosophy at the University of Copenhagen. Publication in the Academy’s Mémoires occurred in 1799, in Danish, and the work languished—unattended by Danish mathematicians and unknown to the wider European mathematical community for nearly a century. In 1895 it was rediscovered and referenced in a thesis by Sophus Andreas Christensen (1861-1943), further explored by Christian Juel (1855-1935) and then reprinted in 1896 at the behest of Sophus Lie (1842-1899). In 1897 it was translated and published in French by H.G. Zeuthen (1839-1920).
As a result, it has the curious fate of being the earliest recorded account of
the geometrization of imaginary numbers—and one that stands at present
as perhaps the most highly regarded of the early accounts—but it played no
role in the historical development of its subject [3].

Wessel’s sole published work was entitled (as translated) *On the Analytical
Representation of Direction. An Attempt Applied Chiefly to Solving Plane
and Spherical Polygons* [143]. In it, he sought primarily to establish an ana-
lytical representation (an algebra) of “straight lines” (directed line segments
or vectors, as later named by Hamilton) and to apply these results to de-
rive Cotes’ Theorem and to obtain other formulae useful for the analysis of
polygons and spherical triangles. In the process of establishing his algebra of
directed line segments, he obtained the means to represent complex numbers
geometrically, a result that was not incidental to the writing of his work:
“The occasion for its composition was that I sought a method whereby the
impossible operations could be avoided . . .” [143, page 104]. To obtain his
analytic representation, Wessel needed to specify the operations of addition
and multiplication of directed line segments. He defined addition in a manner
equivalent to adding vectors ‘tip to tail’, i.e., the lengths and orientations of
the individual vectors are maintained, and the resultant vector extends from
the ‘tail’ of the first vector to the ‘tip’ of the second.

To represent multiplication, Wessel made use of an analogy with real num-
bers, for which multiplication of two numbers, e.g., a and b, can be written
in terms of equivalent proportionalities, i.e., \((a \cdot b) : a :: b : 1\). On the
basis of this relationship, Wessel obtained a product rule for two directed
line segments based on the similarity of triangles formed by a positively di-
rected line segment of unit length, +1, with the two directed line segments
A and B, and the line segment that represented their geometric product,
\(A \cdot B\) (Figure 2(a)).

With a product rule in hand, Wessel noted that the directed line segments
+1 and -1 could be represented as lying in opposite directions on a straight
line, with direction angles of 0° and 180° respectively, and he introduced a
new directed line segment, \(\epsilon\), also of unit length but oriented at 90° (Figure
2(b)). With these he was able to show that the product of \(\epsilon\) with itself, \(\epsilon \cdot \epsilon\),
produced the directed line segment -1; i.e.,

\[-1 : \epsilon :: \epsilon : 1 \text{ hence } \epsilon^2 = -1.\]
In so doing, Wessel had effectively geometrized \( \sqrt{-1} \) as a directed line segment represented by \( \epsilon \), which could be used to precisely locate complex numbers in the plane of a rectangular coordinate system, an achievement that would linger without recognition for more than a century, yet it would be duplicated, at least twice, in less than a decade.

2.6. Argand’s Essai

In the fall of 1806, Adrien Marie Legendre (1752-1833) mentioned in a letter to François Joseph Français (1768-1810) that a person unknown to him had given him a mémoire (apparently an unpublished manuscript not bearing the author’s name) that astonished him by the quality of the ideas that it contained. He disclosed the essentials of the work and encouraged Français to pursue them. Français was not able to complete this before his death in 1810, but the letter was discovered, and the work mentioned was carried out by his brother, Jacques Frédéric Français (1775-1833), a professor of military art at the artillery school at Metz. Français published a paper on the geometric interpretation of imaginary symbols in 1813, and at the end of the paper, he explained the circumstances by which his work arose, and he expressed the hope that the author would come forward to further present and defend it. Before the end of the year, Argand had replied to the journal with a summary of his earlier work and some additional thoughts. A brief debate on complex numbers ensued with Argand and Français on side against François-Joseph Servois (1767-1847) leading to Argand’s second and final publication in 1815 [95].
Very little of Argand’s life has been well documented [126, 3]. What has been stated and previously accepted derives largely from one source, Guillaume-Jules Hoüel (1823-1886), who edited and published Argand’s 1806 and 1813 essays in 1874. His full name as stated by Hoüel is Jean-Robert Argand, and he is said to have been born in Geneva in 1768 and to have died in Paris in 1822. His profession is reported as bookkeeper, though Schubring has argued that he may have been “a scientifically oriented technician, based in the Parisian clock industry.”

In his development of a geometric representation of complex numbers, Argand made use of directed line segments (i.e., vectors), as had Wessel, but whereas Wessel’s intention was to equip himself with a vector algebra to be used for cartographic and other geometric purposes, Argand’s intention was to clarify the algebraic status of $\sqrt{-1}$ by calling for the mean proportional, $+1: x :: x : -1$, with its solutions of $\pm \sqrt{-1}$, in the guise of the geometric mean. If $+1$ and $-1$ are represented by line segments of opposite directions, then there exists a third line segment, “such that the positive direction shall stand in the same relation to it that the latter does to the negative.”

Figure 3: Geometric rendering of complex numbers by Argand in Hoüel, 1874 [3].

Argand arrived at a geometric rendering of these relations by adopting his figure 4 (Figure 3 above), with the additional understanding that every directed line segment originating from $K$ would represent a quantity of the form $\pm a \pm b\sqrt{-1}$ that clearly represents its composition of real and imaginary parts. From the geometric properties of his construction, Argand obtained geometrical operations of addition and multiplication that were consistent with his adopted construction—and which, in fact, would later be seen as equivalent to Wessel’s. However, as Argand readily acknowledged, his geometrical representation of complex numbers and their associated algebraic properties was founded on an analogy that could be only regarded as hypo-
Theoretical, whereas in Wessel’s system, addition and multiplication were defined prior to an equivalent geometrization such as might be expressed by Argand’s figure 4, and, in fact, these defined operations were necessary presuppositions for Wessel to achieve this representation. Argand would proceed with his geometrical construction to present a proof of Ptolemy’s Theorem, to sketch a proof of the fundamental theorem of algebra, and to explore the possibility of extending this representation to three dimensions.

2.7. Buée’s Mémoire
1806 was a remarkable year for the geometric representation of complex numbers; a few months before Argand conveyed his manuscript to Legendre, in London the Royal Society published, Mémoire sur les quantités imaginaires, presented in the previous year by Adrien-Quentin Buée (1748-1826). Buée was a French Catholic priest who had fled to England in 1792 after having refused to give his oath to the French Constitution of 1791. He was not a professional mathematician; his Mémoire [12] would be his sole publication in mathematics. With regard to the soundness of their presentations, Buée’s is considered to be weaker than either Argand’s or Wessel’s, but of the three, certain ideas of Buée’s would receive marginally more immediate attention than those of the other two. Buée divided algebra into a “universal arithmetic” and a “mathematical language,” a division that would appear later in the work of George Peacock (1791-1858). He would distinguish different meanings for + and - ; they played different roles, whether as “signs of operations” or as “signs of qualities.” He also distinguished arithmetical from geometrical operations, but remarkably, he allowed the possibility that they might be re-united as a single “arithmetic-geometric” operation. Buée considered $\sqrt{-1}$ to be a prime example of one such operation, and he declared it to be the sign of perpendicularity.

He provided a figure to illustrate his intention that reveals a conception similar to Wessel’s and Argand’s (Figure 4 below). For mathematicians in England who would have no knowledge of Argand’s or Wessel’s work, Buée’s proposition marked the next provisional development towards a geometric representation of complex numbers.

2.8. The Anticipations of Foncenex, Karsten, and Truel
Wessel, Argand and Buée are given historical recognition, albeit belated, for separately arguing for and presenting geometric representations of complex
numbers that are largely the same conception, but there were still others who had advanced similar approaches [126]. In a postscript to his Mémoire, Buée briefly mentioned that after completing his work, he had become aware of a paper by François Daviet de Foncenex (1733/4-1799), Réflexions sur les Quantités Imaginaires (1759), in which he put forward a geometrical construction of an imaginary quantity by locating it on a perpendicular projected from a line segment of real numbers. Although his construction is valid, Foncenex was at pains to disavow what he had, in fact, correctly shown, stating that, “...imaginary roots do not, hence, admit a geometrical construction, and one cannot infer any advantage in resolving problems by using them ...” [126]. Foncenex was a military officer living in Turin, where he apparently became acquainted with Joseph-Louis Lagrange (1736-1813). His Réflexions had been written in the context of a dispute between Jean le Rond d’Alembert (1717-1783) and Euler about the admissibility and meaning of logarithms of negative quantities, and this assured that it was widely read. One of those who read and made use of Foncenex’s work was Wenceslaus J.G. Karsten (1732-1787), a professor of mathematics, first at Bützow, then later at Halle, who in a paper of 1768 (revised in 1786), constructed imaginary quantities geometrically while exploring a relation between a particular hyperbola and circle. Curiously, he would also follow Foncenex by first
disavowing the possibility and the legitimacy of the geometric construction of imaginary quantities, which he then went on to produce. Also in this era, as Cauchy would later claim, “un savant modeste,” Henri-Dominique Truel, had apparently obtained by 1786 a geometric representation similar to those of Wessel, Argand and Buée, but he concealed his methods until 1810, when he communicated them to a marine engineer, Augustin Normand, with whom Cauchy had become acquainted as a young man at Le Havre [3].

Despite the multiplicity of discoveries of a method to represent complex numbers geometrically, little of this work was immediately carried forward and developed in the practices of the mathematicians of that age; it largely remained unknown, unconsidered or dismissed. In 1817-18, Benjamin Gompertz (1779-1865) published, *The Principles and Application of Imaginary Quantities*, in two volumes, the second containing a geometric interpretation that was loosely inspired by Wallis and Buée. In 1828, geometrical representations were taken up in England by John Warren (1796-1852) in, *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, and in France by C.V. Mourey (c.1791-1830), in, *La vraie théorie des quantités negatives et des quantités prétendues imaginaires*. Seemingly unaware of their predecessors, both worked with directed line segments and introduced rules of addition and multiplication formally, in a manner similar to Wessel [3]. Warren’s work was unknown outside of Britain, though it did catch the attention of Peacock, and decades later, of W.R. Hamilton (1805-1865) [122].

2.9. *A Christening by Gauss*

The tipping point for the acceptance of a geometric representation of complex numbers was reached in 1831 when C.F. Gauss (1777-1855) explicitly advocated for it in the *Commentatio secunda* to his paper, *Theoria residuum biquadraticorum*. In it Gauss remarked that traces of his viewpoint that imaginary numbers should be considered to have as objective an existence as negative quantities were to be found in his first paper, published in 1799. By 1811, in a letter to Bessel, he made it clear that he had taken up the geometrization of imaginary numbers:

> Just like the realm of all real magnitudes can be conceived as an infinite straight line, so can the realm of all magnitudes, real and imaginary, be made meaningful by an infinite plane, in which
every point, determined by abscissa=a and ordinate=b, as it were represents the quantity a+bi. [3, page 89]

Two decades later, Gauss began to advocate openly and forcefully for a geometric representation of imaginary numbers. In his *Commentatio secunda* he started by reviewing the status of negative roots and justifying their use:

The early algebraists called the negative roots of equations false roots, and these are indeed so when the problem to which they relate has been stated in such a form that the character of the quantity sought allows of no opposite. But just as in general arithmetic no one would hesitate to admit fractions, although there are so many countable things where a fraction has no meaning, so we ought not to deny to negative numbers the rights accorded to positive simply because innumerable things allow no opposite. The reality of negative numbers is sufficiently justified since in innumerable other cases they find an adequate substratum. [7, page 45]

Gauss next began to contrast the circumstances of imaginary numbers with those of negative numbers:

. . . the imaginary quantities—formerly and occasionally now, though improperly, called impossible—as opposed to real quantities are still rather tolerated than fully naturalized, and appear more like an empty play upon symbols to which a thinkable substratum is unhesitatingly denied by those who would not depreciate the rich contribution which this play upon symbols has made to the treasure of the relations of real quantities. [7, page 45]

Gauss remarked that he

. . . had for many years considered this highly important part of mathematics from a different point of view, where just as objective an existence may be assigned to imaginary as to negative quantities. . . [7, page 45]

He observed that “positive and negative numbers can only find an application when the thing counted has an opposite which when conceived of as united with it has the effect of destroying it,” and that this will only occur “. . . where the things enumerated are not substances (objects thinkable in themselves), but relations between any two objects.”
Gauss then went on to consider the relations that represent transitions from one object to another object in a single series of objects and the relations that are obtained in an opposite sense by reversing the order of the transition. For relations of a single series, +1 and -1 are sufficient to indicate the order of the transition, but for a series of series, i.e., a two-dimensional manifold, an additional pair of units denoting opposition is required, namely +i and −i. He then remarks that, “these relations can be made intuitive only by a representation in space . . .” He discusses this representation by pointing out that the directions of +1 and +i may be arbitrarily assigned, with -1 and −i determined by their required opposition. Gauss then pointed out that geometrically +i is a mean proportional between +1 and -1, and therefore it corresponds to \( \sqrt{-1} \).

Gauss then concluded:

Here then the demonstrability of an intuitive signification of \( \sqrt{-1} \) has been fully justified and nothing more is necessary to bring this quantity into the domain of objects of arithmetic.

We have thought to render the friends of mathematics a service by this brief exposition of the principal elements of a new theory of the so-called imaginary quantities. If people have considered this subject from a false point of view and thereby found a mysterious obscurity, this is largely due to an unsuitable nomenclature. If +1, -1, \( \sqrt{-1} \) had not been called positive, negative, imaginary (or impossible) unity, but perhaps direct, inverse, lateral unity, such obscurity could hardly have been suggested. [7, page 47]

2.10. Cauchy’s Conversion

With Gauss’s acceptance of their rightful algebraic status and endorsement for their geometrical representation, complex numbers were rapidly brought into the mainstream of mathematics. Even mathematicians such as Augustin Louis Cauchy (1789-1857), who had a long-standing extreme aversion to granting ontological status to imaginary numbers, were eventually won over. Cauchy was a mastermind behind the development of complex function theory, but he did not acknowledge complex numbers as mathematical objects as such, holding that, “... an imaginary equation is only a symbolic representation of two equations between real quantities.” When Gauss wrote (above) about “... imaginary quantities ... still rather tolerated than fully
naturalized . . .” he had Cauchy and allied mathematicians such as De Morgan in mind [98]. Cauchy carried out his work on complex numbers with “a horror of $\sqrt{-1}$”, and as late as 1847, he would claim that his formal symbolic approach had helped him to avoid, “the torture of finding out what is represented by the symbol $\sqrt{-1}$, for which the German geometers substitute the letter $i$.” (Andersen, 1999) Nevertheless, somewhat later in the year, Cauchy would divulge that after “mature reflections” he had adopted a geometric representation of complex numbers that was essentially the same as that of Wessel. By the middle of the 19th century, the geometric representation of complex numbers was commonly accepted and routinely utilized.

2.11. A Source for Further Abstraction

Beyond its immediate conceptual utility, the geometric representation of complex numbers would serve as a precursor and stimulant for the development of abstract algebra (analysis) and of the methods of vector analysis that would progressively emerge in the latter half of the 19th century [63]. With the intention of founding algebra as a science of pure time (a companion to the Kantian notion that geometry was the science of pure space) and having studied Warren’s work on complex numbers, Hamilton (1805-1865) extracted from the two-dimensional model of complex numbers an algebraic representation based on couples (ordered pairs of real numbers deduced from ‘time steps’), accompanied by their associated operations and properties. Later, shorn of Hamilton’s metaphysical intentions but retaining his careful axiomatic and constructive methods, the algebra of couples would signal an early departure from the grounding conception of the algebra of real numbers towards the construction of abstract algebras with widely diverse elements, operations and properties. Hamilton, as well as others, would go on to search for an analogous algebra of triplets that would correspond to a representation of a ‘complex’ three-dimensional space, a quest for a result that later would be proved to be impossible [122]. Having spent more than thirteen years on this futile search, Hamilton came one day to a sudden and dramatic realization that an expansion of his conception to that of a ‘hypercomplex’ four-dimensional space (actually, a three-dimensional vector space joined with an additional scalar quantity) was required. Critical to this discovery was his visualization of the rotation of the complex plane about its real axis in a direction orthogonal to the plane [68]. The algebraic elements of this conception he named *quaternions*, and the algebra that resulted was
characterized by the non-commutativity of its operation of multiplication. Hamilton would spend the remainder of his life developing the theory of quaternions and advocating (along with Tait) for its incorporation in the mathematical development of theoretical physics. While quaternion theory had early successes, notably in Maxwell’s formulation of electrodynamics, it ultimately was superseded by the rival vectorial methods of Gibbs and Heaviside, nevertheless greatly shaping these latter methods in the course of their contention [139, 145].

3. A Cognitive-Historical Analysis of the Conceptualization of Complex Numbers

3.1. Evidence of the Conceptual Discontinuity of Complex Numbers

Cauchy’s life, with its “horror”, “torture”, and “mature reflections”, offers a glimpse of the navigation by one individual in the course of his lifetime through a conceptual discontinuity—a representation in miniature of the journey taken by mathematicians over a span of two millennia from the “impossible”, the “false”, the “imaginary”, to the “complex”. The conceptual development of complex numbers originated with reactions of avoidance and denial, progressed to a begrudging acceptance based on their undeniable utility, found mitigation through a new geometric analogy, then stunningly burst forth in a multitude of similar conceptions that eventually became widely accepted, even commonplace. In the conceptual development of complex numbers, a discontinuity is clearly surmounted, and the bootstrapping and scaffolding processes that gave rise to this circumstance are available for examination.

3.2. Nersessian’s Model-Based Reasoning Approach

The climactic era for the conceptualization of complex numbers coincided with the appearance of a variety of similar geometric models, each of which supported the visualization of the imaginary unit $i$. The imaginary unit

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1 Editors’ note: For more on this story especially emphasizing the geometric interpretation of complex numbers, see the article “What is an Imaginary Number? The Plane and Beyond” by Andrew Powell, also in this issue of the Journal of Humanistic Mathematics.

2 A similar perspective is explored in [73], where this journey is compared to that of a contemporary novice student of mathematics.
consequently had been given an imagistic representation that was both a structural and a dynamical analog available for simulation and for other forms of model-based reasoning. Specifically, $i$ could be visualized as a directed line segment of unit length lying orthogonal to the line joining 1 and -1, and multiplication by $i$ could be seen as equivalent to a $90^\circ$ rotation in the plane containing the line. See Figure 5.

Figure 5: Multiplication by $i$ is analogous to rotation by $90^\circ$.

The geometric plane that was eventually conceptualized was not the initially available two-dimensional Cartesian plane populated by points corresponding to ordered pairs of ‘real’ numbers—i.e. the integers and the accompanying rational and algebraic irrational numbers that together comprised the number system available at the end of the 18th century. Rather, it was the analogous complex plane whose points corresponded to ordered pairs $(a, b)$ representing the complex numbers $a + b\sqrt{-1}$. The revisioning of the real-valued plane as it was familiar to the mathematicians of the early 19th century to one involving complex numbers is an example of an analogical bootstrapping process of the kind identified by Nersessian. It denotes a progression from a previously obtained intermediate model to the final conception of the complex plane. However, this intermediate model, the Cartesian plane, was itself a hybrid progressively developed (as previously discussed) from a succession of models that did not utilize negative axes until their justification and inclusion by Wallis. In fact, the conceptualization of negative numbers presents a second, historically earlier example of the bootstrapping of a conceptualization that, like complex numbers, required recognition of both object and process aspects in order to overcome a conceptual discontinuity. The circumstances of this prior conceptualization reveal its necessity; the conceptualization of com-
plex numbers was unobtainable until negative numbers were first adequately conceptualized, and when this was finally in hand, it partially enabled the subsequent conceptualization of complex numbers.

3.2.1. The Conceptual Discontinuity of Negative Numbers
The earliest appearance of negative numbers and the rules for calculating with them via red and black counting rods appeared in *The Nine Chapters on the Mathematical Art*, the fundamental source of traditional Chinese mathematical knowledge first codified in the Han and Xin Dynasties (206 BCE-220 CE) [132]. Centuries later in India, negative numbers received a systematic treatment, notably in the works of Brahmagupta (598-665) [94] and Bhaskara II (1114-c.1185) [132]. The works of Indian mathematicians were brought to Arabia in the 9th century, and by the 11th century, Islamic mathematicians such as al-Karaji (953-1029) had accommodated negative numbers in their mathematical works. Negative numbers were subsequently introduced to European mathematicians through such Arabic texts, but most mathematicians of the 16th and 17th centuries did not accept them as numbers, particularly as roots of equations [86]. Chuquet and Stifel referred to them as absurd. Cardano considered them to be fictitious, yielding impossible solutions. Viète discarded them, and Descartes regarded negative roots as false. Eventually a growing acceptance of negative numbers emerged in the work of Bombelli, Stevin and Girard. Wallis, as shown above, accepted negative numbers and roots, though he maintained some conceptions about their relationship to zero and infinity (as did Euler) that later would be shown to be incorrect. However, despite their undeniable and growing utility, deep distrust of negative numbers persisted for some mathematicians into the early 19th century, as may be seen in the arguments of Masères, Frend, and De Morgan [122].

3.2.2. The Bootstrapping of Negative Numbers
The conceptual discontinuity that accompanied the negative number concept was transcended by bootstrapping in the way shown by Wallis, whereby an intermediate model that is essentially a positive number line supports the uptake of a new bootstrapping analogy that regards *number as displacement* along that line. Whereas the prevailing analogy of *number as a representation of an object* served as a hindrance to the conceptualization of negative number—i.e., it would signify a “... Magnitude Less than Nothing, [...] Number Less than None”—its association with an analogous
process, a spatial displacement—either “forward”, “backward”, “advanced”, or “retarded”—brought conceptual clarity to the significance of a negative number.

3.2.3. A Succession of Bootstraps
It should be noted that the intermediate model in this instance of bootstrapping, namely the positive number line, had itself arrived on scene via prior bootstrapping scenarios that involved a geometric line as the analogical source domain for several intermediate models, supporting the bootstrapping analogies that different numbers are analogous to different points on a line, that different numerical magnitudes are analogous to different line segments and that different numbers (i.e. the rational numbers) are analogous to the proportions of different line segments. Preceding these models, the representation of number by a geometric point was itself the result of a bootstrapping episode whereby the conceptual model is a spatially distinct geometric point and the associated bootstrapping analogy asserts that a distinct number is analogous to a distinct point. As revealed by this sequence of bootstrapping episodes, many of the signature elements of Nersessian’s model-based reasoning are clearly evident: there is a succession of hybrid, intermediate models issuing from the geometrization of natural numbers onward to complex numbers, and there are prominent roles played by all of the representational modes—imagistic representation, simulation, and analogy.

3.3. Sfard’s Three-Stage Approach
Sfard’s emphasis on object-process complementarity and her three-stage approach to mathematical concept development via processes of interiorization, condensation and reification prefigures numerous aspects of this present work. This is further borne out by her recognition and concern with conceptual discontinuities in the learning processes of mathematics, her formulation of a process of reification to account for the surmounting of these discontinuities, her construction of a hierarchical model of the conceptual development of number systems, her early engagement with the roles of analogy and conceptual metaphor in mathematical conceptualization, and her adoption of a dual historical and psychological perspective [127, 128, 129]. However, her hierarchical account of mathematical concept development proceeds from a lower-level process conceptualization to a higher-level conceptualization of the reified object, and for complex numbers and even for negative
numbers this progression appears to be at variance with what historically occurred. Nevertheless, some congruence between her account and the present one can be found in her later discussion of the vicious circle entailed by reification:

“But here is a vicious circle: on one hand, without an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher-level processes are performed seems indispensable for the interiorization—without such objects the processes must appear quite meaningless. In other words: the lower-level reification and the higher-level interiorization are prerequisite for each other!” [127, page 31]

Here, “higher-level interiorization” connotes processes at the higher level of representation at which the reified object is now brought into play. As she would succinctly point out, “According to the model, reification of a given process occurs simultaneously with the interiorization of higher-level processes.” In other words, Sfard has moved to an understanding that both the reified object and its higher-level processes in some manner must appear together, and this is consistent with her earlier determination that the object and processes are complementary aspects of a holistic conception. Furthermore, this holistic conception, in both of the cases of negative and complex numbers, admits a visualizable representation—in a sense a ‘reified’ image—that contributes greatly to the comprehension and acceptance of the model. The nature of such a holistic conception in the context of mathematical conceptualization has been pursued by Gray and Tall and co-authors [61, 136] who refer to it as a procept, an amalgam of the process with the mathematical object that it produces, along with the symbol used to represent them both.

3.4. Carey’s Bootstraps and Core Representational Resources

Nersessian’s model-based reasoning approach provides the means to re-envision the transition across a conceptual discontinuity in terms of a succession of intermediate hybrid models, each reducing the explanatory gap between the initial conceptualization and the one finally achieved. However, the roles played by a succession of mental models alone may not fully eliminate the conceptual discontinuity in all cases, particularly in the cases at hand of negative and complex numbers. Sfard’s attention to process descriptions as a necessary complement to object descriptions suggests the aspect that has
been missing in the modeling scenario, but her notion of reification ultimately needs to be expanded to include the consideration of the bootstrapped entity as an object-process holism. Yet another notion of bootstrapping seems to be required, one that directly engages the potential for conceptual discontinuity and also that situates bootstrapping in the broader milieu of the cognitive resources available for constructing a higher-level conceptual representation system that is locally incommensurable with a lower-level system but one that is nevertheless obtained despite the presence of this conceptual discontinuity. A fitting candidate for such a notion may be found in Carey's conception of Quinean bootstrapping.

Carey claims that Quinean bootstrapping provides the cognitive means to ascend beyond the conceptual scaffolding provided by the core representational resources utilized in the lower-level conceptual system to attain a higher-level system that is conceptually discontinuous. Seen in light of the foregoing analysis, conceptual bootstrapping involving both object and process analogies that are jointly undertaken—such as occurs with negative and complex numbers—might be characterized as an episode of Quinean bootstrapping.

Furthermore, the concatenation of bootstrapping episodes discussed previously constitutes a prime example of what Heintz identifies as “the scaffolding on core cognition” that as a developmental process leads to conceptual change [72]. Specifically for number, this begins with the analogical bootstrapping of the representation of number as a geometric point, scaffolded by set-based quantification (SBQ) which is implemented to demark a point as distinct from its background. Parallel individuation (PI), along with SBQ, operates at the next level of scaffolding to associate different numbers analogically with different points. Finally, the association of number with spatial magnitudes attributed to the analogue magnitude system (AMS) initiates the bootstrapping that, along with PI and SBQ and the process analogy number is a displacement, ultimately leads to the conceptualization of negative numbers. However, these three core systems alone are insufficient to scaffold the analogies that produce the imagistic representations needed to conceptualize complex numbers, principally because they are unable to represent the process of rotation that accompanied the visual representation of complex numbers. This circumstance suggests that an additional core representational system was required to be brought into play with the scaffolding of the conceptualization of complex numbers.
3.5. Suggestions of a Core Role for Mental Rotation

It is curious that the conceptualization of complex numbers might have been initiated by an instance of the human cognitive capacity to mentally rotate an image, in this case of a geometric object. Although it can never be determined with any certainty what role, if any, might have been played by such an instance of mental rotation (MR) in the bootstrapping of complex numbers, there are some historical indications that, indeed, this may have occurred.

3.5.1. Hints of a Phylogenetic Factor

First of all, the discovery of the means to geometrically represent complex numbers was not the inspired achievement of a solitary mathematical genius. By published accounts, it emerged from the writings of Wessel, Argand and Buée and more tangentially from the attempts of Foncenex and Karsten and from Cauchy’s anecdotal remarks concerning Truel. All of these discoveries occurred within a few years of each other—virtually simultaneously, given the long history of imaginary numbers—and with one minor exception, none of these individuals was acquainted with any of the others or with their work. Each was geographically and culturally isolated from the others, and what is more, their singular discoveries comprised virtually the entirety of their mathematical attainments. Quickly following their work and largely independent of it were the writings of Gompertz, Warren, and Mourey, and altogether this circumstance raises the possibility that a tipping point may have been reached for the conceptual bootstrapping process when, with a salient placeholder at last commonly available (i.e. the Cartesian plane), a cognitive factor possessed by each of these men—perhaps one that is phylogenetic—played a decisive role.

This raises the question, however, why individuals with limited formal mathematical education operating in isolation outside the boundaries of professional mathematical institutions and societies might implement this cognitive factor, while the professional mathematicians of the day, with the apparent exception of young Gauss, could not, or chose not to take up similar thoughts. The answer is perhaps found in the pervading collective attitude of the mathematical community of the era that held that arguments based on analogy, inductive reasoning, and particularly those involving geometrical visualization, were of a much lower stature, if not to be rejected outright, than those achieved by algebraic demonstration (proof by deductive logic) [3, 122]. Given this prevailing professional sentiment, the timidity and reluc-
tance of Foncenex and Karsten to defend their arguments for geometrizing complex numbers, despite the overall soundness of the arguments, begins to be explicable, as does the avoidance of analogy and geometrical methods by the majority of mathematicians of this period, and even Gauss’s more-than-thirty-year delay in openly defending the use of such methods.

Although the foregoing historical considerations are only circumstantial, with no record of the details of the discovery processes employed by any of the co-discoverers of the geometric model of complex numbers, it is noteworthy that there does exist in Hamilton’s publications and correspondence a remarkably detailed account of his thought processes that led to his extension of complex numbers to quaternions, and in that account the bootstrapping role played by mental rotation as a visual modeling process is clearly confirmed [68].

3.5.2. Evidence for a Fourth Core System of Numerical Cognition

If mental rotation were to be accepted as a crucial contributor to the bootstrapping of complex numbers, then a question arises concerning its status. From the initial behavioral experiments by Shepard and Metzler in 1971 to present-day neural studies involving fMRI, EEG, etc., MR has been exceedingly well documented as a cognitive phenomenon, to the extent that it is used as a metric for testing differences in age, gender, and ability. A variety of studies [78, 79, 147, 150] support the conclusion that there are two dissociable types of MR—one concerned with object-based transformations (OR) and another concerned with egocentric transformations (spatial perspective taking, SPT). These two types of MR are associated with different characteristic reaction time profiles, and they are implemented by two different cognitive systems that are correlated with partially distinct neural activations, depending on the task studied [137]. In the discussion that follows, MR will be understood to refer to object-based transformations (OR).

Based on a variety of recent empirical studies, MR can be shown to bear the hallmarks of the six defining criteria set by Carey for core cognitive systems [18, 20]. First of all, MR is innate. Rudimentary MR has been observed in infants as young as 3 months, as well as with 5-, 8- and 10-month-olds [77, 46, 47, 91, 92]. Secondly, MR capabilities are elaborated and strengthened during childhood development and may decline for the aged, but MR remains a continual presence throughout the lifespan of the human [78, 79]. Thirdly, MR has a long evolutionary history. It is observed generally across Homo sapiens and to a lesser extent in great apes [114].
Fourthly, its format is iconic, utilizing analogue representations of objects being subjected to MR [48, 49, 87]. Fifthly, it is domain-specific and neurally involves the dorsal and ventral-caudal regions of the intraparietal sulcus and the superior parietal lobule [140, 124, 149, 125]. Lastly, it is a dedicated perceptual-input analyzer producing conceptual representations that cannot be reduced to perceptual or sensori-motor primitives [84].

As might be suspected from the overlapping regions associated with MR and those attributed to the AMS, performance on MR and on basic numerical representation skills have been shown to be linked [71, 140, 138]. While it has been well established that spatial ability—for which MR plays a central role along with spatial orientation and visualization—is strongly correlated with STEM (science, technology, engineering and mathematics) expertise [141], and a variety of studies have linked MR skills to performance in areas of mathematics ranging from geometry, algebra, and word problems to mathematical logic and computational mathematics [69, 70, 11, 96, 148, 57], the correlation of MR to developmentally early numerical and mathematical attainment is being substantiated as well. Mix, with her team of collaborators [97], have found that MR is the best predictor of math performance among the kindergarten students that they studied. Meanwhile, Lauer and Lourenco have carried out a study of spatial aptitude that provides evidence that spatial-change detection processes present in 6- to 13-month-old infants predict interindividual variation in both spatial and mathematical competence at 4 years, a result that they suggest may be the developmentally earliest predictive relation known to occur between spatial processing and mathematical knowledge [92]. Complementing these results from infant studies, research by Amalric and Dehaene has established that mathematicians engaged in high-level mathematical tasks utilize the same neural networks that are associated with core numerical systems, and they do not recruit language circuits while reflecting on higher-level mathematical statements that do not contain numbers [1, 4]. On the basis of this accumulated and growing evidence, it seems reasonable to consider mental rotation to be a fourth core cognitive system, joining set-based quantification, parallel individuation, and analogue magnitude systems as an innate representational resource with numerical content.
4. The Bootstrapping and Scaffolding of the Conceptualization of Complex Numbers

4.1. Mappings of Analogy and Conceptual Metaphor

The foregoing analysis of the conceptual development of complex numbers draws upon a body of work centered on Nersessian’s cognitive-historical approach to case-study analysis and mental-modeling theory augmented by Gentner’s structure-mapping of analogy, as well as Carey’s research on systems of core cognition and Quinean bootstrapping, and additionally, on the incorporation of process description based on the implications of a handful of diverse research initiatives. A broadly similar corpus, with many congruent features—but with several crucial divergences as well—has already been elaborated in the stream of linguistics research in cognitive semantics that issues from Lakoff and Johnson’s work on embodied cognition and conceptual metaphor [85, 86], Fauconnier’s work on mental spaces [39, 40] and his subsequent development of blending theory with Turner [41, 42, 43], followed by the refinement and application of these approaches to mathematical cognition by Lakoff and Núñez [90]. In this second body of work, the conceptual mapping of the hierarchical structures that support complex numbers has already led to results that will now be reviewed and expanded upon in the following sections.

While metaphor is a more extensive and inclusive category than analogy, the notion of extended conceptual metaphor is in many respects the same as that of analogy [52, 53, 54, 55, 41], particularly in the case of the analogical/metaphorical aspects of the conceptualization of complex numbers. Thus, Gentner’s structure mapping approach, with its base and target domains, mapping rules, and systematicity principle, will be considered here to be largely equivalent to Fauconnier’s mental spaces, frames, roles and connectors [39, 40]. The further development by Fauconnier and Turner of conceptual blending theory [40, 42, 43], with its focus on the processes of conceptual integration, also finds an analogue in Nersessian’s approach to bootstrapping.

4.2. Blending as Bootstrapping

Conceptual blending, which entails the construction of a blended space comprised of elements drawn from the source and target spaces of a conceptual metaphor, seems at first blush to be the direct antithesis of a sudden and dramatic bootstrapping event. Nevertheless, although Fauconnier and Turner
characterize blending as “... dynamic, supple and active in the moment of thinking...” [43, page 133], they are at pains to clarify that blending, “... often performs new work on its previously entrenched products as inputs.” More specifically, they state that, “Conceptual blending is not a compositional algorithmic process and cannot be modeled as such for even the most rudimentary cases. Blends are not predictable solely from the structure of the inputs. Rather, they are highly motivated by such structure, in harmony with independently available background and contextual structure... the blend contains emergent structure not in the inputs. Blends may contain incongruent conceptual structure with respect to the input spaces that makes the blend more visible” [43, pages 136-142]. Conceptual blending thus can be seen to share many of the features generally attributed to Nersessian’s conception of mental-model bootstrapping and as well to Carey’s allied notion of Quinean Bootstrapping.

4.3. Fauconnier and Turner’s Case Study of Complex Numbers

With their theory of conceptual blending in hand, Fauconnier and Turner apply it to a number of case studies, in particular to the conceptual development of complex numbers [40, 42, 43]. In their analysis, they make use of their “Four Space Model” (see Figure 6 below) that includes a “generic” middle space in addition to the two input (source and target) spaces and the resultant blended space. The source space (Input I_1) is the geometric plane, containing points, lines and vectors, the target space (Input I_2) is the real number system, and the blend is the system of complex numbers.

The fourth space in the Four Space Model is the generic space, which in the context of the complex number blend is identified as the “commutative ring operations on pairs of elements.” Initially, Fauconnier and Turner [42, page 14] assert that “...when a rich blended space of this sort is built, an abstract generic space will come along with it,” but they soon clarify that, “...the concept of a complex number is logically and coherently constructed in a blended middle space, on the basis of a (presumably non-conscious) generic middle space structured as a commutative ring,” adding in the relevant footnote that,

23The generic space is not consciously conceptualized as an abstract domain when the full-blown concept of complex number gets formed. It becomes a conceptual domain in its own right, when mathematicians later study it and name it.” [42, page 36]
In a more formal presentation of blending theory published four years later [43], the authors elaborate their notion of generic space, stating that the abstract generic space maps onto each of the input spaces, helping to define the cross-space mapping between the two input spaces.

Figure 6: Four Space model from Fauconnier and Turner, 1998 [43].

Generic spaces and blended spaces are related: blends contain generic structure captured in the generic space, but also contain more specific structure and can contain structure that is impossible for the inputs ... [43, page 143]

4.4. Lakoff and Núñez’s Case Study of Complex Numbers
Building upon the Fauconnier and Turner case study and elaborating it significantly, Lakoff and Núñez [90] have constructed a conceptual metaphor and blend analysis of complex numbers that goes into considerably more
detail than the single-blend analysis of Fauconnier and Turner. Lakoff and Núñez have constructed multiple blends, the first being the Multiplication-Rotation Blend that results from the *Multiplication by -1 is Rotation by 180°* metaphor, and the second blend, the Number-Line Blend, that results from the *Numbers are Points on a Line* metaphor. (See Figure 7 below.)

These two blends are then themselves blended to produce the Rotation-Number-Line Blend. Another blend, the Cartesian Plane Blend, is also constructed by blending the previously obtained Number-Line Blend with the *Euclidean Plane with Line X Perpendicular to Line Y* metaphorical space.
The resulting blend is now blended with the Rotation-Number-Line Blend to produce the Rotation-Plane Blend. From this blend a metaphor is extracted (Rotation by 90° is Multiplication by i), and the resulting metaphorical blend leads to the conceptualization of the complex plane and complex number system.

While the Lakoff and Núñez construction is rich in the details of a succession of conceptual metaphors and blends and it is inferentially correct, it suffers from a lack of groundedness in known cognitive systems and processes that would help it to successfully distinguish itself from other possible constructions. An empirical study [30] has brought the Multiplication by -1 is Rotation by 180° metaphor into question as a suitable depiction of the cognitive process actualizing the conceptualization of negative numbers. In particular, the authors of the study are led “... to question whether the multiplication by (-1) as rotation metaphor is truly a component of the conceptual blend of the complex plane, or whether this metaphor arises as a consequence of experience with the complex plane (or other experience with multiplication of polar coordinates)” [30, page 9].

Additionally, although Lakoff and Núñez build their analysis on The Four Basic Grounding Metaphors of Arithmetic and they do make use of the cognitive phenomena of subitizing (PI/OTS) and mental rotation (MR), these core systems go unrecognized as such. In fact, subitizing is misconstrued to the extent that it is held to provide both specific numerical content and order relations, thus taking on characteristics that are normally associated with the AMS. Similarly, the concept of Set is used in their analysis but is unattributed to a core process such as SBQ. That specific reference to core systems of cognition is lacking is hardly a surprise because Lakoff and Núñez’s preparation and publication of [90] occurred at a time when the central notions of core numerical cognition were just being developed.

4.5. The Complex Number Scaffold and Bootstraps

Although Núñez and Lakoff provided a far more elaborate account of the conceptual blending of complex numbers than the one first presented by Fauconnier and Turner, they did not utilize the notion of a generic space that was an important feature of the earlier account. In fact, the notion of a generic space with abstract content relevant to both input spaces as well as the blended space presents an opportunity to introduce core cognitive systems into the context of conceptual blending. Fauconnier and Turner had
already given expression to the possibility of the existence of a, “...generic space that is not consciously conceptualized as an abstract domain...”, yet one that, “...maps onto each of the input spaces, helping to define the cross-space mapping between the two input spaces.” This role could be seen as an anticipation of one played by a core cognitive system such as any or all of the four systems with numerical content identified in the conceptualization of complex numbers.

Given this circumstance, a possibility for carrying out analogical modeling arises, that, fittingly, is itself the outcome of an analogy between conceptual metaphor and blend theory as the target domain and analogy and bootstrapping theory as the source domain, where, accordingly, the concept of domain is mapped to space, system of core cognition to generic space, placeholder to input space, and bootstrapping to blend.

Figure 8 below illustrates the bootstrapping and scaffolding by core processes that are involved in the conceptualization of the complex number system. Both object- and process-based analogies play prominent roles as the construction of a series of intermediate mental models proceeds from one with a source domain involving collections of points, then to a second model with a source domain involving lines, then onward to a third model involving displacements and lines, followed by a fourth model involving lines and planes. This succession of intermediate hybrid models, each the result of a change of source domain, highlights the cognitive difficulty involved with bootstrapping. A new source domain is invoked in each case that is not logically deducible from the previous source domain, although once identified, the latter and former domains can be seen to bear a natural relation with each other.

Beginning with the first mental model, the systems of set-based quantification (SBQ) and parallel individuation (PI) take up a ‘bridging’ role for the construction of two analogies—the Points are Geometric Objects analogy and the Numbers are Algebraic Objects analogy—which are then progressively bootstrapped to produce the Point-Natural Number Bootstrap. Advancing to the next intermediate model, the analog magnitude system (AMS) is implemented to produce the Line-Natural Number Bootstrap. Next, with the uptake of the process analogy that associates the formation of a geometric ratio with the algebraic operation of division, the intermediate model of the Positive Rational Number Line is bootstrapped and extended to include zero.
With further extension via Dedekind cuts, the Positive Real Number Line Bootstrap is obtained. At this point the next intermediate model appears, featuring the concept of displacement in both its object and process aspects. This model in turn leads to the Real Number Line Bootstrap. A subsequent hybrid model is then obtained from the Real Number Line and Cartesian Plane Bootstraps. At this point, and not before, the core cognitive process of mental rotation (MR) assists with the bootstrapping that results in the Complex Number System and Complex (Argand) Plane Bootstrap by supporting the analogy, \( \text{Multiplication by } i \text{ is Rotation by } 90^\circ \).

In this succession of mental models, the conceptual scaffolding provided by the systems of core numerical cognition, as anticipated by Carey and by Heintz, is clearly evident. The progression of the scaffolding rises upward,
illustrating the cognitive-historical development of the complex number system and indicating why mental rotation was unavailable for bootstrapping purposes until intermediate bootstrapped visual conceptualizations of the real number line and the Cartesian plane were both fully historically available as representations. This structure, as a consequence, is well grounded both in the factuality of the historical development of the complex number system and in the role played by core systems of numerical cognition whose roles are substantiated by a diverse and growing body of empirical research.

5. Recent Empirical Studies

The foregoing cognitive-historical analysis has been an excursion into a realm of abductive, imagistic reasoning where analogy has been given a primary bootstrapping role, scaffolded by core representational resources. Nothing has been proved, neither as historical nor as scientific fact. The interpretations offered are wholly based on circumstance. The sole justification of the interpretations that have been offered is the extent to which they may serve to interpret and integrate a disparate collection of empirical studies that have already been carried out and to stimulate further research.

Presently, the theoretical approaches to object-process complementarity by both Sfard and Dubinsky are playing active roles in mathematics education research, and more to the purposes of this present work, in the exploration of the cognitive aspects of mathematical learning. Drawing upon both of their approaches as well as that taken directly from Piaget, Norton has articulated an action-object perspective where objects are seen as subordinate to actions, and he has used this perspective to analyze the concept of cohomology as developed in abstract algebra [111]. As well, he has recognized that mathematics is a product of psychological action and has begun to inquire about neural and cognitive mechanisms that are associated with actions and objects as well as the conceptual processes of interiorization, condensation and reification, all of which might support students’ construction of new mathematical objects [112].

More specific to this inquiry, there exist at present a handful of studies that are focused on the conceptualizing and learning of complex numbers and rudimentary complex analysis as viewed through the lens of Sfard’s approach and, in some cases, APOS Theory. Danenhower [31, 32] studied twenty undergraduate students in three classes of complex analysis at two Canadian universities. Among his findings, which he analyzed from both Sfard’s framework
and APOS Theory, he observed that many students displayed an inability to flexibly navigate between object and process representations of complex numbers. They had a tendency towards “thinking real—doing complex” and would choose to work with a Cartesian (algebraic) form when a polar (geometric) representation would have led to considerable simplifications. In sum, they tended to regard $i$ as a static object and not as an operator acting on other objects when appearing in the context of multiplication.

In Greece, Panaoura and colleagues [115] studied ninety-five high school students as they adopted either a geometric or an algebraic representation to carry out calculations and word problems. In general, the students that used a geometrical approach were more successful with calculations, observing relevant connections and relations, than those using an algebraic approach, but the former tended to compartmentalize their approach and did not flexibly apply it when solving word problems.

In the United States, Conner and colleagues [30], as previously discussed, carried out a study while conducting a two-and-a-half week instructional unit for ten pre-service high school math teachers. Initially, the prospective teachers largely regarded a complex number as “a pair of things,” but eventually a majority progressed to a view of it as “a single thing.” None of the ten teachers formed a geometric model of multiplication by a complex number as involving rotation.

By far the most active research group exploring complex number cognition from a Sfardian perspective is that of Soto-Johnson. In a series of studies, she and her collaborators have explored the representational fluidity of an expert teacher and researcher [133]; the use of dynamic representations and gestures blended with metaphors by six experts who also displayed evidence of reification when explaining complex variable concepts [134]; the difficulty of attaining capability or comfort with the object/process duality of complex numbers by three in-service high school mathematics teachers [80, 81]; the use of algebraic information supplemented by dynamic gestures to complete geometric representations of complex numbers, elicited through interviews of two undergraduate students [135]; the similarities in use of gesture to explain complex number operations by six experts and four pairs of novices [67], and the issues of conceptualizing continuity, differentiation and integration of complex-valued functions (see contributions to the 2017 RUME Conference).
As a whole, the studies mentioned above yield results that are consistent with the object-process complementarity framework advocated herein. They reveal aspects of the conceptual difficulties faced by complex-number learners as they struggle to progress from an algebraic representational system—where a numerical object, e.g. the negative real number $-1$, is subjected to the algebraic processes of calculating a square root—to that of a geometric representational system—where $\sqrt{-1}$ is associated with a vector-like object and the process of rotation-by $-90^\circ$. In the context of the present analysis, this representational shift is the outcome of multiple episodes of bootstrapping.

The studies above have begun to document the challenges typical of the bootstrapping processes that learners who master complex numbers encounter in the course of their learning. Studies such as these need to be replicated and refined and further studies carried out to substantiate the nature of these bootstrapping processes. As one example, the role potentially played by the Cartesian plane as a placeholder for the bootstrapping of the complex number concept could be assessed by dividing into two groups a cohort of students of generally equal mathematical ability and preparedness but who have no prior background in complex number concepts. One group would be given pre-instructional training only in algebraic concepts and procedures, whereas the other group would receive pre-instruction training that includes both algebraic and analytic geometry concepts and methods, particularly ones that pertained to the Cartesian plane. Pre- and post-tests would accompany their instruction in complex numbers, with the intention to determine possible differences in learning outcomes between the two groups. A variation on the above methodology would be the inclusion of MR exercises along with the analytic geometry instruction afforded to the second group.

6. Limitations, Transgressions and Omissions

It bears repeating that a cognitive-historical analysis by its very nature precludes repeated empirical examination of its immediate concern, a unique historical event or series of historical events. The validity of this approach is only established abductively, by the fecundity of the results that are derived from its claims. Hopefully, the merit of the foregoing cognitive-historical analysis of the complex number system has been demonstrated by its capacity for integrating concepts and methods across a variety of domains and by its capability for generating a novel hypothesis, i.e., that mental rotation
should be considered to be a core system of numerical cognition. The validity of this hypothesis awaits the empirical examination that the means of its generation is unable to provide.

In carrying out this analysis, we have crossed many borders between domains as distinct as the history and philosophy of mathematics, numerical cognition, infant development, neurophysiology, analogical mapping, conceptual metaphor and blend theory, mathematical education research, and even the philosophy of quantum physics. The propriety of these transgressions ultimately may be justified by what has been learned. It has been our perspective throughout this work that more will be gained than lost by respectful disregard for the differences that these territorial boundaries may signify and that differences might be better viewed as potential complementarities than as sources of opposition.

We have also omitted much from this already long and complex discussion. There are both differences of appearance and of substance that must await further development. For example, the common concerns and differences specific to Nersessian’s mental model bootstrapping, Sfard’s theory of reification and Carey’s Quinean bootstrapping theory and more generally to core cognition and embodied cognition approaches need to be articulated in greater detail.

7. Conclusion

The foregoing cognitive-historical analysis of the conceptual development of complex numbers provides a new vantage point of the roles played by analogy and imagistic reasoning as conceptual bootstrapping processes, of the scaffolding provided by core systems of numerical representation, and of the pivotal role played by process descriptions in the conceptualization of the hierarchy of number systems. Through this historical case study, when viewed from Nersessian’s model-based reasoning approach, the agency of a succession of intermediate, hybrid mental models is clearly evident, giving a detailed understanding of the roles played by analogy and analogical change when the conceptual discontinuities that distinguish locally incommensurable systems are surmounted. In particular, the cognitive feat required by the bootstrapping process becomes especially clear when visualized via analogical mapping: initially the target domain is critically underspecified and deeper understanding is sought analogically from a suitable source domain.
But that is the rub, because also at this initial stage, no such source domain is at hand, and furthermore none is obtainable by deductive inference from the target domain that is awaiting clarification. The search may be broken down into a succession of discoveries of intermediate source domains, but each such domain that is subsequently discovered is the result of an abductive or related model-based reasoning process. This discovery process appears to be supported by core numerical representational systems that scaffold the formation of analogical source domains that eventually bootstrap conceptual development across a conceptual discontinuity. Scaffolding roles are played by three core numerical representational systems—identified previously by Carey as set-based quantification, parallel individuation, and the analog magnitude systems—and as the present analysis makes evident, there is likely a fourth core system with numerical content—mental rotation.

In addition, coming into view with this analysis is the prominent role played by process descriptions—particularly by object-process complementarity as foreseen by Sfard and Dubinsky and supported by recent empirical studies—that is clearly seen in the conceptualization of negative and imaginary numbers. Furthermore, using conceptual mapping techniques developed by Gentner for analogy and by Fauconnier and Turner for metaphors and blends, previous mapping analyses of the conceptualization of complex numbers (i.e. those of Fauconnier and Turner and of Lakoff and Núñez) are improved and the scaffolding and bootstrapping of the hierarchy of number systems is given a visual representation (figure 8). In sum, the merit of the cognitive-historical approach as advocated and developed by Nersessian, with mutually beneficial insights gained for multiple branches of cognitive science as well as for the history and philosophy of mathematics, is amply borne out.

As noted by De Cruz, \( i \) as a symbol is semantically opaque, i.e. it represents an idea that is not intuitively accessible [33, 34]. However, as remote an abstraction as complex numbers may seem to be, their conceptualization has been of great consequence to humanity. Beyond the role they have played in the development of mathematics, complex numbers are invaluable for the description of wave phenomena—from fluids, to fields, to quantum phenomena. From quantum theory a new quantum logic has emerged—a logic by which the universe reasons, a logic whose very structure is the result of the properties of complex numbers [121]. As has been recently proposed in the rapidly developing area of quantum cognition, this quantum logic appears to have found its way back to the realm of psychology [14, 5].
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