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A Classification of Musical Scales Using Binary Sequences

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Abstract

Every beginning music student has gone through the four main musical scales: major, natural minor, harmonic minor, and melodic minor. And some might wonder, why those four and not five, or six, or just three? Here we show that a mathematical classification can be used to identify these scales as representatives of certain *scale families*. Moreover, the classification reveals another scale family, which is much less known: the *harmonic major scale*. We find that each scale family contains exactly seven scales, which include the modes (*dorian, phrygian,...*) and other scales such as the *Romanian, Gypsi* or *Hindu*-scales. Besides giving a complete classification of proper heptatonic scales, we also emphasize the pedagogical potential of these $\{0, 1\}$-representations of scales, as these can help students to navigate and remember complicated scales.

Keywords: musical scales, heptatonic scales, binary sequences, modes, classification of scales.

1. Introduction

Musical scales are at the centre of playing, understanding and analysing music. The identification of scales provides context for a melody, it gives direction of flow, and it sets the frame for an improvisation in jazz. A scale is basically a collection of tones which are perceived to sound well together. Some scales are omnipresent in modern western music such as the *major scale*, the *minor scales*, or the *blues scale*. Other scales are obscure and only known to experts such as the *half diminished scale*, the *harmonic major scale*, or the *Hungarian scale*, for example [10, 7].

This paper is about proper seven-tone scales, also called *proper heptatonic scales*, where the scale can be arranged as a sequence of minor and major thirds. The major scale and the natural, harmonic and melodic minor scales are examples of proper heptatonic scales. We give a formal definition later. We show that proper heptatonic scales can be represented as a binary sequence of length 7, where 0 corresponds to a minor third and 1 corresponds to a major third. Group theory will be applied to these sequences and we prove that there are exactly four scale families, where each family contains exactly seven scales. These include the well known *major* and *minor scales*, but also all the modes (*dorian, phrygian, lydian, mixolydian, aeolian, locrian, ionian*), a number of lesser known scales such as the *half-diminished scale*, the *phrygian raised 6th scale* and some obscure scales, such as the *dorian* $\flat 5$, and the *mixolydian* $\flat 2$, for example. A full list of relevant scales is shown in Tables 2-5.

A comprehensive mathematical theory of music has been developed in the seminal works of Tymoczko [8] and Mazello [6]. Those methods go far beyond a simple classification of scales, as they also include voicings, harmonizations, tonalities, production, perception, and many other aspects of music. Nevertheless, the classification presented here, based on simple binary sequences, is not contained in the comprehensive theories mentioned above.

In the next section we develop the representation of proper heptatonic scales as binary sequences and introduce a left-shift and a right-shift operator. These operators allow us to define scale families as scales that can be transformed into each other by a sufficient number of left- or right-shifts. The main result is a classification into four families, which gives a clear systematic for these scales and it ensures that there are no other proper heptatonic scales outside this classification in Tables 2-5. It turns out that the scales of the harmonic-major family are quite strange, and they do not often arise in music. There are exceptions, though: Rimsky-Korsakov [8, Chapter 4] and one of the Beatles songs: Blackbird [3]. When played on the piano the scales of the harmonic-major family sound unpleasant and they contain lots of tension. One signature chord within this scale family is the tension rich VI $\sharp 5, j7$ -chord, which does not arise in any other scale system. It can be resolved to the III m7 chord.

After the classification has been achieved, we consider well-known and lesser known scales and see how and if they fit into this classification. We briefly discuss their use in traditional and modern music and highlight some curiosities of the *harmonic major scale family*. Stimulated by Rick Beato [2], we discuss mirror symmetries of these scales to find some surprising relations. Finally, we consider the pedagogical value of this classification.

2. Scales

The theory developed here uses note intervals of *thirds*. A third is an interval of three notes (for example from C to E). Thirds can arise in two forms, a *minor third* has 3 half-note steps (C to E♭), and a *major third* has four half-note steps (C to E). Any major scale can be arranged as a sequence of thirds. For example, the tones of C-major can be written as

$$C \quad E \quad G \quad B \quad D \quad F \quad A \quad C$$

where the interval between two consecutive tones is a minor or a major third. We assign the symbol 0 for a minor third and 1 for a major third. Then we can write the major scale as

$$\begin{array}{ccccccccccccccc} C & & E & & G & & B & & D & & F & & A & & C \\ & 1 & & 0 & & 1 & & 0 & & 0 & & 1 & & 0 & \end{array} \tag{2.1}$$

and consequently we use the symbol 1010010 to encode any major scale. Given any base-tone, if we build a sequence of major third, minor third, major third, minor third, minor third, major third, minor third, we obtain the major scale for the given base tone.

Let us define this properly.

Definition 1. *A* <u>proper heptatonic scale</u> *contain seven different tones, which can be arranged in a sequence of minor and major thirds, beginning on a base tone and ending on the base tone two octaves higher. The sequence of minor and major thirds can be represented by a* $\{0,1\}$*–sequence of length 7.*

Let $S \subset \{0,1\}^7$ denote the set of all proper heptatonic scales. The major scale is a proper heptatonic scale, hence we write

$$1010010 \in S.$$

Now we introduce group operations on the set S of heptatonic scales. One operation is the left-shift L, such that the last element in the $\{0,1\}$-sequence is moved to the front, and the other is the right-shift, moving the first entry to the back. For example

$$L(1010010) = 0101001, \quad \text{and} \quad R(0101001) = 1010010.$$

We see that applying a left-shift to a major scale gives us a new scale 0101001. But which scale represents 0101001? Looking back at the example in (2.1) then moving the last entry to the front corresponds to moving the interval AC to the front, which gives the scale

$$\begin{array}{ccccccccccccccc} A & & C & & E & & G & & B & & D & & F & & A \\ & 0 & & 1 & & 0 & & 1 & & 0 & & 0 & & 1 & \end{array}, \tag{2.2}$$

which is the A natural minor scale. If we left-shift again $L(0101001) = 1010100$ we can represent the scale starting on F

$$\begin{array}{ccccccccccccccc} F & & A & & C & & E & & G & & B & & D & & F \\ & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 0 & \end{array}, \tag{2.3}$$

which is the *lydian* scale. As we continue to shift to the left we find all the modes, *dorian, locrian, mixolydian, phrygian, ionian, aeolian.* Noting that the major scale is also the *ionian* scale and the natural minor scale is the *aeolian* scale, we find, using the left-shift L, that

$$\begin{aligned} L(\text{ ionian }) &= \text{aeolian} \\ L(\text{ aeolian }) &= \text{lydian} \\ L(\text{ lydian }) &= \text{dorian} \\ L(\text{ dorian }) &= \text{locrian} \\ L(\text{ locrian }) &= \text{mixolydian} \\ L(\text{ mixolydian }) &= \text{phrygian} \\ L(\text{ phrygian }) &= \text{ionian} \end{aligned}$$

until we come full circle back to the ionian scale. Let us summarize the scale symbols in Table 1. These scales can also be generated by applying the right-shift in reverse order.

Looking at the scale symbols in Table 1 we notice that each scale symbol has exactly three ones and four zeroes. This is, in fact, true for all proper heptatonic scales.

scale sequence	mode	common name
1010010	ionian	major
0101001	aeolian	natural minor
1010100	lydian	
0101010	dorian	
0010101	locrian	
1001010	mixolydian	
0100101	phrygian	

Table 1: The major scale family

Lemma 1. *Each proper heptatonic scale has exactly three ones and four zeroes. Hence there are at most 35 proper heptatonic scales.*

Proof. We need to cover 24 half-tones over two octaves. We divide these into x intervals of length 3 half-tones (minor third) and y intervals of 4 half-tones (major third). We cover seven intervals in total, i.e. $x + y = 7$ over a total span of two octaves $3x + 4y = 24$. The only solution to these two equations is $x = 4$ and $y = 3$. To distribute 3 ones in a string of length 7 gives us

$$\binom{7}{3} = 35$$

choices. □

In the example of the major scale above, we found other scales that are related to the major scale by left-shifts (or right-shifts). We call these related scales a *scale family*.

2.1. Scale Families

Definition 2. *Let $s \in S$ be a proper heptatonic scale. The scale family F_s contains all scales that can be generated by one or multiple left-shifts (or right-shifts) from s.*

This means that the major scale family is

$$\begin{aligned} & F_{1010010} \\ = \ & \{1010010, 0101001, 1010100, 0101010, 0010101, \\ & \quad 1001010, 0100101, 1010010\} \\ = \ & \{\text{ionian, aeolian, lydian, dorian, locrian,} \\ & \quad \text{mixolydian, phrygian}\}. \end{aligned}$$

It should be noted that each scale within a scale family can represent the entire family, in other words

$$F_{1010010} = F_{0101001} = \ldots .$$

How many scale families do exist? We give a complete classification of all scale families in our main result.

Theorem 1. *There are exactly four scale families. These are*

$$\begin{aligned} F_{1010010} &= \text{major scale family} \\ F_{0110001} &= \text{harmonic minor scale family} \\ F_{0110010} &= \text{melodic minor scale family} \\ F_{1010001} &= \text{harmonic major scale family.} \end{aligned}$$

We summarize these scale families in Tables 2-5.

Proof. We saw in Lemma 1 that we can have at most 35 scales. Since each scale can produce 6 other scales via left-shifts, we have $35/7 = 5$ potential scale families. Using systematic counting we find that these 5 scale families are represented by

$$F_{1110000}, F_{1101000}, F_{1100100}, F_{1100010}, F_{1010010}$$

all other sequences of three ones and four zeroes can be generated by left-shifts (or right-shifts). The last one, $F_{1010010}$ is our major scale family (see Table 2). We will show that the first one $F_{1110000}$ is not a scale family. If we write the scale 1110000 starting on a C we find

$$\begin{array}{ccccccccccccccc} C & & E & & G\sharp & & B\sharp & & D\sharp & & F\sharp & & A & & C \\ & 1 & & 1 & & 1 & & 0 & & 0 & & 0 & & 0 & \end{array}, \tag{2.4}$$

but $B\sharp$ is the same tone as C, hence the scale tones are not different, and 1110000 is not a proper scale as defined in Definition 1. The same will be true for all left-shifts of 1110000. Hence $F_{1110000}$ is not a scale family.

Let us consider $F_{1100100}$. If we apply a left-shift we get $L(1100100) = 0110010$, which, starting on C, gives

$$\begin{array}{ccccccccccccccc} C & & E\flat & & G & & B & & D & & F & & A & & C \\ & 0 & & 1 & & 1 & & 0 & & 0 & & 1 & & 0 & \end{array}, \tag{2.5}$$

which is the C upward melodic minor scale. Hence we call $F_{1100100}$ the *melodic minor scale family*. (See Table 4.)

If we consider $F_{1100010}$ we apply a left-shift to get $L(1100010) = 0110001$, which, starting on C, gives

$$\begin{array}{ccccccccccccccc} C & & E\flat & & G & & B & & D & & F & & A\flat & & C \\ & 0 & & 1 & & 1 & & 0 & & 0 & & 0 & & 1 & \end{array}, \tag{2.6}$$

which is the C harmonic minor scale. Hence we call $F_{1100010}$ the *harmonic minor scale family*. (See Table 3.)

Finally, the scale family $F_{1101000}$ is quite an obscure family, since no shift of 1101000 leads to a commonly known scale. A right-shift leads to $R(11001000) = 1010001$, which, starting at C, becomes

$$\begin{array}{ccccccccccccccc} C & & E & & G & & B & & D & & F & & A\flat & & C \\ & 1 & & 0 & & 1 & & 0 & & 0 & & 0 & & 1 & \end{array}, \tag{2.7}$$

which is known as the C harmonic major scale [1]. Hence $F_{1101000}$ is the *harmonic major scale family*. (See Table 5.) □

Remarks:

- Note that using the right-shift would lead to the very same classification, where we cycle through the scales in reverse order, but always stay within a scale family.

- Transposition of a given scale does not change its $\{0, 1\}$ sequence, it just changes the base tone. Hence transposition would always stay within the same scale family.

3. From Siberian overtone singing to modern Jazz music

We found four scale families, which contain seven scales each, making 28 proper heptatonic scales. We saw the major scale, the minor scales and the modes. Now let us have a detailed look at the other scales. Some of those are rather special and basically only known by experts and others seem to be completely unexplored. All 28 scales and their scale families are summarized in Tables 2 - 5. Scales are a very old concept of music and many of the scales that arise here in a systematic way have been used for hundreds of years in very different cultures. Good references can be easily found on the internet, for example Wikipedia: heptatonic scales [10] or Pianoscales [7].

Major scale family $F_{1010010}$		
sequence	mode	common name
1010010	ionian	major
0101001	aeolian	natural minor
1010100	lydian	
0101010	dorian	
0010101	locrian	
1001010	mixolydian	
0100101	phrygian	

Table 2: The major scale family

Harmonic minor scale family (Table 3). The abbreviation HMj indicates the scale that arises if we start on the j-th level of the harmonic minor scale.

Harmonic minor scale family $F_{0110001}$		
sequence	mode	common name
0110001	HM1	harmonic minor
1011000	HM6	
0101100	HM4	Misheberak, Romanian minor
0010110	HM2	
0001011	HM7	super locrian ♭♭7
1000101	HM5	phrygian major, Freygish, Spanish gypsi
1100010	HM3	

Table 3: The harmonic minor scale family. HM stands for harmonic minor

For example HM5 in C harmonic minor is G A♭ B C D E♭ F G. This scale 1000101 (HM5) has various names such as *phrygian major, Freygish scale* or *Spanish gypsi*. It is a common scale in India, Middle Eastern, Arabic and Egyptian music and it can also be found in Kletzmer music. Sometimes it is called the Jewish scale. Also in Jewish music we can find the Misheberak scale 0101100 (HM4).

Melodic minor scale family $F_{0110010}$		
sequence	mode	common name
0110010	MM1	melodic minor
0011001	MM6, locrian ♯2	half-diminished
1001100	MM4	acoustic scale, lydian dominant scale
0100110	MM2, phrygian ♯6	
0010011	MM7	altered scale, super locrian
1001001	MM5	major-minor, aeolian dominant, Hindu
1100100	MM3, lydian ♯5	lydian augmented

Table 4: The melodic minor scale family. MM stands for melodic minor.

Melodic minor scale family (Table 4). The abbreviation MMj indicates the scale that arises if we start on the j-th level of the melodic minor scale. The super locrian scale 0010011 (MM7) is a commonly used scale in modern jazz. The Hindu scale 1001001 (MM5) has a distinct Indian flavor, while the acoustic scale 1001100 originates from Siberian overtone singing - a very curious form of vocal expression.

Harmonic major scale family $F_{1010001}$		
sequence	mode	common name
1010001	HMa1, ionian ♭6	harmonic major
1101000	HMa6, ionian ♯2♯5	lydian augmented ♯2
0110100	HMa4, lydian ♭3	Jazz minor ♯4, lydian diminished
0011010	HMa2, dorian ♭5	
0001101	HMa7, locrian ♭♭7	locrian diminished 7
1000110	HMa5, mixolydian ♭2	mixolydian ♭9
0100011	HMa3, phrygian ♭4	superlocrian nat. 5, superphrygian

Table 5: The harmonic major scale family.

Harmonic major scale family (Table 5). The scales of the harmonic major scale family are much less known than the other scales above [9]. For example, in advanced jazz course work, students learn about the scales of the major, harmonic minor and melodic minor scale families, but typically not about the harmonic major scales [4]. One exception is the comprehensive textbook of Charlie Austin from McEwan University in Canada [1], where the harmonic major scale is briefly mentioned.

These scales of the harmonic major family are used in modern jazz music, but they are obscure and they do sound quite strange to the ear of traditional jazz musician. Among those of the harmonic major family I find the 0011010 (HMa2) the most pleasant scale for an improvisation.

If we build chords from the tones of the harmonic major scale then we find one chord, the VI ♯5 j7 – chord, which does not arise in any of the other scale families. This chord has a lot of tension and it can be resolved into a III m7 chord. It can also be used in a VI-II-V-I cadence, from VI ♯5 j7 to a II $\emptyset$ (half-diminished), to a V7 and the tonic Ij7. Try an improvisation in HMa2 (0011010) over this VI-II-V-I sequence.

3.1. The Mirror Image

A prolific musical pedagogist Rick Beato [2] had the idea to relate scales with a certain sequence of ascending semitones and full tones to a mirror scale that has the same sequence of half and full tones, but descending. He found that the *super locrian* ♭♭7 (0001011) has as mirror image the *lydian augmented* ♯2 (1101000) scale. This analysis can be extended to all scales using the $\{0, 1\}$-sequences introduced here. To be formal, we introduce a mirror operator

Definition 3. *Consider a sequence $s \in S$. The mirror operator M turns the sequence around. For example*

$$M(\text{major}) = M(1010010) = 0100101 = \text{phrygian}.$$

In this way we can consider all mirror images of all the scales and we make a quite interesting observation (Table 6). The mirror images of members of the major scale family and the melodic minor scale family stay in their original scale family, where the dorian scale and the MM5 scale are fully symmetric.

major	melodic minor	harm. minor/harm. major
M(ionian) = phrygian	M(MM1) = MM2	M(HM1) = HMa5
M(aeolian) = mixolydian	M(MM3) = MM7	M(HM2) = HMa4
M(lydian) = locrian	M(MM4) = MM6	M(HM3) = HMa3
M(dorian) = dorian	M(MM5) = MM5	M(HM4) = HMa7
		M(HM5) = HMa1
		M(HM6) = HMa2
		M(HM7) = HMa6

Table 6: Mirror images of the scale symbols. .

They are the only two scales that have the same sequence going up or down. However, mirror images of members of the harmonic minor scale family are in harmonic major and vice versa. The relation M(HM7) = HMa6 is the one that was found by Rick Beato [2]. In Table 6 we list all mirror images.

4. Conclusion

The identification of scales through a sequence of minor and major thirds leads to a rich classification of proper heptatonic scales into scale families. Taking a closer look at individual scales we find a surprisingly rich relation to very different musical styles, cultrues and time periods. A major conclusion from Theorem 1 is completeness. There are no other proper heptatonic scales than those listed here in Tables 2-5.

However, there are many more scales, which do not fit into the classification of proper heptatonic scales as discussed here. First of all, there are scales with five tones (pentatonic), six tones (sextatonic), eight tones (octatonic) and more. Here we summarize some of them [10, 7]:

- **Pentatonic scales:** Pentatonic, Byzantine, Chinese, Egyptian, Hirajoshi, Japanese, Yo.
- **Sextatonic scales:** augmented, whole tone, blues.
- **Octatonic scales:** Algerian, diminished, dominant diminished, eight note Spanish, Oriental, bebop.

The scales of length other than 7 tones are an interesting object for further study. For example octatonic scales have 8 tones. The number 8 is not a prime number and it is divisible by 2 and 4. Hence using left–shifts (or right–shifts) on the chord symbol can lead to period 2 or period 4 cycles. This means that scale families of octatonic scales can have different number of members. Further research into octatonic sequences can be found in [5].

There are still many heptatonic scales which are not included in our classification. The reason is that those scales cannot be written as a sequence of minor or major thirds. Often they have three tones that are chromatic (for example $G, G^\sharp, A$), and they are not included in the above classification. Some examples are [10, 7]

- double harmonic, minor gypsy, Hungarian minor, Byzantine, Neapolitanian, heptatonic tertia, Ukrainian dorian, Magam, enigmatic, Arabic

A systematic analysis of those scales requires an extension of the binary sequence representation, since seconds and fourth-intervals need to be included. This might be possible, but it will extend the space of possible scales dramatically.

The identification of scales with binary sequences of length 7 (for example major = 1010010) has great pedagogical value. While students learn different scales they often wonder where these scales come from. In the classification here it is easy to see that *Romanian minor* and the *phrygian major* are from the same scale family, and one can get from the former to the latter by the use of three left-shifts. Or, imagine you are asked to find the super locrian scale starting on $C\sharp$. Not an easy task. However, if you know that super locrian $= 0010011$, then you can easily identify the tones: $C\sharp, E, G, B, D, F, A, C\sharp$ and rearrange, to find a rather simple version of the super locrian scale $C\sharp, D, E, F, G, A, B, C\sharp$.

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