1-1-1982

Sums of z-Ideals and Semiprime Ideals

Melvin Henriksen
Harvey Mudd College

Frank A. Smith
Kent State University

General Topology and its Relations
to Modern Analysis and Algebra V
Proc. Fifth Prague Topol. Symp. 1981
J. Novak (ed.)

272 – 278

SUMS OF z-IDEALS AND SEMIPRIME IDEALS

M. Henriksen, F.A. Smith

Abstract: If B is a ring (or module), and K is an ideal (or submodule) of B, let $B(K) = \{(a,b) \in B \times B : a-b \in K\}$. The relationship between ideals (or submodules) of B and those of B(K) is examined carefully, and this construction is used to find a lattice-ordered subring of the ring $C(\mathbb{R})$ of all continuous real-valued functions on the real line $\mathbb{R}$ with two z-ideals whose sum is not even semiprime.

AMS subject classification: 54C40, 06A70.

1. *Introduction*

In [M1], G. Mason calls an ideal I of a commutative ring B with identity element a *z-ideal* if whenever $a,b \in B$ are in the same set of maximal ideals of B and $a \in I$, then $b \in I$. It was shown in [GJ] and [R] that if B is a solid (or absolutely convex) subring of the ring of all continuous functions on a topological space, then a sum of two z-ideals of B is a z-ideal, but no example is given in any of these papers or in the more recent [M2] of a commutative ring containing two z-ideals whose sum is not a z-ideal. We supply such an example here with the aid of a construction of independent interest. In particular, if B is a ring or a module, and K is an ideal or submodule of B, we consider $A = \{(a,b) \in B \times B : a-b \in K\}$ and examine carefully the relationship between the ideals (or submodules) of B and those of A. This construction enables us also to answer a question posed in [H1], and to give a simpler version of an example given in [HP].

2. *Extensions of modules and rings*

Suppose R is a ring, B is either a left or a right R-module, and K is a submodule of B. Let $A = B(K) = \{(a,b) \in B \times B : a-b \in K\}$ and call A the *extension of* B *by* K. The following properties of B(K) are easily verified.

(1) $A = B(K)$ *is a submodule of* $B \times B$ *and* $D = \{(b,b) : b \in B\}$ *is a submodule of* A *isomorphic to* B. *If* B *is a unital* R*-module, so is* A.

(2) If $B = R$ *is a ring, and* K *is an ideal of* B, *then* A *is a subring of* $B \times B$. Clearly (2) follows from (1) since the latter applies if B is also a right R-module.

Let $p:B \to A$ be defined by letting for any $b \in B$ $p(b) = (b,b)$, let $p_1(b) = (b,0)$ and $p_2(b) = (0,b)$. If I is a (left) submodule of B, let

(3) $I^{(1)} = \{(i+k,i): i \in I,\ k \in K\}$ and let
$I^{(2)} = \{(i,i+k): i \in I,\ k \in K\}$.

Note that for $n = 1,2$,

(4) $I^{(n)}$ *is a submodule of* A, *and if* I *is an ideal of* B, *then* $I^{(n)}$ *is an ideal of* A.

1. <u>Theorem</u> *For* $n = 1,2$, *the map* $I \to I^{(n)}$ *is a bijection of the set of submodules of* B *onto the set* $\mathcal{I}^{(n)}$ *of submodules of* A *that contain* $p_n(K)$. *Moreover, if* I, $J \in \mathcal{I}$, *then* $I^{(1)} = J^{(2)}$ *if and only if* $I = J \supset K$.

<u>Proof.</u> Suppose $n = 1$. It is clear from (3) that if $I \in \mathcal{I}$, then $I^{(1)} \in \mathcal{I}^{(1)}$. Suppose $S^* \in \mathcal{I}^{(1)}$, whence $S^* \supset \{(k,0): k \in K\} = p_1(K)$. Let $I = \{b \in B: (b+k,b) \in S^*$ for some $k \in K\}$. It is routine to verify that I is a submodule of B, and, since $S^* \supset p_1(K)$, we have $S^* = I^{(1)}$. Clearly if I, $J \in \mathcal{I}$ and $I^{(1)} = J^{(1)}$, then $I = J$, so the map $I \to I^{(1)}$ is a bijection of $\mathcal{I}$ onto $\mathcal{I}^{(1)}$. If $n = 2$, the same argument applies with a change in notation.

Suppose $I^{(1)} = J^{(2)}$. Then for any $i \in I$, there is a $j \in J$ and a $k \in K$ such that $(i,i) = (j,j+k)$. So $I \subset J$. Similarly $J \subset I$, and $I = J$. Since for every $i \in I$ and $k \in K$, there is an $i' \in I$ and $k' \in K$ such that $(i+k,i) = (i',i'+k')$, we must have $K \subset I$.

If $i \in I \setminus J$, then $(i,i) \in I^{(1)} \setminus J^{(2)}$. If $I = J$ and $k \in K \setminus I$, then $(k,k) \in I^{(1)} \setminus I^{(2)}$. This completes the proof of the theorem.

Our next result illustrates that for $n = 1,2$, the map $I \to I^{(n)}$ of $\mathcal{I}$ onto $\mathcal{I}^{(n)}$ preserves a number of algebraic properties of ideals. Recall that a ring B is called *prime* if whenever $a,b \in B$ and $aBb = \{0\}$, then $a = 0$ or $b = 0$. If the intersection of all the prime ideal of B is $\{0\}$, then B is called *semiprime*. Thus B is semiprime if $aBa = 0$ implies $a = 0$. An ideal P of a ring B is called *prime* (resp. *semiprime*) if B/P is a prime (resp. semiprime) ring. We call a ring B *reduced* if $B \neq \{0\}$ and if $b \in B$ and $b^2 = 0$ imply $b = 0$. A reduced ring B is semiprime, and the converse holds if B is commutative. For background, see [H2], [K1], or [M3].

2. <u>Theorem.</u> *Suppose* B *is a ring,* K *is an ideal of* B, $A = B(K)$, *and* $n = 1$ *or* 2. *For any proper ideal* P *of* K

(a) P *is a prime ideal if and only if* $P^{(n)}$ *is prime ideal of* A.

(b) P *is a semiprime ideal of* B *if and only if* $P^{(n)}$ *is a semiprime ideal of* A.

(c) *If* B *is reduced, then* P *is a minimal prime ideal of* B *if and only if* $P^{(n)}$ *is a minimal prime ideal of* A.

(d) P *is a maximal ideal of* B *if and only if* $P^{(n)}$ *is a maximal ideal of* A.

Before giving the proof of Theorem 2, we introduce some notation. For any ring R, let Sp(R) denote the family of proper prime ideals of R, MinSp(R) = {P ε Sp(R):P} is a minimal prime, and MaxSp(R) denote the family of maximal ideals of R.

<u>Proof</u>. We assume that n = 1. If $P \in Sp(B)$, then $P \times B \in Sp(B \times B)$, so $(P \times B) \cap A = P^{(1)} \in Sp(A)$. Suppose conversely that $P^{(1)} \in Sp(A)$ for some ideal P of B, and $axb \in P$ for some $a \in P$, $b \in B \setminus P$ and all $x \in B$. Suppose $k_1, k_2, k_3 \in K$. Then $\alpha = (a+k_1,a)(x+k_2,x)(b+k_3,b) = (axb + k_4,axb)$ for some $k_4 \in K$. Thus $\alpha \in P^{(1)}$ and $b \notin P$. Hence $a \in P$, so $P \in P(B)$. Thus (a) holds.

Part (b) follows from a routine modification of the proof of (a). Recall from [K] that:

(5) *A prime ideal* P *of a reduced ring* R *is a minimal if and only if* $a \in P$ *implies there is a* $b \notin P$ *such that* $ab = 0$. See also [HJ].

Observe that A is reduced if and only if B is. Suppose $P \in MinSp(A)$ and $\alpha = (a+k,a) \in P^{(1)}$ for some $a \in P$ and $k \in K$. Note that since B is reduced, $\{b \in B:ab = 0\} = A(a) = \{b \in B:ba = 0\}$ is a (two-sided) ideal for any $a \in B$.

We consider three cases.

(i) Assume $K \subset P$. Then since P is minimal, there are $b,c \notin P$ such that $ab = kc = 0$. Since $bc \notin P$, $\beta = (bc,bc) \notin P^{(1)}$, while $\alpha\beta = 0$. Hence $P^{(1)} \in MinSp(A)$ by (5).

(ii) Suppose that for each $a \in P$ there is a b in K but not in P such that $ab = 0$; that is assume $A(a) \cap K \not\subset P$. Then $\beta = (0,b) \in A \setminus P^{(1)}$ and $\alpha\beta = 0$. So $P^{(1)} \in MinSp(A)$ by (5).

(iii) Suppose there is an $a \in P$ such that $A(a) \cap K \subset P$. By (5) since P is minimal, $A(a) \not\subset K$. So $K \subset P$ and $P^{(1)} \in MinSp(A)$ by case (i).

Suppose conversely that $P^{(1)} \in MinSp(A)$ and $a \in P$. Since $(a,a) \in P^{(1)}$, there is a $b \notin P$ and a $k \in K$ such that $(a,a)(b+k,b) = (0,0)$ then $ab = 0$, so $P \in MinSp(A)$ by (5). This completes the verification of (c).

Suppose $P \in \mathrm{MaxSp}(B)$ and $a \in A \setminus P$. Then there is a $k \in K$ such that $(a+k,a)$ is not in $P^{(1)}$. For, otherwise, since $(k,0) \in P^{(1)}$, it would follow that $(a,a) \in P^{(1)}$ contrary to the fact that $a \notin P$. Since the smallest ideal of B containing P and a is all of B, the ideal I generated by $\{(m,m): m \in P\}$ and (a,a) contains $\{(b,b): b \in B\}$. But $P^{(1)} \supset p_1(K)$, so $I = B$ and $P^{(1)} \in \mathrm{MaxSp}(A)$. The proof of the converse is an exercise. This completes the proof of (d) and Theorem 2.

An element of a ring that is neither a left nor a right divisor of 0 will be called *regular*.

3. Corollary. *Suppose* B *is a ring,* K *is an ideal of* B, *and* $A = B(K)$.
   (a) P^* *is a (minimal) prime ideal of* A *if and only if* $P^* = P^{(1)}$ *or* $P^* = P^{(2)}$ *for some (minimal) prime ideal* P *of* B.
   (b) *An element* (a,b) *of* A *is regular if and only if both* a *and* b *regular in* B.

Proof. Suppose $P^* \in \mathrm{Sp}(A)$. Since $p_1(K) \cap p_2(K) = 0$, $p_1(K) \subset P^*$ or $p_2(K) \subset P^*$. So (a) follows from Theorems 1 and 2 (a, c).

Clearly if a and b are regular in B, then (a,b) is regular in A. By (5), if (a,b) is regular, then it is in no minimal prime ideal of A. So, by Theorem 1 and part (a), neither a nor b can be in any minimal prime ideal of B. Using (5) again, we condlude that both a and b are regular B. This completes the proof of the corollary.

4. Remarks.

(A) This argument of Theorem 2 (d) applies to maximal one-sided ideals, and this may be used to show that the (left) primitive ideals of A are of the form $P^{(j)}$ for P a primitive ideal of B. For definitions see [K1].

(B) It is well known that for any ring B with identity element, the sets of ideals, $\mathrm{Sp}(A)$, $\mathrm{MinSp}(B)$, and $\mathrm{MaxSp}(B)$ are topological spaces under the Zariski (or hull-kernel) topology. It is clear from Theorem 2 and Corollary 3 that, at least if B is reduced, that $\mathrm{Sp}(A)$, resp. $\mathrm{MinSp}(A)$, resp. $\mathrm{MaxSp}(A)$, is the quotient space of $\mathrm{Sp}(B \times B)$, resp. $\mathrm{MinSp}(B \times B)$, $\mathrm{MaxSp}(B \times B)$ obtained by identifying $P^{(1)}$ and $P^{(2)}$ whenever P is a prime, resp. minimal prime, resp. maximal ideal of B that contains K. See for example [G].

If R is a totally-ordered ring, $B(+)$ is an abelian lattice-ordered group, and $rb \geq 0$ whenever $r \geq 0$ in R and $b \geq 0$ in B, then we call B an *ℓ-module* over R. Thus, every lattice-ordered abelian group is an ℓ-module over the ring Z of integers with its usual order. See [BKW] for background.

A submodule I of B such that $a \varepsilon I$ and $|b| \leq |a|$ imply $b \varepsilon I$ is called *solid*. A submodule I of B is solid if and only if it is the kernel of an R-module homomorphism of B that preserves the lattice operations. If B is a lattice-ordered ring (= *ℓ-ring*) and I a solid submodule and an ideal, then I is called an *ℓ-ideal* of B. An ℓ-module over the real field is called a *Riesz* space. If $a \geqq 0$ and $na \leqq b$ $n = 1,2,\ldots,$ imply $a = 0$, then B is said to be an *archimedean* ℓ-module. As usual, if $a \varepsilon B$, we let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. It follows that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. For background material, see [BKW] and [LZ].

5. Proposition. *If* B *is an ℓ-module over a totally ordered ring* R, *and* K *is an ℓ-submodule of* B, *then* $A = B(K)$ *is an ℓ-module of* $B \times B$ *if and only if* K *is solid submodule of* B.

Proof. Suppose K is solid, $b \varepsilon B$ and $k \varepsilon K$. Now $|(b+k,b)| = (|b+k|,|b|)$ is in $B \times B$, $||b+k| - |b|| \leq |(b+k)-b| = |b| \leq |k| \varepsilon K$, and it follows from the solidity of K that $(|b+k|,|b|) \varepsilon A$. Hence A is an ℓ-submodule of $B \times B$.

Conversely, suppose A is an ℓ-submodule of $B \times B$ and $|\ell| \leq |k|$, where $k \varepsilon K$ and $\ell \varepsilon B$. Then

$$(6) \qquad (|\ell| + |k|,|\ell|) \wedge (|k|,|k|) = (|k|,|\ell|) \varepsilon A.$$

Also $|(k,0)| = (|k|,0) \varepsilon A$, so $|k| \varepsilon K$. Hence $|\ell| \varepsilon K$ by (6). Since $(\ell,\ell) \vee (|\ell|,0) = (\ell^+ + \ell^-,\ell^+) \varepsilon A$, we know that $\ell^- \varepsilon K$. Replacing ℓ by $(-\ell)$ in the last argument yields $\ell^+ \varepsilon K$. Thus $\ell \varepsilon K$ and we know that K is solid.

Generalizing a notion introduced for rings of continuous functions in [GJ], G. Mason calls an ideal I of a commutative ring R with identity element a *z-ideal* if whenever $a \varepsilon I$ and b is in every maximal ideal of R that contains a, we also have $b \varepsilon I$. See [M1] and [M2]. He notes that every z-ideal is semiprime and that any intersection of maximal ideals is a z-ideal. He calls an ideal with this latter property a *strong z-ideal*. In [M2], he uses D. Rudd's result [R] that if S is a solid subring of the ring $C(X)$ of all continuous real-valued functions on a topological space X, then a sum of two z-ideals of S is a z-ideal. Neither G. Mason nor D. Rudd give any example of a commutative ring with identity with two z-ideals whose sum is not a z-ideal. Next, we provide a large class of such examples. Recall that a ring R with identity is called *semisimple* is the intersection of all the maximal left ideals of R is $\{0\}$.

6. Proposition. *Suppose* B *is a commutative semisimple ring with identity,* K *is an ideal of* B, *and* $A = B(K)$. *Then:* $p_1(K)$ *and* $p_2(K)$ *are strong z-ideals. If* K *fails to be semiprime, then neither is* $p_1(K) + p_2(K)$*; in particular this sum is not a z-ideal.*

Proof. By Theorem 2 (d), if $M \in \mathrm{MaxSp}(B)$, then $M^{(1)} = \{(m+k, m): m \in M, k \in K\}$ $\in \mathrm{MaxSp}(A)$ and $p_1(K) = \cap\{M^{(1)}: M \in \mathrm{MaxSp}(B)\}$ is an intersection of maximal ideals. Similarly, $p_2(K)$ is also a strong z-ideal. But $p_1(K) + p_2(K)$ $= \{(k,k)': k,k' \in K\} = K^{(1)}$ fails to be semiprime by Theorem 2 (b) since K is not semiprime. Hence K is not a z-ideal.

An ℓ-ring that is a subdirect product of totally ordered rings is called an f-*ring*. In [H1], M. Henriksen gave a necessary and sufficient condition for a sum of two semiprime ideals of an f-ring B to be semiprime, and asked if the latter could fail to occur is B were archimedean. The following example will apply Proposition 6 to give a negative answer to this latter question.

7. Example. Let X denote a topological space such that the ring C(X) of all continuous functions contains an ℓ-ideal K that is not semiprime, let $B = C(X)$, and let $A = B(K)$. (For example, we could take $X = [0,1]$ and $K = \{f \in C(X): |f| \leq ki$ for some $k \in C(X)\}$, where i is the identity function). Then A is an ℓ-ring by Proposition 5. But A is a subring of the archimedean f-ring $B \times B$ and hence is an archimedean f-ring. By Proposition 6, $p_1(K)$ and $p_2(K)$ are two strong z-ideals whose sum is not semiprime.

We close with one more application of our extension technique.

In [HP], C. Huijsmans and B. de Pagter call a subspace L of a Riesz space R a d-*ideal* if $a \in L$ implies $\{\{a\}^d\}^d \in L$, where, for any subset T of R, $T^d = \{b \in R: |b| \wedge |t| = 0$ for all $t \in T\}$. They give sufficient conditions for the sum of two d-ideals to be a d-ideal, and give an example of an archimedean Riesz space where this latter fails.

We observe that Example 7 also serves this latter purpose. For, since A is reduced, T^d is the annihilator of T whenever $T \subset B$. By Theorem 2 and Corollary 3, $p_1(K)$ and $p_2(K)$ are each the intersections of all the minimal prime ideals containing it, while their sum is not even semiprime.

REFERENCES

[BKW] A. Bigard, K. Keimel, and S. Wolfenstein, *Groups et Anneaux Reticules*, Lecture Notes in Mathematics 608, Springer-Verlag, 1977.

[G] L. Gillman, *Rings with Hausdorff structure space*, Fund. Math. 45 (1957), 1-16.

[GJ] L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. Nostrand, New York, 1960.

[H1] M. Henriksen, *Semiprime ideals of f-rings*, Symposia Math. 21 (1977), 401-409.

[H2] I. Herstein, *Noncommutative Rings*, Mathematical Association of America, John Wiley & Sons, 1968.

[HJ] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. 115 (1965), 110-130.

[HP] C. Huijsmans and B. de Pagter, *On z-ideals and d-ideals in Riesz spaces* I, Indag Math. 42 Proc. Netherl. Acad. Sci A 83, (1980), ~~263-2~~79. 133 –

[K1] I. Kaplansky, *Fields and Rings*, University of Chicago Press, Chicago, Ill. 1969.

[K2] K. Koh, *On functional representations of a ring without nilpotent elements*, Canad. Math. Bull. 14 (1973), 349-352.

[LZ] W. Luxemburg and A. Zanen, *Riesz spaces I*, North Holland Publ. Co., Amsterdam, 1971.

[M1] G. Mason, *z-ideals and prime ideals*, J. Algebra 26 (1973), 280-297.

[M2] ________, *Prime z-ideals of* C(X) *and related rings*, Canad. Math. Bull. 23 (1980), 437-443.

[M3] N. McCoy, *The Theory of Rings*, Chelsea Publ. Co., Bronx, NY, 1973.

[R] D. Rudd, *Two sum theorems for ideals of* C(X), Mich. Math. J. 17 (1970), 139-141.

Harvey Mudd College
Claremont, CA 91711
USA

Kent State University
Kent, Ohio 44240
USA

Pitzer College
Claremont, CA 91711
USA