Journal of Humanistic Mathematics

Volume 14 | Issue 1 January 2024

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Peter J. Rowlett
Sheffield Hallam University

Recommended Citation

Peter J. Rowlett, "Seating Groups and 'What a Coincidence!': Mathematics in the Making and How It Gets Presented," *Journal of Humanistic Mathematics*, Volume 14 Issue 1 (January 2024), pages 229-238. DOI: 10.5642/jhummath.BDZK3303. Available at: https://scholarship.claremont.edu/jhm/vol14/iss1/12

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | http://scholarship.claremont.edu/jhm/

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Cover Page Footnote

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Seating Groups and 'What a Coincidence!': Mathematics in the Making and How It Gets Presented

Peter J. Rowlett

Sheffield Hallam University, Sheffield, UNITED KINGDOM
p.rowlett@shu.ac.uk

Synopsis

Mathematics is often presented as a neatly polished finished product, yet its development is messy and often full of mis-steps that could have been avoided with hindsight. An experience with a puzzle illustrates this conflict. The puzzle asks for the probability that a group of four and a group of two are seated adjacently within a hundred seats, and is solved using combinatorics techniques.

1. Introduction

In the opening of the documentary about his work on Fermat's Last Theorem [1], Sir Andrew Wiles describes the process of mathematical discovery by analogy with stumbling around in a series of dark rooms. He describes spending time bumping into furniture and slowly learning where things are, then "you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were."

Pólya, in his famous manual on problem solving *How To Solve It* [2], describes this process of discovery to readers he does not expect to be aware of it. "Yes," he asserts on page *vii*, "mathematics has two faces: it is the rigorous science of Euclid but it is also something else."

The first of these faces is the form in which mathematics is often presented. We start with a collection of axioms, simple statements we take to be true. For Euclid's geometry, these are permissible constructions like: if you have two points, you can draw a line between them with your straightedge, and if you have a line segment, you can draw a circle with that radius using your pair of compasses. We then proceed to make logical deductions, results which are proven using the axioms and previously proved results. Euclid, in his *Elements*, deduces results in plane and solid geometry and other areas.

For Pólya, the other face is "mathematics in the making" [2, vii], which he describes as experimental and inductive. This is mathematics as creative thought, messy and non-linear. It is Wiles' stumbling around a dark room. Pólya feels the need to point out to his readers that this side of mathematics exists because then, as now, the presentation of mathematics so often hides its making. Having discovered the light switch, mathematicians see a way to skip most of the stumbling and, naturally, they choose to write down the simplest possible description of the layout of the room. As a consequence, students are often presented with the products of doing mathematics rather than being shown the process by which it was created.

This can cause confusion for those reading or listening to the presentation of a piece of mathematics. A clever leap at the start pays huge dividends by the end. Students might reasonably throw their hands up in despair: how did they know to make that leap? Some might feel they are not clever enough to develop new mathematics because they didn't see the leap coming. This arises because what is being presented is not mathematics in the making, and the person who developed the mathematics they are learning about likely made false starts, took mis-steps and found themselves following dead-ends that taught them what clever leap they needed to make.

Not to compare my work to any of the great mathematicians mentioned so far, but I had an experience recently playing around with a puzzle that I feel neatly illustrates the difference. Similar to the approach of an earlier JHM paper by VanHattum [4], I share my experience of solving this puzzle here — mis-steps and all — in hopes of casting a light on the different faces of mathematics.

2. A funny coincidence at the theatre

It all started when a friend wrote me an email. He, his wife and their family, a group of four, had visited the theatre. He explained that the seats were allocated from 28 performances in the theatre that has over a thousand seats. Arriving in their allocated seats on the allotted night, they found themselves seated next to my friend's wife's ex-husband and his new wife. My friend enquired about the probabilities involved.

I wrote back explaining that coincidences are difficult beasts. I said it is important to think about how many people you might have met at the theatre and found it a funny coincidence, and how many other circumstances you might be in where a similar coincidence might occur. I pointed out that ex-spouses likely have similarities in social class and shared interests, which increases the chances of being at the same event. I think the coincidence is unlikely enough to be striking, but I am aware such matters are hard to think about.

Still, my interest was piqued thinking about the probability of a group of four and a group of two being seated adjacently. I found the whole problem too much to think about at once, and I didn't have detailed information about the layout of the theatre, so I made myself a simpler puzzle:

> A group of four and a group of two are seated on a row of one hundred seats. What is the probability that one group is seated next to the other?

I saw that the hundred seats must hold ninety-four strangers and six special people in the two groups, who I numbered 1–4 and 5–6. I decided to write the groups as [1|2|3|4] and [5|6], with the interchangeable strangers as [X]. The coincidence is equally striking whether the group of two are seated as [5|6] or [6|5], and similarly for the group of four, so I ignored rearrangements of the individuals within the groups.

I could see that seating one special person among a larger group was simple enough. What seemed to complicate this problem is the way the groups cannot be seated just anywhere; for example, the group of four cannot be seated in the last three seats of the row, the group of two cannot be seated in one seat between the group of four and a stranger, etc.

3. Smaller examples

My first instinct was to try some smaller examples and see how the problem behaved.

The minimum problem size is six seats, since I have six people who must be seated. There are two ways to seat my six people in six seats, and both lead to adjacent groups.

[1|2|3|4|5|6]

5	6	1	2	3	4

There is therefore a probability of 1 that the two groups would be seated adjacently on six seats.

Adding a seat, we seek to position our six people and one stranger in seven seats. I drew out the cases, grouped in mirrored pairs for convenience of counting, like this:

1	2	3	4	5	6	X
1	2	3	4	X	5	6
X	1	2	3	4	5	6

X	5	6	1	2	3	4
5	6	X	1	2	3	4
5	6	1	2	3	4	X

The two groups are adjacent in four of these, so the probability of being seated adjacent is $\frac{4}{6} = \frac{2}{3}$.

With eight seats, I counted twelve ways to seat the people.

1	2	3	4	5	6	X	X
1	2	3	4	X	5	6	X
1	2	3	4	X	X	5	6
X	1	2	3	4	X	5	6
X	1	2	3	4	5	6	X
X	X	1	2	3	4	5	6

X	X	5	6	1	2	3	4
X	5	6	X	1	2	3	4
5	6	X	X	1	2	3	4
5	6	X	1	2	3	4	X
X	5	6	1	2	3	4	X
5	6	1	2	3	4	X	X

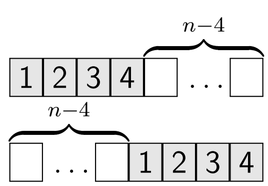
The two groups are adjacent in six of these. The probability is $\frac{6}{12} = \frac{1}{2}$.

4. Seeking a pattern

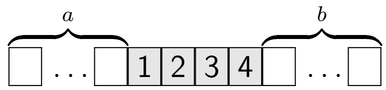
Feeling I had a handle on the problem, I started to think about how to generalise this, realising I could work on the version where there are n people and then set $n = 100$.

I started playing around with what I thought were the different cases:

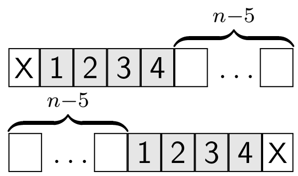
- I put the group of four on an end, leaving $n - 4$ empty seats on the row in which to seat the group of two.



- I put the group of four in the middle, with a seats to the left and b seats to the right and $a + b = n - 4$. The other group could be seated among the a seats or among the b seats.



- I realised there was a special case where either $a = 1$ or $b = 1$, in which case the group of two would not fit.



After a while, I realised this was over-complicating matters. The basic structure I was looking for was like this:

strangers, one group, strangers, the other group, strangers

where any of the three sets of strangers could be empty (up to two if $n > 6$).

At that point, I realised this had turned into a classic combinatorics problem called stars and bars.

5. Stars and bars

Stars and bars seeks to separate n indistinguishable stars ($*$) into $k + 1$ distinguishable bins (any of which may be empty), which can be thought of as separating the stars using k bars ($|$). For example, here we place eight stars into four bins (using three bars):

$$\{ * \mid * * * \mid * * \mid * * \}.$$

Counting these is a matter of choosing k symbols from $n + k$ to be our bars (or equivalently n symbols to be our stars), so the number of ways is

$$\binom{n+k}{k} = \binom{n+k}{n}.$$

Here, I realised we could consider the strangers to be our stars and the two special groups to be our bars. Then I seek the number of ways to distribute two bars among $n-6$ stars. The number of ways to distribute $n-6$ stars using two bars is

$$\binom{n-4}{2}.$$

The bars in stars and bars are indistinguishable, but in the puzzle one is the group of four and the other is the group of two. There are therefore 2! ways to arrange our two bars, for a total of

$$\begin{aligned} 2!\binom{n-4}{2} &= \frac{2!(n-4)(n-5)(n-6)!}{2!(n-6)!} \\ &= (n-4)(n-5) \end{aligned}$$

ways to seat our two groups among our strangers.

To calculate the probability, we need to know how many of these are adjacent. We can think of this as being the same $n-6$ stars (strangers) but this time just one bar, which is the six people sitting together. The number of ways to distribute one bar among $(n-6)$ stars is

$$\binom{n-5}{1} = (n-5).$$

This bar consists of two groups who can be seated two different ways, with the group of four on the left or the right of the group of two. Thus there are

$$2(n-5)$$

ways to seat the people with our two groups adjacent.

The number of ways of seating the two groups adjacent as a proportion of the number of ways of seating the two groups at all is therefore

$$\frac{2(n-5)}{(n-4)(n-5)} = \frac{2}{(n-4)}$$

and this is the probability of our two groups being seated adjacent.

6. Solution to smaller examples

As a sense check, I used this formula to solve the small cases I had enumerated by hand.

For $n = 6$, we have $(n-4)(n-5) = 2 \times 1 = 2$ ways of seating people, of which $2(n-5) = 2 \times 1 = 2$ are adjacent. The probability is $\frac{2}{(n-4)} = \frac{2}{2} = 1$.

For $n = 7$, we have $(n-4)(n-5) = 3 \times 2 = 6$ ways of seating people, of which $2(n-5) = 2 \times 2 = 4$ are adjacent. The probability is $\frac{2}{(n-4)} = \frac{2}{3}$.

For $n = 8$, we have $(n-4)(n-5) = 4 \times 3 = 12$ ways of seating people, of which $2(n-5) = 2 \times 3 = 6$ are adjacent. The probability is $\frac{2}{(n-4)} = \frac{2}{4} = \frac{1}{2}$.

7. Solution for a hundred seats

Since these small cases match what I got by hand, I proceeded to try the $n = 100$ case. Here, we have $(n-4)(n-5) = 96 \times 95 = 9120$ ways of seating people, of which $2(n-5) = 2 \times 95 = 190$ are adjacent. The probability is $\frac{2}{(n-4)} = \frac{2}{96} \approx 0.02$.

Note that this assumes we can always find a configuration of strangers to sit around our two groups. It might be that the strangers are actually in groups. Since we have allowed a group of two, I worked out a version of the problem with minimum group size of two, which was a little more complicated and did not change the first significant digit of the probability, so I spare you the details here.

8. Communicating my findings

Having a nice problem and a solution I was confident in, I decided to write it up for my students to try in an introductory problem-solving class later that week (taking the place of a puzzle I had got from a book and did not quite remember how to solve). I also decided it was time to write back to the friend who had sent the original email.

Writing the reply, I found it quite hard to express what was going on with the stars and the bars. I think this was because the technique of stars and bars is about putting the stars into groups using the bars, but here I was considering the bars as groups themselves and the double-meaning of the word ‘groups’ was unravelling the clarity of what I was trying to write.

I thought about this for a while and decided on a simpler way to explain my solution: Imagine you cut out ninety-six pieces of paper and you want to write labels on these. You want to write 'group of four' on one, 'group of two' on another, and 'stranger' on the other ninety-four.

Reframing the problem like this, essentially ignoring the chairs and disregarding the sizes of the two groups, the solution appears much more quickly.

There are ninety-six pieces of card on which you can write 'group of four'. Having done so, there are ninety-five pieces of card on which you can write 'group of two', and the rest are labelled 'stranger'. Therefore there are 96×95 ways to seat the groups among the strangers.

When the two groups are adjacent, we can imagine this time we have cut out ninety-five pieces of card and want to label ninety-four of them 'stranger'. There are ninety-five ways to choose the non-'stranger'. Having chosen, there are two ways to label this piece of card, with the group of four on the left or the right of the group of two, so there are 2×95 ways to seat the groups adjacently.

Seeing this, I realised what had come before was overcomplicating. Worrying whether the group of four were too close to the end of a row was a false start; the stars and bars was overkill. Viewed as labelling two special pieces of card from a pile of ninety-six pieces of card, the solution appears immediately.

9. Discussion

Perhaps you saw this simpler method much earlier than I did, and you have been reading in increasing frustration as I complicated matters. The important thing, in terms of 'mathematics in the making', is that I did not see this. I over-complicated the problem and did not see the simpler method until I had worked through the complicated one and thoroughly understood the problem.

Seeing the simpler method, I suddenly worried about my problem-solving class. The idea of the class is to give the students something they can try to solve using Pólya's heuristic. Would the students see right through the problem and solve it too quickly? Should I revert to the puzzle I had used last year?

Even as I write this, I can feel the desire to not expose the mis-steps I took along the way. I feel now that I should have seen this way of solving the problem much sooner, and do not want to appear silly for not doing so. I could scroll up and delete all the mis-steps, just leaving the simple method: reframe the puzzle to ignore the group sizes and outline the two paragraphs of simple combinatorics from section 8. I could state and solve the problem in around half a page. If I did so, my readers might reasonably throw up their hands in despair, asking "how did you know to do that?"

I was reminded of a joke included by Renteln and Dundes in their sampling of mathematical folk humour [3]:

> A mathematics professor was lecturing to a class of students. As he wrote something on the board, he said to the class "Of course, this is immediately obvious." Upon seeing the blank stares of the students, he turned back to contemplate what he had just written. He began to pace back and forth, deep in thought. After about 10 minutes, just as the silence was beginning to become uncomfortable, he brightened, turned to the class and said, "Yes, it IS obvious."

I trusted that my experience with the problem had rendered something I initially wrestled with into something I now found obvious, and gave the problem to my class. They explored it for a suitable amount of time, trying out Pólya's advice on it. One group solved it, much quicker than I had, but not so quickly as to render the exercise pointless.

I am glad I did not let the mathematicians' desire to present a solution neatly get in the way of giving my students an experience of mathematics in the making.

10. Acknowledgements

Thanks to John Read, whose coincidental visit to the theatre inspired this paper. Thanks also to Alex Corner for playing with the puzzle, too, and commenting on a draft of this paper. Alex showed me how to treat the arrangements of people as a pattern of letters and build an automaton to count the ways, which was fun to see and reassuringly came up with the same numbers.

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[1] British Broadcasting Corporation (BBC), *Horizon: Fermat's Last Theorem*, documentary first broadcast on January 15, 1996. Available in the United Kingdom at https://www.bbc.co.uk/programmes/b0074rxx.

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