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Dagan Karp
Harvey Mudd College

Dhruv Ranganathan
Yale University

GROMOV-WITTEN THEORY OF $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

DAGAN KARP AND DHRUV RANGANATHAN

Abstract. We prove equivalences between the Gromov-Witten theories of toric blowups of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^3$. In particular, we prove that the all genus, virtual dimension zero Gromov-Witten theory of the blowup of $\mathbb{P}^3$ at points precisely coincides with that of the blowup at points of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, for non-exceptional classes. It follows that the all-genus stationary Gromov-Witten theory of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ coincides with that of $\mathbb{P}^3$ in low degree. We also prove there exists a toric symmetry of the Gromov-Witten theory of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ analogous to and intimately related to Cremona symmetry of $\mathbb{P}^3$. Enumerative applications are given.

1. Introduction

While $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$ are birational, it is too much to expect their Gromov-Witten (GW) theories coincide. Indeed, GW invariants are not preserved by birational transformation in general. Instead, one may hope to study special birational maps, such as crepant transformations or blowups.

We prove the equivalence of the all-genus virtual dimension zero non-exceptional GW theories of four spaces, illustrated in the following diagram.

$$\begin{tikzcd} GW(\hat{X}) \arrow[rr, dash, "\text{Isomorphism}"] \arrow[d, dash, "\text{Blowup}"] & & GW(\hat{\tilde{X}}) \arrow[d, dash, "\text{Blowup}"] \\ GW(X) \arrow[rr, dash, "\text{Crepant}", "\text{Transformation}"'] & & GW(\tilde{X}) \end{tikzcd}$$

Here, X is the blowup of $\mathbb{P}^3$ at k points $p_1, \ldots, p_k$, and $\hat{X}$ is the blowup of X at six lines. Also, $\tilde{X}$ is the blowup of $(\mathbb{P}^1)^{\times 3}$ at $k-2$ points $\tilde{p}_1, \ldots, \tilde{p}_{k-2}$, and $\hat{\tilde{X}}$ is the blowup of $\tilde{X}$ at six lines. The equivalence of the all genus virtual dimension zero GW theories of X and $\hat{X}$ for nonexceptional classes was proved by Bryan-Karp in [4, Lemma 7]; in this work we complete the square.

In more detail, let $h \in H_2(X; \mathbb{Z})$ denote the class of the proper transform of a general line in $\mathbb{P}^3$, let E_i denote the exceptional divisor above p_i, and let e_i be the class of a general line in E_i.

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We decorate classes on $\tilde{X}$ with $\sim$. Accordingly, let $\tilde{h}_j$ for $1 \leqslant j \leqslant 3$ be the classes of the proper transforms of the three lines in $(\mathbb{P}^1)^{\times 3}$, and $\tilde{e}_i$ be the class of a general line in the exceptional divisor above the point $\tilde{p}_i$.

Theorem 1. *As above, let X be the blowup of $\mathbb{P}^3$ at k points and let $\tilde{X}$ be the blowup of $(\mathbb{P}^1)^{\times 3}$ at $k-2$ points. If $\beta = dh - \sum_{i=1}^{k} a_i e_i \in A_1(X)$ with $a_i \neq 0$ for $i > 4$, then for any genus g, we have*

$$\langle\ \rangle^X_{g,\beta} = \langle\ \rangle^{\tilde{X}}_{g,\tilde{\beta}},$$

where $\tilde{\beta} = \sum_1^3 \tilde{d}_i \tilde{h}_i - \sum_{i=1}^{k-2} \tilde{a}_i \tilde{e}_i$ and the coefficients of β and $\tilde{\beta}$ are related by

$$\begin{aligned}
\tilde{d}_1 &= d - a_2 - a_3\\
\tilde{d}_2 &= d - a_1 - a_3\\
\tilde{d}_3 &= d - a_1 - a_2\\
\tilde{a}_1 &= a_4\\
\tilde{a}_2 &= d - a_1 - a_2 - a_3\\
\tilde{a}_i &= a_{i+2} \text{ for } i \geqslant 3\ .
\end{aligned}$$

Remark 1. Note that the birational map $X \dashrightarrow \tilde{X}$ is crepant. In general, GW invariants are not preserved under crepant transformations. This is the subject of the Crepant Transformation Conjecture; see [3, 7, 17]. However, Theorem 1 shows equality does indeed hold in the case considered here.

Remark 2. Also note that we use the term nonexceptional rather strongly. Let $\pi : \hat{Y} \to Y$ be the blowup of the variety Y centered at $Z \subset Y$. We say $\beta \in H_2(Y;\mathbb{Z})$ is *nonexceptional* if any stable map to Y representing β has an image with empty set theoretic intersection with Z, and, moreover, any stable map to $\hat{Y}$ representing $\hat{\beta} = \pi^! \beta$ has image disjoint from the exceptional divisor $E \to Z$. We prove that the classes considered in Theorems 1 and 3 are nonexceptional in this strong sense.

Now, let l_{ij} denote the line through p_i and p_j, and let $\hat{X}$ denote the blowup of X at the proper transform of the six lines l_{ij}, where $1 \leqslant i < j \leqslant 4$. Let $\hat{h}$ and $\hat{e}_i$ denote the the proper transform of h and e_i respectively.

Theorem 2 ([4], Lemma 7)**.** *Let $d, a_1, \ldots, a_k \in \mathbb{Z}$ be such that $2d = \sum a_i$ and $a_i \neq 0$ for some $i > 4$. Then*

$$\langle\ \rangle^X_{g,\beta} = \langle\ \rangle^{\hat{X}}_{g,\hat{\beta}},$$

where $\beta = dh - \sum a_i e_i$ and $\hat{\beta} = d\hat{h} - \sum a_i \hat{e}_i$.

Similarly, for $(i,j) \in \{1,2\} \times \{1,2,3\}$, let $\tilde{l}_{ij} \subset \tilde{X}$ denote the line containing the point $\tilde{p}_i$ and representing one of the three line classes $\tilde{h}_j$; see Figure 1 below. Let $\hat{\tilde{X}}$ be the blowup of $\tilde{X}$ along the six lines $\tilde{l}_{ij}$. Also, let $\hat{\tilde{h}}_j$ and $\hat{\tilde{e}}_i$ denote the proper transforms of $\tilde{h}_j$ and $\tilde{e}_i$ respectively.

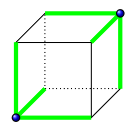

FIGURE 1. The six lines in $\tilde{X}$ blown up to obtain $\hat{\tilde{X}}$

We prove that the all-genus virtual dimension zero GW theory of $\hat{\tilde{X}}$ is equivalent to that of $\tilde{X}$, in the nonexceptional case.

Theorem 3. *Let $\tilde{X}$ and $\hat{\tilde{X}}$ be as above, and let $\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{a}_1, \dots, \tilde{a}_k \in \mathbb{Z}$ be such that $\sum \tilde{d}_j = \sum \tilde{a}_i$ and $\tilde{a}_i \neq 0$ for some $i > 2$. Then for $\tilde{\beta} = \sum \tilde{d}_i \tilde{h}_i - \sum \tilde{a}_j \tilde{e}_j$, we have*

$$\langle\ \rangle^{\tilde{X}}_{g,\tilde{\beta}} = \langle\ \rangle^{\hat{\tilde{X}}}_{g,\hat{\tilde{\beta}}},$$

where $\hat{\tilde{\beta}} = \sum \tilde{d}_i \hat{\tilde{h}}_i - \sum \tilde{a}_j \hat{\tilde{e}}_j$.

In Section 3 we show that $\hat{X}$ and $\hat{\tilde{X}}$ are isomorphic. Indeed, they are each isomorphic to a blowup of the permutohedral variety. Since GW invariants are functorial under isomorphism, this result, combined with those above, completes the square.

We immediately point out two additional implications of this square of equivalences: enumerative calculations and toric symmetry.

Remark 3. One may extract enumerative information directly from invariants of X and $\tilde{X}$, or one may relate invariants of X and $\tilde{X}$ to invariants of (the convex spaces) $\mathbb{P}^3$ and $(\mathbb{P}^1)^3$. To accomplish the later, one may use the following result of Bryan-Leung [5], which generalizes a result of Gathmann [12]. Let Y be a smooth algebraic variety and $\pi : \hat{Y} \to Y$ the blowup of Y at a point. Let $\beta \in A_1(Y)$ and $\hat{\beta} = p^!(\beta)$. Then we have,

$$\langle p\rangle^{Y}_{g,\beta} = \langle\ \rangle^{\hat{Y}}_{g,\hat{\beta}-\hat{e}},$$

where $\hat{e}$ is the class of a line in the exceptional locus, and $p^!(\beta) = [p^\star[\beta]^{\mathrm{PD}}]^{\mathrm{PD}}$.

Example 4. *How many rational curves in $(\mathbb{P}^1)^{\times 3}$ of class $h_1 + h_2 + h_3$ pass through three general points? We compute*

$$\begin{aligned}
\langle p^3\rangle^{(\mathbb{P}^1)^{\times 3}}_{0,h_1+h_2+h_3} &= \langle\ \rangle^{\tilde{X}}_{0,h_1+h_2+h_3-e_1-e_3-e_4} \\
&= \langle\ \rangle^{X}_{0,3h-e_1-\cdots-e_6} \\
&= \langle p^6\rangle^{\mathbb{P}^3}_{0,3h} \\
&= 1.
\end{aligned}$$

To illuminate, the first equality holds via Remark 3; we have blown up $(\mathbb{P}^1)^{\times 3}$ along four points, and simply not used $\tilde{p}_2$. The second equality follows from

Theorem 1. The third equality again follows from Remark 3. The final equality holds as the invariant $\langle p^6\rangle^{\mathbb{P}^3}_{0,3h}$ counts the number of degree-3 rational curves in $\mathbb{P}^3$ through six general points. There is only one such curve, the rational normal curve.

Additionally, the invariants on $\tilde{X}$ above satisfy a symmetry given by the following theorem.

Theorem 5. *Let $\tilde{X}$ be as in theorem 1. Then if $\tilde{\beta} = \sum_{1\leqslant j\leqslant 3} \tilde{d}_j\tilde{h}_j - \sum_{i=1}^4 \tilde{a}_i\tilde{e}_i$, and $\{a_3, a_4\} \neq \{0\}$, we have*

$$\langle\ \rangle^{\tilde{X}}_{g,\tilde{\beta}} = \langle\ \rangle^{\tilde{X}}_{g,\tilde{\beta}'},$$

where $\tilde{\beta}' = \sum_{1\leqslant i<j\leqslant 3} \tilde{d}_j'\tilde{h}_j - \sum_{i=1}^4 \tilde{a}_i'\tilde{e}_i$ has coefficients given by

$$\begin{aligned}
\tilde{d}_1' &= \tilde{d}_1 + \tilde{d}_3 - \tilde{a}_1 - \tilde{a}_2\\
\tilde{d}_2' &= \tilde{d}_2 + \tilde{d}_3 - \tilde{a}_1 - \tilde{a}_2\\
\tilde{d}_3' &= \tilde{d}_3\\
\tilde{a}_1' &= \tilde{d}_3 - \tilde{a}_2\\
\tilde{a}_2' &= \tilde{d}_3 - \tilde{a}_1\\
\tilde{a}_3' &= \tilde{a}_4\\
\tilde{a}_4' &= \tilde{a}_3.
\end{aligned}$$

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2. Gromov-Witten theory

We now briefly recall GW theory and fix notation. Let X be a smooth complex projective variety, and let $\beta \in A_1(X)$ be a curve class. We denote by $\overline{M}_{g,n}(X, \beta)$ the moduli stack of isomorphism classes of stable maps

$$f : (C, p_1, \ldots, p_n) \to X,$$

where C is an n-marked, possibly nodal genus g curve. This stack admits a virtual fundamental class $[\overline{M}_g(X, \beta)]^{\mathrm{vir}}$ of algebraic degree

$$\mathrm{vdim}(\overline{M}_{g,n}(X, \beta)) = (\dim X - 3)(1 - g) - K_X \cdot \beta + n.$$

We denote by ev_i evaluation morphisms $ev_i : \overline{M}_{g,n}(X, \beta) \to X$ defined by

$$(f, C, p_1, \ldots, p_n) \mapsto f(p_i).$$

Let $\gamma_1, \ldots, \gamma_n \in H^*(X)$ be a collection of cohomology classes. The genus-g class β Gromov-Witten invariant of X with insertions γ_i is defined by

$$\langle \gamma_1, \ldots, \gamma_n\rangle^X_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n ev_i^*\gamma_i.$$

For further details regarding the fundamentals of Gromov-Witten theory see, for example, the wonderful text [14].

3. Toric Blowups and the Permutohedron

In this section we construct $\hat{X}$ and $\hat{\hat{X}}$. To do so, we first consider the case $k = 4$, i.e. we blowup $\mathbb{P}^3$ at four points and $(\mathbb{P}^1)^{\times 3}$ at two points. In this case, we are in the toric setting. We prove that $\hat{X}$ and $\hat{\hat{X}}$ are isomorphic, and in fact are both isomorphic to the permutohedral variety. It follows that $\hat{X} \cong \hat{\hat{X}}$ for general $k > 4$ by simply blowing up along additional points, which need not be fixed.

From the viewpoint of the dual polytopes of these varieties, to construct $\hat{X}$ we realize the permutohedron as a truncation of the simplex, which is classical. However the permutohedron is also constructible by truncation of the cube, yielding $\hat{\hat{X}}$. This construction is not original; for example Devadoss and Forcey [9] use this truncation of the cube to construct the permutohedron.

Notation. Let Y be a toric variety with fan Σ_Y. We will denote torus fixed subvarieties in multi-index notation corresponding to generators of their cones. For instance, $p_{i_1 \dots i_k}$ will denote the torus fixed point which is the orbit closure of the cone $\sigma = \langle v_{i_1}, \dots, v_{i_k} \rangle$, for $v_i \in \Sigma_Y^{(1)}$. Similarly $\ell_{i_1 \dots i_r}$ will denote the line which is the orbit closure of $\sigma = \langle v_{i_1}, \dots, v_{i_r} \rangle$, and so on. Further, $Y(Z_1, \dots, Z_s)$ will denote the iterated blowup of Y at the subvarieties $Z_1, \dots, Z_s$. By abuse of notation, we will denote by $Y(k)$ the blowup of Y at k points.

3.1. **The fans of $\hat{X}$ and $\hat{\hat{X}}$.** The fan $\Sigma_{\mathbb{P}^3} \subset \mathbb{Z}^3$ of $\mathbb{P}^3$ has 1-skeleton with primitive generators

$$\begin{aligned} v_1 &= (-1,-1,-1) & \quad v_2 &= (1,0,0) \\ v_3 &= (0,1,0) & \quad v_4 &= (0,0,1), \end{aligned}$$

and maximal cones given by

$$\begin{matrix} \langle v_1, v_2, v_3 \rangle & \langle v_1, v_2, v_4 \rangle \\ \langle v_1, v_3, v_4 \rangle & \langle v_2, v_3, v_4 \rangle. \end{matrix}$$

Also note that the fan $\Sigma_{(\mathbb{P}^1)^{\times 3}} \subset \mathbb{Z}^3$ of $(\mathbb{P}^1)^{\times 3}$, has primitive generators

$$\begin{aligned} u_1 &= (1,0,0) & \quad u_3 &= (0,1,0) & \quad u_5 &= (0,0,1) \\ u_2 &= (-1,0,0) & \quad u_4 &= (0,-1,0) & \quad u_6 &= (0,0,-1), \end{aligned}$$

and maximal cones given by

$$\begin{matrix} \langle u_1, u_3, u_5 \rangle & \langle u_1, u_2, u_4 \rangle & \langle u_1, u_2, u_3 \rangle & \langle u_1, u_2, u_4 \rangle \\ \langle u_2, u_4, u_6 \rangle & \langle u_2, u_3, u_4 \rangle & \langle u_1, u_2, u_3 \rangle & \langle u_1, u_2, u_4 \rangle. \end{matrix}$$

The three dimensional permutohedron Π_3 is precisely realized as the dual polytope of the blowup of $\mathbb{P}^3$ at its 4 torus fixed points and the 6 torus invariant lines between them,

$$X_{\Pi_3} \cong \mathbb{P}^3(p_{123}, p_{124}, p_{134}, p_{234}, \ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34}).$$

It is also realized as the dual polytope of a blowup of $(\mathbb{P}^1)^{\times 3}$. In particular,

$$X_{\Pi_3} \cong (\mathbb{P}^1)^{\times 3}(p_{135}, p_{246}, \ell_{13}, \ell_{15}, \ell_{35}, \ell_{24}, \ell_{26}, \ell_{46}).$$

This blowup of $(\mathbb{P}^1)^{\times 3}$ can be viewed as the blowup of two antipodal vertices on the 3-cube and the 6 invariant lines intersecting these points, as shown in Figure 1. This common blowup yields an isomorphism $\hat{\tau} : \hat{X} \to \hat{\tilde{X}}$ and a birational map $\tau : \mathbb{P}^3(4) \to (\mathbb{P}^1)^{\times 3}(2)$. The situation is depicted in Figure 2.

$$\begin{tikzcd}
\hat{X} \arrow[r, "\hat{\tau}"] \arrow[d, "\pi"] & \hat{\tilde{X}} \arrow[d, "\tilde{\pi}"] \\
\mathbb{P}^3(4) \arrow[r, dashed, "\tau"] & (\mathbb{P}^1)^{\times 3}(2)
\end{tikzcd}$$

FIGURE 2. The variety X_{Π_3} as a blowup.

Remark 4. These constructions can be generalized to higher dimensions. The permutohedron Π_n is the dual polytope corresponding to the blowup of $\mathbb{P}^n$ at all its torus invariant subvarieties up to dimension $n-2$. Note that $\Delta_{(\mathbb{P}^1)^{\times n}}$, the dual polytope of $(\mathbb{P}^1)^{\times n}$ is the n-cube. Then Π_n is the dual polytope of the variety corresponding to the blowup of $(\mathbb{P}^1)^{\times n}$ at the points corresponding to antipodal vertices on $\Delta_{(\mathbb{P}^1)^{\times n}}$, and all the torus invariant subvarieties intersecting these points, up to dimension $n-2$.

3.2. **Chow Rings.**

Notation. We will use D_α for the divisor class corresponding to v_α or u_α. For blowups, we will label a new element of the 1-skeleton, introduced to subdivide the cone $\sigma = \langle v_i, \ldots, v_j \rangle$, by $v_{i \cdots j}$. Foundations of this material may be found, for instance, in Fulton's canonical text [11].

As above, classes on $\mathbb{P}^3(k)$ remain undecorated, tilde classes, such as $\tilde{H}_i$ or $\tilde{e}_{ijk}$ signify classes on $(\mathbb{P}^1)^{\times 3}(k)$, and classes pulled back via the blowup to the variety X_{Π_3} will be decorated with a hat.

Finally, we report abuse of notation already in progress. We often denote subvarieties and their classes using the same notation. When we need care, we will use brackets. For instance the divisor H is of class $[H]$.

3.2.1. X_{Π_3} *as a Toric Blowup of* $\mathbb{P}^3$. The Chow ring of $\mathbb{P}^3$ is generated by the first Chern class of hyperplane bundle on $\mathbb{P}^3$. Let $\hat{H}$ be the pullback of this class to X_{Π_3} and let $\hat{h} = \hat{H} \cdot \hat{H}$ denote the class of a general line in $A_1(X)$. Let $\hat{E}_\alpha$ be the class of the exceptional divisor above the blowup of p_α, and $\hat{e}_\alpha$ be the line class in the exceptional divisor. Let $\hat{F}_{\alpha'}$ denote the class of the exceptional divisor above the blowup of the line $\ell_{\alpha'}$. Note that that this divisor is abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so we let $\hat{f}_\alpha$ and $\hat{s}_\alpha$ be the section and fiber class respectively. Observe that

$$A_2(X_{\Pi_3}) = \langle \hat{H}, \hat{E}_\alpha, \hat{F}_{\alpha'} \rangle, \quad A_1(X_{\Pi_3}) = \langle \hat{h}, \hat{e}_\alpha, \hat{f}_{\alpha'} \rangle.$$

The divisor classes corresponding to $\Sigma^{(1)}_{X_{\Pi_3}}$, are written in terms of this basis as

$$\begin{aligned} D_i &= \hat{H} - \sum_{i \in \alpha} \hat{E}_\alpha - \sum_{j \in \alpha'} \hat{F}_{\alpha'} \\ D_{ij} &= \hat{F}_{ij} \\ D_{ijk} &= \hat{E}_{ijk}. \end{aligned}$$

3.2.2. *As a Toric Blowup of* $(\mathbb{P}^1)^{\times 3}$. Let $\hat{\tilde{H}}_1$, $\hat{\tilde{H}}_2$ and $\hat{\tilde{H}}_3$ be the 3 hyperplane classes pulled back from the Künneth decomposition of the homology of $(\mathbb{P}^1)^{\times 3}$. We let $\hat{\tilde{h}}_{ij}$ be the line class $\hat{\tilde{H}}_i \cdot \hat{\tilde{H}}_j$ and $\hat{\tilde{E}}_\alpha$, $\hat{\tilde{e}}_\alpha$, $\hat{\tilde{F}}_{\alpha'}$, $\hat{\tilde{f}}_{\alpha'}$ and $\hat{\tilde{s}}_{\alpha'}$ be as above. These classes generate the Chow groups in the appropriate degree. The divisor classes corresponding to $\Sigma^{(1)}_{X_{\Pi_3}}$ are given by

$$\begin{gathered} D_1 = \hat{\tilde{H}}_1 - \hat{\tilde{E}}_{135} - \hat{\tilde{F}}_{13} - \hat{\tilde{F}}_{15}, \quad D_2 = \hat{\tilde{H}}_3 - \hat{\tilde{E}}_{246} - \hat{\tilde{F}}_{24} - \hat{\tilde{F}}_{26} \\ D_3 = \hat{\tilde{H}}_2 - \hat{\tilde{E}}_{135} - \hat{\tilde{F}}_{13} - \hat{\tilde{F}}_{35}, \quad D_4 = \hat{\tilde{H}}_2 - \hat{\tilde{E}}_{246} - \hat{\tilde{F}}_{24} - \hat{\tilde{F}}_{46} \\ D_5 = \hat{\tilde{H}}_3 - \hat{\tilde{E}}_{135} - \hat{\tilde{F}}_{13} - \hat{\tilde{F}}_{25}, \quad D_6 = \hat{\tilde{H}}_3 - \hat{\tilde{E}}_{246} - \hat{\tilde{F}}_{26} - \hat{\tilde{F}}_{46} \\ D_{ijk} = \hat{\tilde{E}}_{ijk}, \qquad\qquad D_{ij} = \hat{\tilde{F}}_{ij}. \end{gathered}$$

The map $\hat{\tau} : \hat{X} \to \hat{\tilde{X}}$, introduced in Figure 2, is an isomorphism induced by a relabeling of the fan $\Sigma_{X_{\Pi_3}}$. In particular, the action of $\hat{\tau}_\star$ on $A_1(X_{\Pi_3})$, is given by

$$\begin{aligned}
\hat{\tau}_\star \hat{h} &= \hat{\hat{h}}_{12} + \hat{\hat{h}}_{13} + \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} \\
\hat{\tau}_\star \hat{e}_{123} &= \hat{\hat{h}}_{13} + \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} \\
\hat{\tau}_\star \hat{e}_{124} &= \hat{\hat{h}}_{12} + \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} \\
\hat{\tau}_\star \hat{e}_{134} &= \hat{\hat{h}}_{12} + \hat{\hat{h}}_{13} - \hat{\hat{e}}_{246} \\
\hat{\tau}_\star \hat{e}_{234} &= \hat{\hat{e}}_{135} \\
\hat{\tau}_\star \hat{f}_{12} &= \hat{\hat{s}}_{46} = \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} + \hat{\hat{f}}_{46} \\
\hat{\tau}_\star \hat{f}_{13} &= \hat{\hat{s}}_{26} = \hat{\hat{h}}_{13} - \hat{\hat{e}}_{246} + \hat{\hat{f}}_{26} \\
\hat{\tau}_\star \hat{f}_{14} &= \hat{\hat{s}}_{24} = \hat{\hat{h}}_{12} - \hat{\hat{e}}_{246} + \hat{\hat{f}}_{24} \\
\hat{\tau}_\star \hat{f}_{34} &= \hat{\hat{f}}_{35} \\
\hat{\tau}_\star \hat{f}_{24} &= \hat{\hat{f}}_{15} \\
\hat{\tau}_\star \hat{f}_{23} &= \hat{\hat{f}}_{13}.
\end{aligned}$$

4. Toric Symmetries of $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$

The classical Cremona transformation is the rational map

$$\xi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

defined by

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_1x_2x_3 : x_0x_2x_3 : x_0x_1x_3 : x_0x_1x_2).$$

Note that ξ is undefined on the union of the torus invariant points and lines, and is resolved on the maximal blowup of $\mathbb{P}^3$, $\pi : X_{\Pi_3} \to \mathbb{P}^3$. The resolved Cremona involution on X_{Π_3} is a toric symmetry induced by the reflecting Π_3 through the origin. Note that the resolved Cremona map, $\hat{\xi} : \hat{X} \to \hat{X}$ acts nontrivially on $A^\star(\hat{X})$. For a more detailed treatment of toric symmetries in general and Cremona symmetry in particular, see [4, 12, 15]. Cremona symmetry is given as follows.

Lemma 6 (Bryan-Karp [4], Gathmann [12]). *Let $\hat{X}$ be the permutohedral blowup of $\mathbb{P}^3$. Let β be given by*

$$\beta = d\hat{h} - \sum_{i=1}^{4} a_i \hat{e}_i - \sum_{\substack{1\leqslant i<j\leqslant 6 \\ 8}} b_{ij}\hat{f}_{ij} \in H_2(X;\mathbb{Z}).$$

There exists a toric symmetry $\hat{\xi}$*, resolving* ξ*, such that* $\hat{\xi}_\star\beta = \beta'$*, where* $\beta' = d'\hat{h} - \sum_i a_i'\hat{e}_i - \sum_{ij} b_{ij}'\hat{f}_{ij}$ *has coefficients given by*

$$\begin{aligned} d' &= 3d - 2\sum_{i=1}^{4} a_i \\ a_i' &= d - a_j - a_k - a_l - b_{ij} - b_{ik} - b_{il} \\ b_{ij}' &= b_{kl}, \end{aligned}$$

where $\{i,j,k,l\} = \{1,2,3,4\}$.

In similar vein, the blowup $\hat{\hat{X}} \to (\mathbb{P}^1)^{\times 3}$, also has a nontrivial toric symmetry analogous to Cremona involution. Consider the rational map

$$\zeta : (\mathbb{P}^1)^{\times 3} \dashrightarrow (\mathbb{P}^1)^{\times 3}$$

defined by

$$((x_0 : x_1), (y_0 : y_1), (z_0, z_1)) \mapsto ((x_1 y_0 z_0 : x_0 y_1 z_1), (y_0 : y_1), (z_0, z_1)).$$

Lemma 7. *Let* $\beta = \sum_1^3 d_j \hat{\hat{h}}_j - a_1\hat{\hat{e}}_1 - a_2\hat{\hat{e}}_2 - \sum_{i=1}^6 b_i\hat{\hat{f}}_i \in A_\star(X_{\Pi_3})$*.* $\hat{\hat{X}}$ *admits a nontrivial toric symmetry* $\hat{\zeta}$*, which is a resolution of* ζ*, whose action on homology is given by*

$$\hat{\zeta}_\star\beta = \beta'$$

where $\beta' = \sum_1^3 d_j'\hat{\hat{h}}_j - a_1'\hat{\hat{e}}_1 - a_2'\hat{\hat{e}}_2 - \sum_{i=1}^6 b_i'\hat{\hat{f}}_i$ *has coefficients given by*

$$\begin{aligned} d_1' &= d_1 + d_3 - a_1 - a_2 - b_2 - b_5 \\ d_2' &= d_2 + d_3 - a_1 - a_2 - b_1 - b_4 \\ d_3' &= d_3 \\ a_1' &= d_3 - a_2 - b_4 - b_5 \\ a_2' &= d_3 - a_1 - b_1 - b_2 \\ b_1' &= b_5, \quad b_2' = b_4 \\ b_3' &= b_3, \quad b_4' = b_2 \\ b_5' &= b_1, \quad b_6' = b_6. \end{aligned}$$

Remark 5. In [15], it is shown that X_{Π_3}, as a blowup of $\mathbb{P}^3$, admits a unique nontrivial toric symmetry. Indeed, although the permutohedron admits many symmetries, in the cohomology basis induced by the isomorphism $X_{\Pi_3} \cong \hat{X}$, each symmetry is either acts trivially, or is equal to the Cremona symmetry above. Here we have a new toric symmetry of the permutohedron, nontrivial in the cohomology basis induced by $X_{\Pi_3} \cong \hat{\hat{X}}$.

Proof. Observe that choosing $\hat{\zeta}$ to be the toric symmetry

$$\hat{\zeta} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$\zeta_\star$ on $A_\star(X_{\Pi_3})$ has the desired action on homology, and the natural blowup-blowdown composition with $\hat{\zeta}$ gives the birational map ζ. $\square$

$$\begin{tikzcd}
X_{\Pi_3} \arrow[r, "\hat{\zeta}"] \arrow[d] & X_{\Pi_3} \arrow[d] \\
(\mathbb{P}^1)^{\times 3} \arrow[r, dashed, "\zeta"] & (\mathbb{P}^1)^{\times 3}
\end{tikzcd}$$

Figure 3. The rational map ζ and a resolution.

Remark 6. Note that to resolve the map ζ it is sufficient to blowup a subset of the six lines described in Section 3.1. However by blowing up these extra lines, we prove both Theorems 1 and 5 simultaneously.

5. Proof of Main Results.

We established the isomorphism between $\hat{X}$ and $\hat{\hat{X}}$ in Section 3. Thus, Theorem 1 will follow from Theorem 2 and Theorem 3. Note also that the classes $\hat{f}_\alpha$ in Lemma 7 form an orbit under $\zeta_\star$. Thus Theorem 5 follows from Lemma 7 and Theorem 3. Therefore, in order to establish Theorems 1 and 5, it suffices to now prove Theorem 3.

Proof of Theorem 3. Let $\tilde{\pi} : \hat{\hat{X}} = X_{\Pi_3}(k-2) \to \tilde{X} = (\mathbb{P}^1)^{\times 3}(k)$ as before. That is, we follow the constructions of Section 3, and blowup at $k-2$ additional points. Let $\hat{\beta} = \sum_1^3 d_j \hat{\hat{h}}_j - \sum_{i=1}^k a_i \hat{\hat{e}}_i$ with $a_i \neq 0$ for $i > 2$. We argue that any stable map in the isomorphism class $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ has an image disjoint from $F = \cup \hat{\hat{F}}_{jk}$ where the union is taken over all the exceptional divisors above line blowups. We similarly show that any stable map $[f] \in \overline{M}_g(\tilde{X}, \beta)$ has an image disjoint from $\ell = \cup \ell_{jk}$. It then follows that the map on moduli stacks induced by $\tilde{\pi}$ is an isomorphism of stacks, obstruction theories, and virtual fundamental classes.

Let $[f : C \to \tilde{X}] \in \overline{M}_g(\tilde{X}, \beta)$. Suppose that $\mathrm{Im}(f) \cap \ell_{rs} \neq \emptyset$ where ℓ_{rs} is one of the six lines in the exceptional locus. Without loss of generality, since $\tilde{a}_i \neq 0$ for some $i > 2$, $\mathrm{Im}(f) \not\subseteq \ell_{rs}$. As a result we may write the class of the image as

$$f_\star[C] = C' + b\ell_{rs}, \quad (b \geqslant 0).$$

Here C' meets ℓ_{rs} at finitely many points for topological reasons. Let $\hat{C}'$ be the proper transform via $\tilde{\pi}$ of C'. Since $C' \cap \ell_{rs} \neq \emptyset$, $\hat{C}' \cdot \hat{F}_{rs} = m > 0$. Thus, we may write

$$\hat{C}' = \hat{\beta} - b(\hat{\hat{h}}_j - \hat{\hat{e}}_\alpha) - m\hat{\hat{f}}_{rs}.$$

Here $\alpha \in \{1, 2\}$, or in other words, e_α is the exceptional line above one of the torus fixed points, and $[\ell_{rs}] = \tilde{h}_j$. Now push forward this class $\hat{C}'$ via the inverse of the map $\hat{\tau}_\star$ described in Section 3.2.2. Observe then that we obtain a curve in X_{Π_3}, whose class is given by

$$\hat{\tau}_\star^{-1}\hat{C}' = d\hat{h} - \sum_{i=1}^{6} a_i\hat{e}_i - b(\hat{h} - \hat{e}_\gamma - \hat{e}_\delta) - m\hat{f}_{pq},$$

where $\{\gamma, \delta\} \subset \{1, 2, 3, 4\}$. In particular, via $\hat{\tau}_\star$, we see that $dh - \sum_{i=1}^{6} a_i e_i$ must have virtual dimension zero since $\tilde{\beta}$ and $\hat{\beta}$ have virtual dimension zero. Further, $\hat{\tau}_\star\hat{f}_{pq} = \hat{\hat{f}}_{rs}$. Now consider the divisor

$$\hat{D}_{pq} - 2\hat{H} - (\hat{E}_1 + \cdots + \hat{E}_6) - \hat{F}_{pq} - \hat{F}_{p'q'},$$

where $\{p, q, p', q'\} = \{1, 2, 3, 4\}$. Bryan-Karp prove in [4] that $\hat{D}_{p,q}$ is nef. However, clearly $\hat{D}_{pq} \cdot \hat{\tau}_\star^{-1}\hat{C} = mF_{pq} \cdot f_{pq} = -m < 0$, which is a contradiction. Thus, $f_\star C \cap \ell_{rs} = \emptyset$.

We argue in similar fashion for $\overline{M}_g(\hat{X}, \hat{\beta})$. Let $[\hat{f} : C \to \hat{X}]$. Suppose $\mathrm{Im}(\hat{f}) \cap \hat{\hat{F}}_{rs} \neq \emptyset$. Since $\hat{\beta} \cdot \hat{\hat{F}}_{rs} = 0$, $f_\star C$ must have a component C'' completely contained in $\hat{\hat{F}}_{rs}$, where we have

$$f_\star C = C' + C'',$$

where C' is nonempty since $\hat{\beta} \cdot \hat{\hat{E}}_4 \neq 0$. Since $C'' \subset \hat{\hat{F}}_{rs}$ is an effective class in $\hat{\hat{F}}_{rs} \cong \mathbb{P}^1 \times \mathbb{P}^1$, it must be of the form $C'' = a\hat{f}_{rs} + b\hat{s}_{rs}$ for $a, b \geqslant 0$ and $a + b > 0$. We compute $\hat{\tau}_\star^{-1}(\hat{D}_{pq}) \cdot C' = -a - b$, contradicting the fact that $\hat{D}_{pq}$ is nef. Thus, $\mathrm{Im}(\hat{f}) \cap \hat{\hat{F}}_{rs} = \emptyset$. □

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E-mail address: dk@math.hmc.edu

E-mail address: dhruv_ranganathan@hmc.edu

Department of Mathematics, Harvey Mudd College