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Book Review: *Algebra the Beautiful: An Ode to Math's Least-Loved Subject* by G. Arnell Williams

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Synopsis

In his book *Algebra the Beautiful*, G. Darnell Williams has undertaken a challenging job — to show the importance, deep structure, intellectual connections, and sheer beauty of classroom algebra. This review describes some of the questions the book raises, the historical and cultural context it provides, and the intellectual apparatus it deploys.

> **Algebra the Beautiful: An Ode to Math's Least-Loved Subject.** By G. Arnell Williams, New York: Basic Books, 2022. (ISBN: 978-1541600683, 416 pages.)

Overheard at various times during my long teaching and tutoring career:

> When I was in high school, the teacher put a bunch of hieroglyphics on the board, and it was too much. "I'm leaving," I said. "Just give me an F." And I walked out.
>
> In ninth grade, we had just one day to learn how to translate from words into algebraic symbols. One day! At least I was there. Imagine how algebra was for any kids who were absent that day.
>
> Yes, I know the quadratic formula. Our teacher taught us to sing it to the tune of "Pop Goes the Weasel."
>
> Hey, I took two years of high-school algebra. I've never, ever used any of it in real life.

The author of a book called *Algebra the Beautiful* has his work cut out for him.

G. Arnell Williams has undertaken a challenging job — to show the importance, deep structure, intellectual connections, and sheer beauty of classroom algebra. He doesn't shy away from questions like why, or even whether, we should teach algebra in high school. Anybody who teaches introductory mathematics, whether in fourth grade or at a university, will find much food for thought in this book. And maybe some current or former students will come to appreciate what the standardized tests, unimaginative textbooks, and underprepared or burned-out teachers failed to get across to them.

So let me talk about some of the questions the book raises, the historical and cultural context it provides, and the intellectual apparatus it deploys. And while you are thinking about these things, reflect on how much, or even whether at all, your introductory algebra course talked about them, and recall what it spent time on instead.

Elementary algebra has in it two powerful ideas, whose importance to human progress is hard to overestimate. What are these two ideas? "Let x equal" and "For any a and b."

The first idea lets you solve problems when the answer is unknown to you. How is this even possible? Plato has the title character in his dialogue *Meno* ask [2, page 363], "But how will you look for something when you don't in the least know what it is? How on earth are you going to set up something you don't know as the object of your search?" But algebra makes it easy. You do it by *naming* the unknown and *treating it as though it is known.* Once you've said, "let x equal" the unknown thing, you can calculate with it: add it to itself, multiply something by it or divide it by something, take its cosine or raise e to its power. And then you can express all the properties you want it to have in terms of an equation. Now you're halfway home. And then you can use the basic rules of arithmetic to convert that equation into another one that has x all alone on one side, and only quantities you already know on the other.

The idea itself, with various symbols or abbreviations for "equal" and "unknown," long predates general algebraic symbolism. That we call our subject by the over-thousand-year-old Arabic word "algebra" demonstrates that symbols are not the only essential part. Further, assuming that you have the problem solved and then working backwards to reduce the solution to something you already know is how the Greeks discovered many constructions in geometry; they called this method "analysis," meaning "solution backward," and François Viète, the 16^{th}-century father of general symbolic algebra, called his book "Introduction to the Analytical Art" [3].

The second powerful idea is that problems which look quite different can all be solved in exactly the same way, if only we have a way to reveal their common structure. And the way we do this is to use general symbols to represent all possible numbers. We're so used to this that we may forget to share, with our beginning students, how amazing it is. But those who were present at the creation of alphabetic symbolic reasoning in algebra, like Viète and René Descartes, had a lot to say that is worth sharing.

For instance, we learn in arithmetic that $1 + 2 = 2 + 1$. Textbooks say "This is called the commutative law" as though this were just a matter of terminology. But if we introduce two distinct symbols a and b, the first standing for "any number you choose" and the second for "any other number you choose," you can write the *infinite* number of statements of the form $1 + 2 = 2 + 1$ in *just one line*: $a + b = b + a$. Wow! Furthermore, in an algebraic expression using general symbols, as Viète said, *the operations leave their footprints behind*. Descartes provided a nice example: If you are told that the hypotenuse of a right triangle with legs 9 and 12 is $\sqrt{225}$, you have the answer, but you don't know how it was obtained. However, if you call the legs of the triangle a and b, then the answer $\sqrt{a^2 + b^2}$ tells you exactly how to find it [1].

The power and beauty of algebra comes from uniting these two ideas — general symbols and the finding of unknowns. And this is what the heart of Williams's book is about. Using a variety of numerical examples, he shows that a large number of significant problems turn out to have the same algebraic structure. And therefore, he shows how once you have solved a problem that has a particular structure, you have solved *all* such problems. For instance, he shows that calculating weighted sums symbolically lets you find grade point averages, slugging percentages in baseball, the ratings of colleges in the *US News* survey (a survey he criticizes), and the composition of cocktails. All these examples involve only linear multiples of the unknowns, and so are accessible to high-school algebra students.

Williams explains our two powerful ideas, but in a less theoretical way, one more focused on doing algebra as an activity. He identifies two key algebraic activities, which he calls "the first drama" and "the second drama."

The first activity, "the first drama," is creating an algebraic expression that shows you how a variable (or unknown) is related to various other parameters. This activity embodies the "let x equal" move, and can also involve the use of general parameters. Given this expression, he then describes the second activity, "the second drama," which is choosing the value you want the expression involving the variable to have. Once you've set up the equation that equates the expression and its chosen value, you've reached what he calls the "jump for joy" step — a lovely image for the student.

There's more to the book, which is full of nice touches and historical insights. For instance, in various word problems which use people's names, the author chooses the names of major figures in the history of mathematics, such as the Greek female mathematician Hypatia, the Islamic algebraist Abu Kamil, the Renaissance Italian Ferrari. He tells how the 12$^{\text{th}}$-century mathematician Bhaskara II from India used different colors to represent multiple variables. To illustrate how notation can stand for something beautiful and profound, he gives the example of musical notation. When discussing what he calls "classroom word problems," he argues that it's okay if they're artificial, as long as you're honest about why you're using them and not pretending that they are real-life applications.

Still, the book is not free from flaws. For instance, the first chapter introduces the basic ideas with an example of the "take some numbers, don't tell me what they are, do various manipulations with them, tell me your answer, and I'll tell you what numbers they are." But it's a complicated example, and if I were a high school student in introductory algebra, I would find it hard to wade through. There is a bit too much terminology. Words like "polysemous" and "homonymous" are not necessary, and some terms that are the author's own coinage could have been avoided. And sometimes the "why?" of rules is not provided for potential student readers, as in "negative times negative is positive," or reducing a fraction, not by saying that dividing top and bottom by the same number preserves their ratio, but by saying "cancel out 5 from top and bottom." The discussion of radioactive decay is accompanied by the statement that the book won't explain solving exponential equations. And the author sometimes repeats the same points, I think unnecessarily, making the book pretty long.

But, flaws aside, the reader will appreciate phrases like these: "In order to foster a greater appreciation of algebra, we need to understand why the equations we're solving matter." "If we want students to experience the wonder, satisfaction, and appreciation of algebra, then we should probably put these into our teaching right from the start." "Many things which don't look or seem alike at all can still be tied together by common mathematical expressions, equations, and reasoning." "Algebra weaponizes metaphorical and analogical reasoning, rendering it more precise and operational." "When we encounter algebraic ideas outside of the classroom... they rarely come at us in the x's, y's, and z's of school algebra. The critical 'tell' of algebraic possibilities in the wild is the presence of numerical variation of some sort." The author amply provides examples illustrating these generalizations. And my personal favorite: "[The] ability to flirt with the eternal from home is... present right here in elementary algebra."

One last point, responding to what we've all heard: "Hey, I took two years of high-school algebra. I've never, ever used any of it in real life." Maybe you didn't solve quadratic equations, and never again had to simplify ratios of polynomials. But to analyze a problem and figure out its structure, to reason symbolically about sales taxes and interest rates and public opinion surveys, to express the relationship of a set of particulars in general terms — yes, you did use algebra. And if you teach algebra, Williams's book will stimulate your thinking about how to get the key ideas — the ones that matter — across to your students. Williams's book can also help you help your students enjoy the aesthetic kick of seeing that two things that they thought were different are, in some profound sense, the same. And then the students too can say, this is algebra the beautiful.

References

[1] René Descartes, *Regulae ad directionem ingenii*, [Rules for the direction of the mind, published posthumously 1701], XVI. In Adam, Charles, and Tannery, Paul, eds., *Oeuvres de Descartes*, vol. X, Paris: J. Vrin, 1983, p. 455-458.

[2] Plato, *Meno* 80D, translated by W. K. C. Guthrie, in Edith Hamilton and Huntington Cairns, eds, *Plato: The Collected Dialogues*, New York: Pantheon, 1961.

[3] Franciscus Viète, *In artem analyticem isagoge*, 1591. An English translation, "Introduction to the Analytical Art," by Rev. J. Winfree Smith is available on pages 315–353 of Jacob Klein's *Greek Mathematical Thought and the Origin of Algebra* (MIT Press, Cambridge, 1968).