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Cover Page Footnote

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The Long Search for Collatz Counterexamples

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Synopsis

Despite decades of effort, the Collatz conjecture remains neither proved, nor refuted by a counterexample, nor formally shown to be undecidable. This note introduces the Collatz problem and probes its logical depth with a test question: can the search space for counterexamples be iteratively reduced, and when would it help?

Keywords: Collatz conjecture, dynamical systems, unsolved problems.

1. The Collatz problem

The Collatz or Syracuse problem (which has also been called Ulam's problem) has been described as notorious and dangerous. The notoriety of the problem increases as the years go by. It is still unsolved, yet any primary school child can grasp it:

Apply recursively to any positive integer the Collatz rule or map, "if odd, triple the number and add one, else halve the number". The challenge: name just one positive integer for which the iteratively repeated application of the Collatz map never meets a power of two, and thus never gets pulled down by a halving cascade to the endless limit cycle $4, 2, 1$ as its ultimate fate. Or else prove that all natural numbers must have this fate.

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All observations of actual Collatz runs to date suggest that the ultimate 2^k cascade pulling down to 1 awaits every positive starting integer, which is today known as the Collatz conjecture. Despite several computational attempts, no counterexample has ever been found. One line of progress in understanding the Collatz map has recently been presented by Terence Tao ([30], see also [3]); earlier articles and a book reviewing aspects of the challenge have been published [19, 10, 21, 20].

This note probes the logical depth of the Collatz problem with a test question: can the search space for counterexamples be iteratively reduced, and when would it help? Along the way, we provide an extensive overview of the problem itself and what is known about it to date.

2. Currently studied forms of the Collatz map

Together with the simplest '$3n + 1$' form of the conjecture just stated, and which we will use for much of the following, two other forms (maps, representations) exist in the current literature. Though these two alternative forms may not be quite as easy to summarize for a child, they each have their advantages. Both make use of the fact that for any odd natural number n, $3n + 1$ must be even. Thus, every upward move in a Collatz '$3n + 1$' run must trivially be followed by at least one halving (downward) move.

The three maps map n to (respectively) an even number m, a number l of either parity, or an odd number k, where

$$\begin{aligned} 3n + 1 &= m \\ &= 2\,l \\ &= 2^j k. \end{aligned}$$

More precisely, the second of these three maps again halves even numbers, but maps odd numbers to $(3n+1)/2$ instead of $3n+1$. The third map sends any starting (positive) integer n to an odd number, namely the largest odd number dividing $3n + 1$ [30]. Figure 1 illustrates the three maps graphically in an example.

For all three maps, the conjecture is that iterative application of the map will eventually lead to 1, and in this sense they can be considered equivalent.

The recent article by Tao [30] and the overview article by Allouche on its main results [3] consider initially the first map, denoted Col, and/or the second map, denoted Col_2 or f or T in the literature, and finally also the third map, denoted Syr.

Figure 1 shows two graphical representations of (parts of) Collatz runs. Panel A shows a run starting at 104, indicating the differences between the different forms of the Collatz map. The plot emphasizes the leaping from one halving cascade to another that is prompted whenever the bottom of a cascade (an odd number) is reached. Panel B shows, for the $3n + 1$ form of the Collatz map that we shall use in the sequel, the run starting at 77671 (an unusually long run, of length 231), as a plot of height (value) versus number of steps, or iterations of the Collatz map. The number of steps can also be interpreted as time or horizontal distance, if one imagines an airplane flying in a constant horizontal direction but frequently changing its altitude.

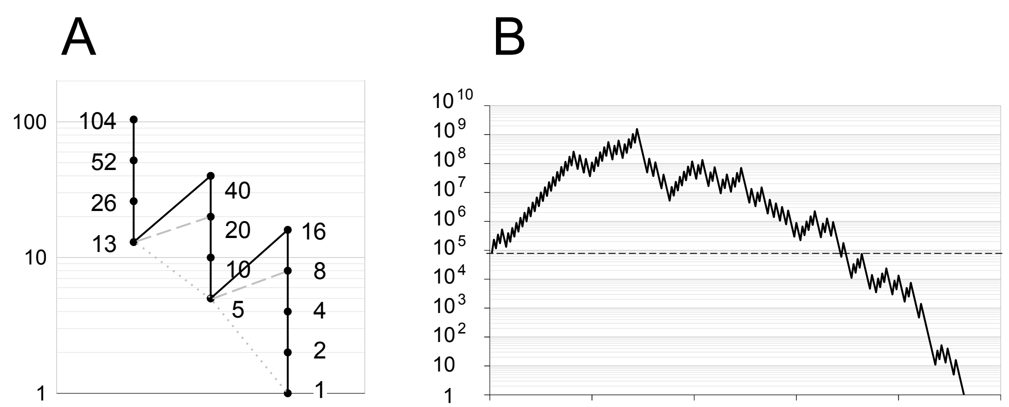

Figure 1: Two runs of the Collatz process, from initial starting values or altitudes 104 *(A) and* 77671 *(B), illustrated via two different graphical display types. Both runs ultimately descend to the integer 1 (i.e., they 'land'); the Collatz conjecture is that this destiny awaits any starting integer. The dashed and dotted diagonal grey lines in panel A indicate two alternative forms of the Collatz map that are sometimes used in the literature, where the former is denoted T, f or Col_2 and the latter is denoted Syr. The horizontal dashed line in panel B marks the starting value ('fog' line); for an induction proof of the Collatz conjecture the main question is whether or not the run (or 'flight') from any starting value will ultimately dip below the line, i.e., below the flight's starting height. (Both plots are the original work of the author.)*

3. Collatz airplanes above the fog

The Collatz problem has the danger that its fascinating dynamics has the power to lure those away who had originally set out to prove or disprove the conjecture. We thus focus completely on the conjecture, and attempt induction.

The image: a deterministic Collatz airplane flying forward, above a level sea of fog, and periodically changing (integer) altitude as it advances in its horizontal flight. We use induction, a base verification at 3 or 5 as starting integer, and then by the rules of induction on the natural numbers we assume validity of the Collatz conjecture from there up to and including n, the fog altitude: if the plane starts from any altitude in or below the fog line, it will always land.

Can we now show (induction step) that such a Collatz airplane starting from height $n+1$, just above the fog, will also ultimately land?

The Collatz process is fully deterministic, so any vertical position of an airplane after departure will have the same future trajectory as another airplane that starts there. Thus, all that has to be shown is that an airplane starting from height $n+1$ will eventually descend to an altitude in the fog ($\leqslant n$). We then know, by our assumption, that a previous airplane that had started at that altitude landed successfully, so we know our airplane will do so too.

For all even values of $n+1$, the first move is right away an altitude-halving dip down into the fog. So for those even starts, the induction step is already done, and we have only 'half' the natural numbers $\mathbb{N}$ (the odd ones) left to take care of. The numbers 1 mod 4 are similarly dispatched because they also have a constant, short time above the fog (leaving '1/4' of $\mathbb{N}$), then the numbers 3 mod 16 are likewise eliminated (leaving '3/16'). With the help of a printout of the Collatz runs for the first hundred or so starting altitudes, in which each run (or flight) is followed (printed out) only until it has dropped to its first altitude in/below the fog, one recognizes further patterns.[2]

[2] I initially tested this approach before screening the literature, and then found that many others had previously followed the same route, beginning around 1973.

For example, the length (or time) up to and including that first dipping event is easily shown to be always (a) 2 if the starting altitude is even, (b) 4 if it is $\equiv 1$ mod 4, and (c) 7 if it is $\equiv 3$ mod 16.[3]

In the literature, the length just mentioned, i.e., the minimum number of iterates of the Collatz map needed to reach a value less than the original starting n, is called the *stopping time* of the map or trajectory for the starting value n (see [31], [19, pages 4-5]). A starting value from which the Collatz process never ultimately dips down to lower value(s) is assigned an infinite stopping time ($+\infty$), and in this way a *stopping time function* is defined. The Collatz or '$3x + 1$' conjecture is thus equivalent to the conjecture that every positive integer greater than 1 has a finite stopping time [19, page 5].

The recursive shrinking of the apparent fraction of $\mathbb{N}$ (or, rather, of any finite interval along the natural number line) that is occupied by potential counterexamples to the Collatz conjecture does not stop at 3/16. In contrast to the 13/16 of flights posing no such threat and having (class-wise) constant, short flight lengths before their first dive into the fog, the remaining 3/16 appear to exhibit substantial length variation, including occasional runs that look 'dangerous' (longer/higher), such as those starting at 27, or 77671. However, as we print out the runs for greater heights, we continue to discover, and then prove by elementary arithmetic, other classes (mod 32, mod 128, mod 256 and so on) that obviously pose no threat of containing potential counterexamples to the Collatz conjecture.[4] For example, examining flights starting at increasing altitudes and allowing inference of rules mod 32, mod 128 and mod 256 lets us reduce to 7.4% the proportion of 'untamed' starting altitudes, apparently mimicking a fractal process, but in any case iteratively reducing the perilous set. A question that then naturally comes to mind is whether or not this successive crowding out of the perilous by the tame can be shown to continue until 'in the end' *none* of the perilous are left.

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[3] For natural numbers k, (a) $2k \mapsto k < 2k$; (b) $4k + 1$ (odd) $\mapsto 12k + 4 \mapsto 6k + 2 \mapsto 3k + 1 < 4k + 1$; (c) $16k + 3$ (odd) $\mapsto 48k + 10 \mapsto 24k + 5$ (odd) $\mapsto 72k + 16 \mapsto 36k + 8 \mapsto 18k + 4 \mapsto 9k + 2 < 16k + 3$.

[4] Repeated patterns suggest, for example, that one may further exclude as 'harmless' the numbers in the classes 11 and 23 mod 32; 7, 15 and 59 mod 128; and 39, 79, 95, 123, 175, 199 and 219 mod 256. (These classes were already tabulated by Dunn in 1973 [13].)

Thoughts along such lines are visible in independent publications beginning in the 1970s; for example, [19, 21] and [3] give lists of authors and their works in the decade 1970–1980 who all appear to have arrived at essentially the same approach independently of each other. The references compiled some years later by Lagarias [19] suggest that the first report may have been that of Dunn in 1973 [13], "On Ulam's Problem"; Dunn had been directed to the problem by Andrzej Ehrenfeucht, professor of computer science at the University of Colorado and a professional acquaintance or friend of Stanislaw Ulam.[5] Attempts to use this general strategy, or class of strategies, to make progress toward solving the Collatz problem have been reviewed in further articles [19, 10, 20, 3] and a book [21].

4. Two periods of intense interest

There is some consensus on the opinion that so far there have been two major phases of intense progress on the road to understanding the dynamics of the Collatz map. The first is perhaps best known by a paper of Riho Terras from 1976 [31], and complemented by an additional proof and comments by the same author in 1979 [32], though some of those results were also derived independently by others, including Everett [14] and Allouche [2]. The results of that period have occasionally been regarded as the last major results achieved in the direction of solving the Collatz problem until 2019 (the next three decades or so saw much less progress on it).

In [31], Terras presented an expanded version of the table found in Dunn's report of 1973 [13], together with a rigorous proof that the proportion of possible starting values of the form $a\,2^j + b$ (where $j, a, b \in \mathbb{N}$) that provably arrive at 4, 2, 1 in a finite number of iterations of the Collatz map tends to 1 as j tends to infinity; Lagarias [19] showed that it does so at an exponential rate. It was soon recognized, however, that such asymptotic results are not strong enough to prove the Collatz conjecture, as the part of $\mathbb{N}$ that was being gradually (asymptotically) crowded out might still allow counterexamples.

[5] Richard Dunn, Personal communication, October 6, 2022.

Another important paper published during that fertile first period of the 1970s was an analysis by Crandall in 1978 [12] that approached the Collatz problem from a slightly different angle. There are two possible scenarios by which the Collatz conjecture could in principle fail to be true. A Collatz run might never enter into a cycle, not even into the typical final cycle passing through 1. This scenario is sometimes referred to as ‘divergence’: meandering or reaching ever higher altitudes, the Collatz plane would never enter into any kind of strict periodicity of its altitudes, but would forever explore new altitudes. The second scenario is that the Collatz plane would never land because its altitudes get locked into a strict, deterministic periodicity, i.e., a cycle, that is different from the trivial cycle for landing planes that passes through 1. Crandall focused on this second scenario, and deduced new conditions that would need to be met by the final, non-trivial cycle that would be the ultimate fate of such a Collatz run. The main theorem of [12], obtained via the Diophantine equation $2^x - 3^y = p$, is its Theorem 7.2; see also [19], where it is also included as Theorem I, and where later results are discussed that were able to exclude a class of particularly simple cycles.

The statement of Crandall’s theorem, as proved and motivated in [12, 19], is essentially as follows, formulated here for the simplest, $3n+1$ form of the Collatz map. We would ideally like to find the smallest possible length of a nontrivial cycle (i.e., the smallest possible period of the strictly periodic part of a Collatz trajectory, other than the trivial period passing through 1). We would like to do this given the latest, current knowledge of the smallest possible starting value of a Collatz run that might violate the Collatz conjecture. The theorem states that for the denominator q_j of the j-th convergent of the continued fraction expansion of $\log_2 3$, and a known lower bound m for the value of the minimal element of a cycle (excluding the usual limit cycle through 1, $j \geqslant 4$ and $m > 1$), the cycle length is greater than $\min(q_j, 2m/(q_j + q_{j+1}))$.[6]

[6] To obtain a good bound, it is advisable to look for the pair $(j, j+1)$ giving the highest value of this expression; in the three estimates mentioned below (from 1978 to present), the respective maxima were attained for j between 10 and 21. Convergents’ denominators can be calculated using `Sage` (e.g., `t=log(3)/log(2);a=continued_fraction(t);denominator(a.convergent(21))`) or looked up from tables (e.g., at https://oeis.org/A005664).

By establishing a connection between a possible cyclic trajectory and the continued fraction for the constant $\log_2 3 = \log 3/\log 2$, Crandall's theorem established, as a result that is easy to apply, "a lower bound for the period of an infinite cyclic trajectory in terms of its smallest member"; for a given period there can be only finitely many cyclic trajectories having that period. Early estimates of minimal possible cycle sizes calculated from Crandall's formula [12], using the upper frontiers of the counterexample-free Collatz runs that had been verified (for starting values $\approx 2^{30} - 2^{40}$), were of the order of 10^4 or 10^5 [12, 19]; calculations using the most recent counterexample-free frontier cited by Tao [30], namely at starting values $\approx 2^{68}$, would lead to a minimal possible cycle size of the order of 10^9 to 10^{10}. Incidentally, the Syr form of the Collatz map appears to have also been first introduced in Crandall's paper (see [30]).

What is considered by many as a novel and major step toward solving the Collatz problem is a recent proof by Terence Tao, strengthening some of the results of Terras. This new proof was first communicated in 2019 in a 49-page preprint and has since been published [30]. The result is indeed stronger than those of the 1970s, and along the same lines, though it shares the need to restrict the scope of results using clauses including "almost", which are however defined in a very precise sense. In other words, in its present form it, too, cannot rule out a counterexample. According to Allouche [3], the main result of Tao's paper "leaves little hope that the function tending to infinity as slowly as one likes in his statement can be replaced by a constant."

The main statement of Tao's paper, and its title, is: "Almost all Collatz orbits attain almost bounded values". In other words [30, 3], for *any* real-valued function $\varphi(n)$ of the positive integers that tends to infinity as the integer argument n tends to infinity, *no matter how slowly*, the minimal value of the Collatz sequence generated by iteratively applying the Collatz map to an initial integer n_{start} will be less than or equal to the function's value at that starting integer, $\varphi(n_{\text{start}})$. The "almost" in "almost bounded" indicates that the minimal value of the orbit is not less than some constant, as one might have expected, but only less than a (possibly slowly increasing, yet ultimately unbounded) function. The "almost" in "almost all" means, in this theorem, that the logarithmic (not the natural) density is maximal; this does not exclude a set of exceptions that, taken together, are negligible in density.

In a brief summary of an intricate proof such as Tao's, there is an inherent danger of emphasizing some important parts of the proof while not giving due attention to others, so we will not include such a summary here. Tao's paper [30] includes a good introductory section, presenting his main theorem, embedding it in the context of other work on the conjecture, and summarizing the key steps of his proof. For another introduction to Tao's paper that keeps a focus on the results and how they improve upon previous findings, we refer to the recent review by Allouche [3], who uses also his own earlier results and those of Korec [17] to set the stage and convey the novelty of Tao's theorem.

5. Interlude: Alice and Achilles running

If a proof of the Collatz conjecture were to be found (perhaps an embarrassingly simple one?), then `arXiv` and similar repositories promising wide visibility, indelible time-stamps, and secure seriation might be a safe haven. Otherwise authors wary of having their idea stolen might, in a thought experiment, agree with a journal on a tested zero-knowledge protocol for verifying a proof before needing to see it [7].[7] This route would, however, have a drawback (which may be a fundamental one; [24, page 255]): that such a protocol could only be used to verify "a 'standard' proof, in a given logical system" [7]. Most of the strategies mentioned in the present exposition contain some step involving 'playing with infinity' [24]. It is not yet clear if one could altogether avoid such steps, although as Terras pointed out in one of his Collatz-related proofs [32], "[t]he trick involved is to get this formula without forming any infinite sums."

[7] Zero-knowledge interactive proof systems or protocols, which were given much attention in the 1980s [7, 22, 16] and have been recently revisited [1], aim to offer a way of demonstrating knowledge of secret information without revealing other details that could be used to deduce the secret information itself. Examples include verifying a claim that a given graph admits a 3-coloring without facilitating the finding of an actual 3-coloring of the graph [7, 22, 16, 1], or verifying that a Hamiltonian cycle of a graph exists without divulging any additional information about such a cycle [7]. Blum [7] proposed (and sketched a proof for) a further methodological step: from zero-knowledge verification of properties to zero-knowledge verification of bounded-length proofs of theorems that are provable within a logical proof system (e.g., Russell and Whitehead's system).

In our imagined induction-step scenario, the set of starting integers that could conceivably generate a Collatz counterexample via the Collatz map appears to be a dwindling subset of the natural numbers $\mathbb{N}$. In any finite range of altitudes, the subset of 'viable' starting altitudes seems to shrink further and further, as one recognizes yet more classes of integers (each of them countably infinite) that would need to be *forever* avoided in order for the trajectory to remain eligible as a candidate counterexample to the Collatz conjecture. In addition, the trajectories are further constrained by the effective blocking of one 'third' of $\mathbb{N}$, the multiples of 3: this is a class that can never be visited by any Collatz trajectory except possibly transiently at the start of the run.[8] Such a radically confined run that never reaches the number 1 would, on peril of failure, need to dodge or avoid 'most' natural numbers for all eternity, and at the same time it would be denied access to a large portion of the remaining numbers. It may be hard to imagine that the proverbial jittering molecules in a room will one day find themselves all in one corner of a room; how much harder would it be to imagine that they will then remain in that corner, forever? Is that not simply too much to ask for?

We return to the title of this note and back to the search for counterexamples. For a moment, let us try to keep two, dual views in focus at the same time (which is not always easy), namely the strict-subset or measure view in which one can make progress removing parts, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, and the cardinality view in which one cannot, $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$. We may think of Alice running with the Red Queen just to keep in the same place, or Achilles chasing after the tortoise in vain until Newton and Cauchy finally let him close the gap with rigor. We may wonder if the Collatz problem is perhaps one of those true but unprovable statements mentioned by Gödel [24, page 255], or in another known class of problems documented to be in some sense 'unsolvable' [21, Part I, and references therein]. We may wonder if there should actually be two (or more) Collatz conjecture variants, differing in stringency.

[8] If a run starts at an even multiple of 3 then it can remain in that class during an initial, limited halving cascade only until all of the start altitude's powers of 2 have been used up and the cascade ends in an odd number. After that, the run must turn upwards and leave the class forever. Indeed, if it were possible to enter the class $3k$, $k \in \mathbb{N}$ from outside that class, then the previous number would need to have been $(3k-1)/3 = k - \frac{1}{3}$, which is not an integer, thus impossible.

And we may be reminded of a rhyme from *Through the Looking-Glass*,

> They wept like anything to see
> Such quantities of sand
> ...
> 'If seven maids with seven mops
> Swept it for half a year,
> Do you suppose,' the Walrus said,
> 'That they could get it clear?'

6. Another system with "almost all" theorems

We now leave behind ways of trying to prove the Collatz conjecture via initially promising results for "almost all" scenarios in a measure-theoretic sense. A possibly unspoken hope may have been to be able to one day jump from such results, using similar methods, to corresponding, stronger results that are valid for *all* scenarios. Such a hope has not yet been fulfilled though of course this does not mean that it never will.

Before moving on, however, we note that the considerable interest in the Collatz conjecture, as seen in papers of the 1970s, occurred partly in parallel with an independent wave of intense interest, during the 1960s and 1970s, in quasiperiodic motions and whether or not they persist after experiencing a small perturbation. This interest, also in the general public, was summarized by the simple question "Is the solar system stable?" [23]. The interest was motivated by the results of two papers, published in 1962 and 1963 by J. K. Moser and V. I. Arnold, respectively, extending a theorem of Kolmogorov from 1954 (see, e.g., [6], [23] and references therein); the theory or theorems created were referred to as "KAM", an acronym that represented the three authors by their initials. We mention this because, as in the Collatz attempts described above, KAM theory was also forced to rest content with statements that included the weakening clause "almost all". Furthermore, as in the case of the Collatz problem, no corresponding result without such a restricting clause has been proven until now. In his overview article in 1975, Jürgen Moser [23] wrote (italics in the original):

> "... it is entirely possible that an arbitrarily small perturbation in the initial values will change a quasi-periodic stable orbit to an unstable one. One can show, however, that the unstable orbits are much rarer, or, as one would say more technically, in phase space have relatively small measure. This means that one is led to a *new concept of stability* in which the restriction applies only to the *majority* of certain orbits. Whether the relatively rarer unstable exceptional orbits actually exist is still an open problem."

Interestingly, the word "orbit" was used also by Terence Tao in the title of his article *Almost all orbits of the Collatz map attain almost bounded values* [30]. The suggested convergence of concepts is not complete, however. In Tao's theorem, the orbits are not conic sections or other continuous orbits as in astronomy, but orbits consisting of discrete integers in a discrete dynamical system.

7. The *p*-adic angle

Following some of the trains of thought mentioned above, we come to the question of whether or not an algorithm that can be *proved to work as intended*, delivered together with a very small, simple and transparent computer program or script that implements it (e.g., written in a simple 'commodity' language that supports symbolic manipulation, such as `Sage` or its predecessor `Macsyma`), might be acceptable as a valid proof.[9]

We return briefly to the simple induction proofs, or 'prooflets' [9], that showed with elementary arithmetic *how* the numbers belonging to certain classes of starting values of the form $a2^j + b$, with a and b constant, always dip 'below the fog' after the same (initially small) number of steps or moves.[10]

[9] Current opinions may vary. They include, for example, the statement that "any proof of the Collatz $3n + 1$ conjecture must have an infinite number of lines; therefore, no formal proof is possible" [15].

[10] Terras [31] derived recursion formulae to calculate the numbers of distinct classes of this type that can be found for each value of j, i.e., to calculate the fractions of $\mathbb{N}$ that could still harbor potential counterexamples surviving after step j of the recursive 'crowding out' of the perilous fraction by the tame.

Verifying prooflets is easy with `Sage`. Furthermore, unlike the equally easy eliminating of individual starting values that might serve as counterexamples, each prooflet now also eliminates, at once, an entire infinite set of starting values that might have harbored counterexamples. Conversely, `Sage` can be used to visualize and explore with examples *how* any integer starting value could be shown to be successful in leading to 1. An idea is to help a starting value find its own counterexample-free class $a2^j + b$, i.e., the j that indexes the counterexample-free component of the (infinitely long) 2-adic sequence that is naturally assigned to the starting value [28, Chapter 2]. A sample script is included in Appendix A.

The strategy seems to work well, and the basic reason is as follows. An expression of the form $ak+b$, with a and b integer constants and a *even*, lets us know if that entire expression is even or odd regardless of the value of the variable k. However, as soon as a becomes *odd*, we can no longer do this: we lose the handle on the information we need. If one no longer knows if an expression is even or odd, the Collatz map cannot be applied, and the Collatz run of the class cannot be predicted any further in this simple way (decision forks are needed). In our airplane induction proof metaphor, for the entire class $ak + b$ to be proven 'tame' (i.e., without Collatz counterexamples), we need to keep the parity information for the expression visible until we reach the first altitude value $ak + b$ that is lower than our (candidate) starting altitude.

Let us look at a concrete example. We choose the integer 3 and help it in a systematic, automatable way to find its counterexample-free class. (We already know that that class is 3 mod 16, i.e., the class of polynomials of the form $16k + 3$ with k a positive integer, but we pretend we do not.) The simplest possible candidate is 3 mod 4, i.e., the class $4k + 3$ with k a positive integer. We apply the usual prooflet *Ansatz* for this candidate: $4k + 3$ (odd) $\mapsto 12k + 10 \mapsto 6k + 5$ (odd) $\mapsto 18k + 16 \mapsto 9k + 8$. This is undecided, and in order to avoid decision forks we now simply choose 3 mod 8 (since 3 mod 4 did not work) as the next candidate. We do this by doubling the coefficient of k on the fly, tallying doublings, and continuing: $18k + 8 \mapsto 9k + 4$ which is again undecided, and a final coefficient doubling brings success, $18k + 4 \mapsto 9k + 2 < 16k + 3$. 3 mod 16 is indeed the correct answer.

Simple doublings of the coefficient ($2^j \mapsto 2^{j+1}$) correspond to shifts along the relevant infinite p-adic sequence, i.e., expansions of the substrate (much like 'plating out' of colonies in experimental microbiology). However, we have followed the rule that this emergency maneuver is allowed where and only where the unmodified Collatz process leads to a polynomial of undecidable parity.[11]

A scheme corresponding to the strategy just sketched is displayed in Figure 2.

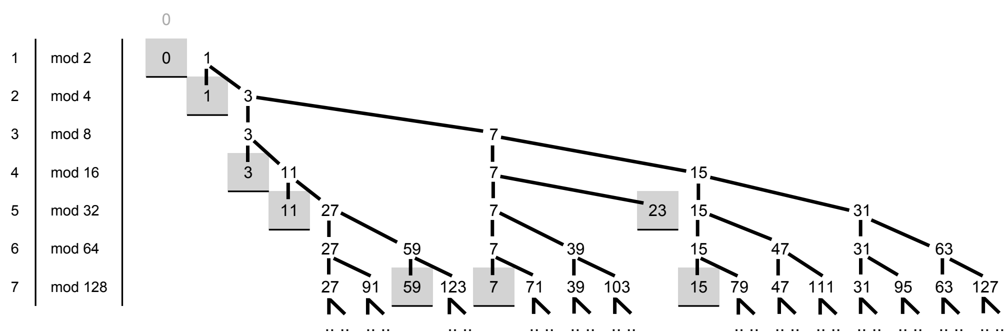

Figure 2: Tree-shaped scheme of initial segments of 2*-adic (dyadic) sequences in* $\mathbb{Z}_2$ *leading downwards to classes of Collatz starting values that cannot contain counterexamples to the Collatz conjecture (nodes represented in gray shading, with bar denoting termination of the branch). Every time a* 2*-adic integer that does not terminate is shifted onwards by one level (e.g.,* 3 *mod* 8 *to* 3 *mod* 16*), its branch bifurcates (e.g., because both* 3 *mod* 16 *and* 11 *mod* 16 *correspond to* 3 *mod* 8*). All of the verifiably counterexample-free classes up to mod* 128 *are (from top to bottom):* 0 *mod* 2*,* 1 *mod* 4*,* 3 *mod* 16*,* 11 *and* 23 *mod* 32*, and* 7*,* 15 *and* 59 *mod* 128*. (The image is the original work of the author.)*

This figure shows an iteratively bifurcating tree (or tree-shaped sieve), in which paths from the top downwards correspond to 2-adic (dyadic) integers, i.e., sequences of $\mathbb{Z}_2$ [28]. The nodes at which the paths are truncated (shaded grey and with a stopping symbol) are those representing classes that can be shown to be free of counterexamples to the Collatz conjecture. The path of the example above is thus $(1, 3, 3, 3)$, which is a beginning segment of the infinite sequence $(1, 3, 3, 3, 3, \ldots)$ representing the integer 3 in $\mathbb{Z}_2$.

[11] A similar change of scale or length when one reaches an impasse/bottleneck as determined by a previously fixed criterion is characteristic of some early 'biological' or 'evolutionary' optimization algorithms; see e.g. [25, p. 123] and [8].

Some integers (such as 5, which is congruent to 1 mod 4) do not appear in the tree because they belong to classes that were eliminated as counterexample-free higher up, i.e., at an earlier stage of the exclusion process. Other integers are not in the figure because their first appearance is in a mod 256 class or in another class further down the scheme. The scheme gives an idea of the concept of a number (such as 3) in search of its counterexample-free class (such as 3 mod 16), with perhaps similarity to the idea of Luigi Pirandello's theater play *Six Characters in Search of an Author*.

It might be tempting to again imagine some exhaustive procedure to sweep out remaining counterexamples from 'under the bed' in this way, and then try to show in some global way that it is somehow efficient — but the problems mentioned in earlier sections do not inspire confidence that it would be successful. Even if Lewis Carroll's "seven maids with seven mops" contemplated by the Walrus and the Carpenter could be replaced, for example, by seven computers doing symbolic computation or computer algebra with seven strategies, such an approach would again run the risk of playing with infinities and "only-almosts" without solving the Collatz problem in question.

A more fruitful way might be to attempt to formally show that *each* integer in $\mathbb{N}$ will eventually be able to find 'its' class of the form $ak+b$ that can in turn be shown to contain *no* counterexamples to the Collatz conjecture. In this sense one would not be at the mercy of having to wait for counterexample-free classes mod 2^j to trickle in as one increases j. Results such as the following by Serre [28] may help. Serre's result is stated in the context of p-adic integers, in which successive expansions of the embedding class correspond to a homomorphism, defined "in an obvious way", that generates a projective system indexed by the integers > 0:

Lemma [28, section II.2] *Let* $\ldots \to D_n \to D_{n-1} \to \ldots \to D_1$ *be a projective system. If the* D_n *are finite and nonempty, then the projective limit* D *of the projective system is nonempty.*

Furthermore, the decreasing family of finite nonempty subsets formed by the images of D_{n+p} in D_n, n fixed, is *stationary*, i.e., such an image is independent of p for p large enough [28].

A result of the desired type would then need to assure the existence of a non-vanishing limit product of an iterative procedure, or to assure that after a finite number of steps a hard and thenceforth stable property can be ensured, and should not fall into the rails of the "almost all" category of assertions mentioned above.

8. Listening to the Collatz map

The music of a Collatz run is worth a note.[12]

After converting values (altitudes) to frequencies (in Hertz) and multiplying them by a constant (i.e., transposing them) to bring them into an audible range, a Collatz run becomes a musical score. The short `R` script in Appendix B, using the `audio` package, offers one possible interpretation of that score for the starting value of 63, played *backwards* as a variation, so that the final descent of the Collatz trajectory becomes the opening bars of an ascent.

The music of this and other Collatz runs is constructed entirely from fifths (with a small correction) and octaves. Fifths (e.g., in C major, going from C up to G), thirds (e.g., C to E) and octaves (e.g., C to C) are perhaps the most familiar intervals or elements of musical composition and harmony (e.g., in the consonances of simple triads [27, 26]). Going down an octave halves the frequency, going up an octave plus a fifth triples it. A fourth can be implicit because it is the interval complementing a fifth to complete the octave; a (major or minor) third is also implicit as the simple (imagined or remembered) stepping-stone within a fifth. The uncertainty zone where music theory and perception psychology overlap was explored in detail by Schoenberg [27, 26].

The small correction mentioned above is because of the "+1" in $3n+1$. In the audible range played by instruments of an orchestra or a piano, this would typically correspond to tuning or tempering, i.e., a very small adjustment, although some of the lower intervals might sound as if the instrument were

[12] Although Conway [10] has referred to a variant of the Collatz map (a permutation) as 'amusical', we here address only the Collatz rule in its simplest '$3n+1$' form that we have used so far in this exposition, and see why it might be justified to describe it as musical.

slightly out of tune (flat or sharp).[13] However, the "+1" is obviously essential to create a musical score of any interest (or mathematical dynamics of any interest), and allows the cascades of drops by an octave to alternate with modulations to another key in accordance with one of the recommendations of traditional musical harmony (the fifth; [27, 26]). The balance between these two movements, the upward and the downward, and the uncertainty as to what the composer has been inspired to write in the next bars of the score, keeps the listener 'tuned' (as also the 'singularity' of the Collatz question has kept mathematicians mesmerized, for perhaps too long). The maintaining of this balance that inspires continuing interest may remind one, in the visual arts and architecture, of the effect on the psyche of the golden section or golden ratio, or its use in architecture in the form of the modulor [11].

Upon first listening to the rendering of a backwards Collatz run (Appendix B), I was instantly reminded of the mysterious orchestral prelude at the beginning of Richard Wagner's opera *Das Rheingold*. The opening bars, in E♭ major, begin with a very low E♭ for four bars, which is then joined by a slightly higher B♭, i.e., a fifth higher; both remain suspended (and creating suspense or anticipation in the listener) for the next twelve bars, followed by a two-bar ascent spanning first a fifth and then the first full octave from E♭ to E♭. Repeated ascents from the depths to heights are then developed in increasing detail, and with more rapid scales like rippling of the river joining in the last part, finally leading to up to the rising of the curtain and the first voice of a Rhine river maiden calling.[14]

[13] According to a `Wikipedia` entry [34], a (practical) tuning or tempering was proposed some years ago that is related to the Collatz map idea, but using "-1" instead of "$+1$"; apparently the author of the idea became interested in it also because he saw it as a possible interpretation of handwritten loops shown on the title page of Bach's first book of *The Well-Tempered Clavier* from 1722.

[14] Wagner wrote that he was inspired for this prelude in a state between sleeping and waking, while relaxing after a long walk near La Spezia, by the coast of the Mediterranean in Italy. Wagner later described the experience as in a "somnambulant state", with a "feeling as if I were sinking into strongly flowing water." The sound of the rushing water was soon transformed, in his perception, into the tones of E♭ major, and there were "melodic figurations with increasing movement, but what never changed was the pure triad of E♭ major, as if its constant presence wanted to give infinite meaning to the element into which I sank" [33].

9. From the Rhine to Everest: a binary code

Let us mention one more approach. We shift our main focus away from the driving map or *primum movens* that gives rise to Collatz trajectories, and instead explore a known, fundamental property of the Collatz map that we left unmentioned until now. The main idea and proofs were presented in the seminal first article [31] of Riho Terras, who referred to the property as "a remarkable periodicity phenomenon". It imposes a simple constraint on the local binary sequences of 'up' and 'down' directions of the movements that can occur along a Collatz trajectory or flight.

We again return to the Collatz plane metaphor, but for the first time we now use a variant of the basic Collatz map, mentioned in Section 2 (and illustrated in Figure 1A) and denoted by T as in Terras [31]:

$$T(n) = \begin{cases} (3n+1)/2 & \text{if } n \text{ is odd;} \\ n/2 & \text{otherwise.} \end{cases}$$

The new halving in the case of an odd n simply 'short-cuts' the trajectory after every odd number (Figure 1A), so that 'up' moves are no longer always trivially followed by a 'down' move. As a consequence, the number of steps, or changes of altitude, becomes shorter. For example, the flight starting at 77671 that we considered earlier, which actually reaches the highest altitude of any flight starting from initial heights between 1 and $100,000$ (Figure 1B), now 'lands' (reaches 1) after only 149 steps, moves, or changes of altitude (Figure 3), instead of 231 as before. It is to this shorter sequence of 'up' (`1`) and 'down' (`0`) moves that the periodicity theorem of Terras [31, Theorem 1.2] applies.

If we consider two boolean up/down sequences or 'bit strings' (also referred to as *encoding vectors* [31] or *parity vectors* [19, pp. 6−7]), of length k, and beginning at heights (vertical positions) n and m, then the periodicity theorem tells us that *they are equal if and only if* $n \equiv m \bmod 2^k$.

Flight 77671 presents one nice example of this, for $k = 11$. In Figure 3 it can be seen that the eleven-bit string `01110001011` occurs twice during the descent. The two occurrences of this bit string begin at heights $20,991,662$ and $22,190$, respectively. Thus, as the theorem predicts, $22190 \bmod 2^{11}$ and $20991662 \bmod 2^{11}$ are equal, namely 1710.

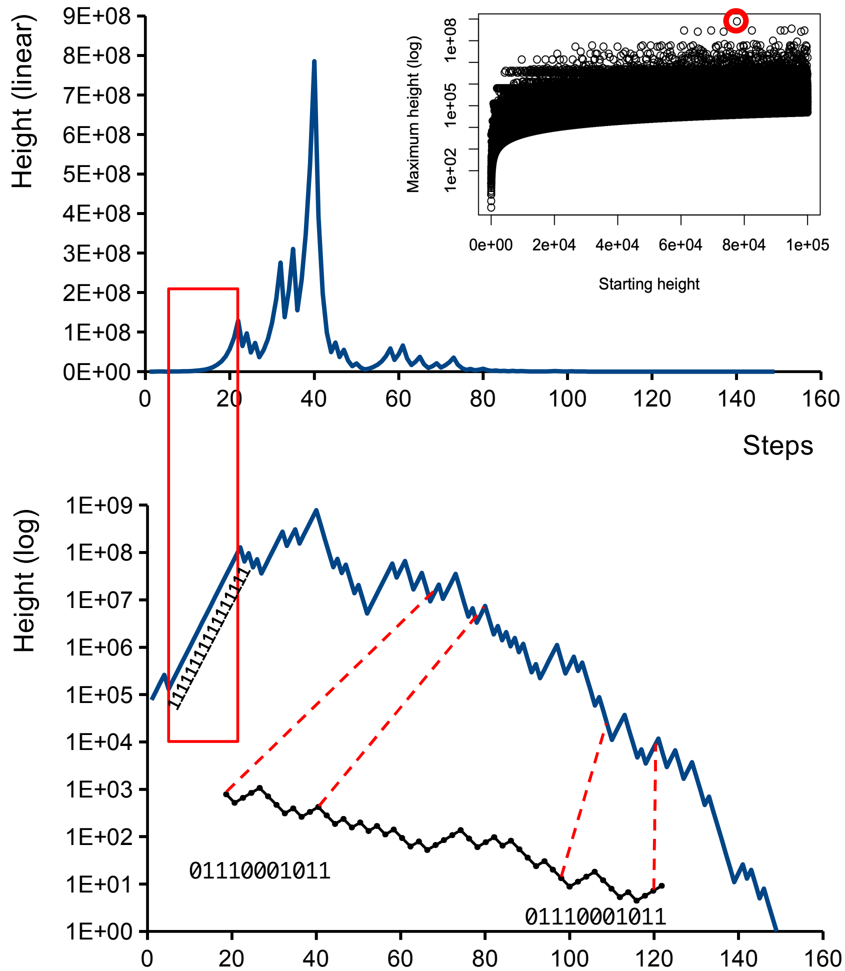

Figure 3: Plots and regions characterizing the Collatz flight starting from height 77671*. In contrast to Figure 1B, the alternative variant T of the Collatz map is used here (see text, and Figure 1A). Height versus step number (vertical position) is shown in the top (linear height scale) and bottom panels (logarithmic height scale). The flight starting at* 7761 *(circled) reaches the highest altitude of all flights having start heights between* 1 *and* 100,000 *(inset). Highlights boxed or shown with dashed lines are the uninterrupted seventeen-step ascent left of the summit (seventeen successive 'up' or* $(3n+1)/2$ *moves), and the existence of an eleven-step repeat in the descent. The binary code shown for these subsequences uses* `1` *for an 'up' move and* `0` *for a 'down' move. (The image is the original work of the author.)*

The plots (linear and logarithmic) in Figure 3 illustrate that although the periodicity constraint itself is remarkable, the resulting coherence along a single flight may be subtle rather than dramatic, at least for flights with starting heights in the beginning range $1 \leqslant n \leqslant 10^5$, where even high-soaring flights land after about one or two hundred steps. For example, in the descent of flight 77671 shown, the match of the two eleven-bit strings was the only match of its kind, all other matches being much shorter. Nevertheless, a corollary of the periodicity theorem [31, Corollary 1.3] assures us that for any given k the function that assigns to heights n their up/down k-bit strings is periodic with period 2^k, and assumes all of the possible 2^k bit strings of length k in each period.

In contrast to the apparently subtle effect on coherence along individual flights of the periodicity phenomenon, the effect of coherence across different flights is strong and clearly visible.

Stacking (left-aligning) of the `1`/`0` bitstrings that characterize successive Collatz flights, and truncating those bit strings after 1 is reached and the trivial final cycle `101010...` begins, reveals obvious and exact periodicities, as demonstrated by the periodicity corollary (here for starting values between 16 and 32, with spaces inserted to highlight periodicities on the left):

```
16:  0 0 0 0

17:  10 1 001000
18:  01 0 11101001000
19:  11 0 01101001000
20:  00 1 000

21:  10 0 000
22:  01 1 01001000
23:  11 1 00001000
24:  00 0 11000

25:  10 1 1001101001000
26:  01 0 01000
27:  11 0 1111101011011101111010011101101111110011110001010100010011100001000
28:  00 1 1101001000
```

```
29:   10 0 1101001000
30:   01 1 1100001000
31:   11 1 1101011011101111010011101101111110011110001010100010011100001000
32:   00 0 00
```

More generally, there is substantial coherence among flights because of the overlap or redundancy of Collatz flights that is seen when they are ordered by their starting height, i.e., flights from different starting heights are often long parts of some longer flight. For example, the two starting values 27 and 31 both have unusually long flights (for such low starting heights), but it is quickly seen (also in the bit string alignment above) that the flight from 31 is just part of the flight from 27:

27, 41, 62, 31, 47, 71, 107, 161, 242, 121, 182, 91, 137, 206, 103, 155, 233, 350, 175, 263, 395, 593, 890, 445, 668, 334, 167, 251, 377, 566, 283, 425, 638, 319, 479, 719, 1079, 1619, 2429, 3644, 1822, 911, 1367, 2051, 3077, 461, 2308, 1154, 577,866, 433, 650, 325, 488, 244, 122, 61, 92, 46, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1.

Such strong cross-coherence, which is not just some vague statistical tendency but is firmly (deterministically) anchored in the structure of the Collatz map, must have an implication for any counterexample to the Collatz conjecture: there cannot just be an individual counterexample or rebellious flight that breaks away from all of the others with which it shares subtrajectories (subflights) and periodicities. If there were to exist a counterexample flight that never 'lands', it would, by such coherence, force many other flights with which it shares stretches or periodicities to also never land. A strategy, reasoning via the absurd, might be to show that if there were a counterexample its collateral or coherence ramifications would have been noted by now.[15]

[15] The degree of redundancy indicated here is obviously reduced if one 'runs time backward' from 1, and considers what would be a tree (if the Collatz conjecture were true), growing 'leftward' from 1 and passing leftward through all even and odd positive integers that one knows would have led to 1 via iteration of the Collatz map (e.g., 3 and 10 both map directly to 5, which maps to 8 and thence to 4, 2, and 1). Such a time-reversed perspective (systematically revealing all the 'pre-images' or precursors of 1 and seeing if

What simple implication(s) of the periodicity constraint could help to solve the Collatz problem? The relatively rebellious, high-soaring flight 77671 might be naively expected to indicate one or two properties that any never-landing, runaway counterexample flight would need to have. In Figure 3, there is a relatively long, monotonous rise enduring for seventeen 'up' steps that is not interrupted by any 'down' step (`11111111111111111`, i.e., 1_{17}, in Figure 3). The seventeen-bit string starts at altitude $131,071$. The exceptional boost it provides allows the plane to rapidly gain altitude because (in this form of the Collatz map) it multiplies the altitude by more than $3/2$ at every step, without any backsliding. Although we found this monotonous seventeen-step rise by looking for the highest record among the Collatz flights for the first $100,000$ starting altitudes (Figure 3, inset), the periodicity corollary lets us find it analytically. Indeed, it tells us that the period that is relevant for such a seventeen-step rise is 2^{17}, which is 131072, so this instance, $2^{17} - 1$, is the lowest one possible, the next opportunity being at $2 \times 2^{17} - 1$. Applying the periodicity theorem to two copies of $\mathtt{1}_8$ with the additional constraint that they must be adjacent tells us that $\mathtt{1}_{16}$ occurs at heights $65,535$ and $131,071$: inspection of the two then shows that only the latter is extended to $\mathtt{1}_{17}$, which completes the analytical identification of this run.

Simple calculations along similar lines could then allow us to find starting heights at which all-`1` bit strings of any desired length m are found. In other words, for any desired finite length m (number of steps along the horizontal axis), no matter how long, we could calculate and locate corresponding starting heights n where the flight will be guaranteed to swiftly gain height during m steps. This is presumably a strengthening of the property (which it includes) of simply not landing in the next m steps. However, just as in the other approaches explored above, it is not obvious how one could harness this strategy to create a bridge to infinity, e.g., via a 'chain reaction' that, once

those integers might ultimately cover all of $\mathbb{N}$) was in essence addressed by Krasikov and Lagarias in 2003 [18]: the cardinality of the set of such precursor integers with values lower or equal to a threshold x was shown by the authors to be at least $x^{0.84}$. However, so far the 'time-reversed' perspective sketched here has not even facilitated a corresponding direct proof of a statement that "almost all" positive integers (in the measure-theoretic sense of a 'density' of 1) are precursors of 1 [3].

started, could continue without needing manual re-'stoking' of the reaction every time one desires a longer duration $m' > m$. And the slow progress one would make toward a zone where a Collatz counterexample might live, while tearing through recursively higher numbers that are soon staggering even by astronomical standards, and that soon leave naive help from commodity supercomputers far behind, is evident.

10. Conclusion

After this exposition, the reader may wish to consider whether or not the Collatz problem, or some of the research directions that tried to elucidate this problem, might be a futile pursuit, or even a time sink.

The attraction of the Collatz conjecture is partly motivated by a wish to 'solve' it. However, decade-long efforts that have still not reached the desired proof make one wonder if this apparently innocent problem might be pointing to a singularity in our mathematical reasoning or in the structure of arithmetic or elementary algebra, and perhaps coming closer to primary school curricula than we had ever imagined.

There may be hopes that to find a definitive proof one just needs to continue along the same direction of asymptotic reasoning that led to the impressive body of results we now have. However, some of the most important of those results apply only to "almost all" possibilities.

A parallel case may be the KAM problem(s) mentioned above. The statements of theorems published in the Collatz and KAM problem domains share the use of "almost all", and of "orbits", and other concepts from the study of dynamical systems. Interestingly, shared elements of their proofs may also indicate some deeper links. For example, original papers and monographs that explain or expand upon elements of the KAM proofs clearly reveal the important role of continued fractions and Diophantine approximations in obtaining key bounds for those proofs (e.g., [4], [29, pages 109-112] and [5, pages 112-117]). The underlying question was how well irrational numbers can be approximated by rational numbers, in a context of small denominators that could destabilize celestial systems. Correspondingly, the 1978 article by Crandall [12], which explored conditions for non-trivial cycles of the Collatz process, again made strong use of continued fractions and Diophantine equations. There, the context was to see if, as the Collatz conjecture would imply, "powers of two and three tend to be poor approximations of each other", i.e., to see how difficult it is to approximate $\log_2 3$ with rational numbers (see [12, page 1288] and [19, page 14]). Thus, in this sense it can be said that the stability of model celestial systems and the 'stability' of the Collatz conjecture both depend upon the existence of a sufficiently clear and robust difference or contrast that is maintained between two systems of numbers. Indeed, related methods have been employed to demonstrate the existence of those respective, parallel contrasts.

If a new strategy for either the Collatz or the solar system problem were to lead to a proof that "almost all" is all one needs (e.g., because the coherence of a system would not allow any set of apparent, potential counterexamples of measure zero to furnish even one viable counterexample for the question posed, even as a thought experiment), then one might imagine implications also for other systems that have remained earmarked with the "almost all" tag.

On a more general note, as Jürgen Moser concluded already in 1975 [23]:

> Is the solar system stable? Properly speaking, the answer is still unknown, and yet this question has led to very deep results which probably are more important than the answer to the original question.

Note added in proof

It should be mentioned that even if there exists a proof that there is a counterexample(s) to a conjecture, thus formally refuting the conjecture, it can still be very difficult to find and specify even one actual counterexample. A well-known instance of such a situation is given by a proof of J. E. Littlewood in 1914 ("Sur la distribution des nombres premiers", *Comptes Rendus* **158**:1869–1872; see also the account and comments in G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by his Life and Work*, Cambridge University Press, 1940, page 17), which refuted a long-believed inequality concerning the distribution of prime numbers $\pi(x)$ and its relation to the logarithmic integral function $\mathrm{li}(x)$, namely

$$\pi(x) < \mathrm{li}(x) ,$$

where $\mathrm{li}(x) = \int_0^x \frac{\mathrm{d}t}{\ln t}$, in the sense of a Cauchy principal value to avoid the singularity at 1. Incidentally, the proof used Kronecker's theorem, which concerns Diophantine approximations (an area that is entered also by Collatz- and KAM-related proofs, as mentioned above in this exposition). To this day, no concrete counterexample to the originally believed inequality has been found, even though the proof establishes that there must exist an infinite number of such counterexamples that reverse the sign of the inequality above. Indeed, even finding an good upper bound for a lowest counterexample (i.e., for a smallest value of x for which the above inequality fails) has not been easy: the first published bounds by Skewes (in 1933 and 1955) were of the form

$$10^{10^{10^c}} ,$$

where the constant c has the value 34 if one can assume that the famous Riemann conjecture is true, and a higher value if one cannot. Recently published improvements on these upper bounds are still astronomically high.

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A. Generating elementary proofs for integer classes in `Sage`

Proofs involving elementary algebra and arithmetic can be generated by symbolic calculation (computer algebra). The following short program/script in `Sage` (`Python`) investigates if the induction step of elementary induction proofs ('prooflets') is in agreement with the Collatz conjecture, for a given class of integers mod 2, 4, 8, and 16 (or higher, if one augments the `range`).

```
k = var('k')
assume(k>0)
assume(k,'integer')
for j in range(0,5):
    print("------> Results for ",2**j,":")
```

```
for c in range (1,2**j):
    if c%2==0:
        continue
    m=((2**j)*k)+c
    a=m
    for i in range(1,100):
        ap=a.polynomial(GF(2))
        if ap==0:
            a=a/2
        elif ap==1:
            a=3*a+1
        else:
            print("decision fork - ",c,":",i,a)
            break
        d=a-m
        if not (d.is_positive()):
            print(c,":",i,d)
            break
```

B. Playing Collatz trajectories

The following script in `R` generates the values of the Collatz map for the starting value 63, and then plays them backwards, in an audible frequency range.

```
d1<-Sys.time(); library(audio);
myplay <- function(x) {play(sin(1:10000/20),rate=110*x);wait(0.7)};
n<-63;v<-63;
while (n>1) {if (n%%2==0) {n<-n/2} else {n<-3*n+1}; v[length(v)+1]<-n}
v<-rev(v); sapply(v,myplay); d2<-Sys.time();d2-d1
```