1-1-2006

Combinatorial Interpretations of Spanning Tree Identities

Arthur T. Benjamin
Harvey Mudd College

Carl R. Yerger
Davidson College

Recommended Citation

A. Benjamin, C. Yerger, Combinatorial Interpretations of Spanning Tree Identities, Bulletin of the Institute for Combinatorics and its Applications, Vol. 47, 37-42, 2006.

Combinatorial Interpretations of Spanning Tree Identities

Arthur T. Benjamin and Carl R. Yerger

November 14, 2004

Abstract

We present a combinatorial proof that the wheel graph W_n has $L_{2n} - 2$ spanning trees, where L_n is the nth Lucas number, and that the number of spanning trees of a related graph is a Fibonacci number. Our proofs avoid the use of induction, determinants, or the matrix tree theorem.

1 Introduction

Let G be a graph and let $\tau(G)$ be the number of spanning trees of G. In this paper we will present combinatorial proofs that determine $\tau(G)$ for the wheel graph and a related auxiliary graph. Two simple bijections will provide a direct explanation as to why the number of spanning trees for these graphs are Fibonacci and Lucas numbers.

Definition 1.1. *For $n \geq 1$, The wheel graph W_n has $n+1$ vertices, consisting of a cycle of n outer vertices, labeled $w_1, \ldots, w_n$, and a "hub" center vertex, labeled w_0, that is adjacent to all the n outer vertices.*

For example, W_8 is presented in Figure 1. The *Lucas numbers* are recursively defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Theorem 1.2. *For $n \geq 1$, $\tau(W_n) = L_{2n} - 2$.*

This result was first proved by Sedlacek in [5] and later by Myers in [3]. As part of Myers' proof, he employs an auxiliary graph, denoted by A_n, that is similar to the wheel graph and presented in Figure 2. For $n \geq 2$, A_n has $n+1$ vertices and $2n+1$ edges, consisting of a path of n outer vertices, labeled $a_1, \ldots, a_n$, and a hub vertex a_0 that is adjacent to all n outer vertices. In addition, a_0 has an extra edge connecting to a_1 and an extra edge connecting to a_n. We label the two edges from a_0 to a_1 as red and blue, and do the same for the edges from a_0 to a_n. Let f_n denote the nth Fibonacci number with initial conditions $f_1 = 1$, and $f_2 = 2$.

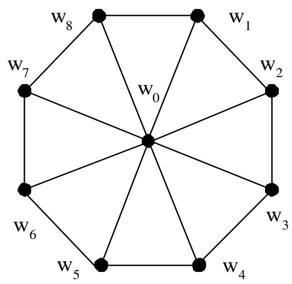

Figure 1: The wheel graph W_8.

Theorem 1.3. *For $n \geq 2$, $\tau(A_n) = f_{2n+1}$.*

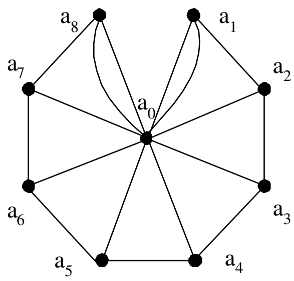

Figure 2: The auxiliary graph A_8.

One way to determine $\tau(A_n)$, as shown by Koshy [2], is to apply the matrix tree theorem [6], first proved by Kirchhoff, by computing the determinant of the n-by-n tridiagonal matrix

$$A_n = \begin{bmatrix} 3 & -1 & 0 & \dots & & 0 \\ -1 & 3 & -1 & \dots & & 0 \\ 0 & -1 & 3 & \dots & & 0 \\ & & & \vdots & & -1 \\ 0 & 0 & 0 & \dots & -1 & 3 \end{bmatrix}.$$

Expanding along the first row, and proceeding inductively, it follows that $\tau(A_n) = |A_n| = 3|A_{n-1}| - |A_{n-2}| = 3f_{2n-1} - f_{2n-3} = f_{2n+1}$.

The matrix tree theorem also indicates that $\tau(W_n)$ equals the determinant of the following matrix n-by-n circulant matrix

$$B_n = \begin{bmatrix} 3 & -1 & 0 & \dots & & -1 \\ -1 & 3 & -1 & \dots & & 0 \\ 0 & -1 & 3 & \dots & & 0 \\ & & & \vdots & & -1 \\ -1 & 0 & 0 & \dots & -1 & 3 \end{bmatrix}.$$

Expanding $|B_n|$ along its first row, we obtain $|A_n|$ as one of its subdeterminants. Proceeding by induction and with a bit more computation (see [2]), $\tau(W_n) = L_{2n} - 2$ can then be obtained. In the next two sections, we give combinatorial proofs of Theorems 1.2 and 1.3 that are much more direct.

2 Combinatorial Proof of $\tau(W_n) = L_{2n} - 2$

The Lucas number L_n counts the ways to tile a bracelet of length n and width 1 using 1×1 squares and 1×2 dominoes [1]. Equivalently, L_n is the number of matchings in the cycle graph C_n. Observe that *even* cycle graphs C_{2n} have exactly two perfect matchings and thus $L_{2n} - 2$ *imperfect* matchings, such as the one in Figure 3.

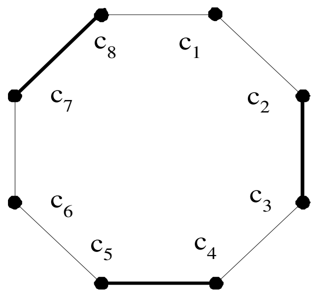

Figure 3: An imperfect matching of C_8.

Given an imperfect matching M (a subgraph of C_{2n} where every vertex c_i has degree 0 or 1), we construct a subgraph T_M of W_n as follows:

1. For $1 \le i \le n$, an edge exists from w_0 to w_i if and only if c_{2i-1} has degree 0 in M.
2. For $1 \le i \le n$, an edge exists from w_i to w_{i+1} (where w_{n+1} is identified with w_1) if and only if c_{2i} has degree 1 in M.

The bijection is illustrated in Figure 4.

To see that T_M is a spanning tree of W_n, suppose that M has x vertices of degree 1 and y vertices of degree 0; thus $x + y = 2n$. Observe that vertices of degree 1 come in adjacent pairs and that if

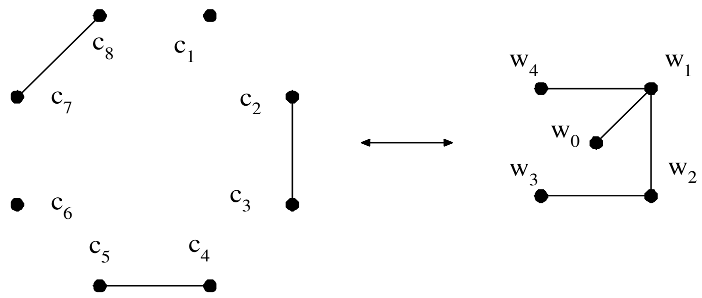

Figure 4: An example of the bijection for $n = 4$.

v_j has degree 0, then the next vertex of degree 0, clockwise from v_j, must be v_k, where k and j have opposite parity. Thus, T_M will use exactly $x/2 + y/2 = n$ edges of W_n. Since W_n has $n+1$ vertices, we need only show that T_M has no cycles. Suppose, to the contrary, that T_M has a cycle C. Then C, denoted by $w_0 w_i w_{i+1} \cdots w_k w_0$, must use two edges adjacent to w_0 (otherwise M would be a *perfect* matching). Thus, c_{2i-1} and c_{2k-1} have degree 0 in M and hence some vertex c_{2j} must also have degree 0 where c_{2j} is strictly between c_{2i-1} and c_{2k-1} on C. But when c_{2j} has degree 0, there is no edge in T_M from w_j to w_{j+1}, yielding a contradiction. Hence no cycle C exists on T_M and so T_M is a tree.

The process is reversible since a spanning tree T of W_n completely determines the degree sequence $d_1, d_2, d_3, \ldots d_{2k}$ where $d_i \in \{0, 1\}$ is the degree of the vertex c_k in a subgraph of C_{2n}. Since w_0 is not an isolated vertex of T, not all d_k are equal to 1. We show that C_{2n} has a unique matching that satisfies this degree sequence by showing that every string of 1s has even length; i.e., if $d_k = 0$, $d_{k+1} = d_{k+2} = \cdots = d_{k+j} = 1$, and $d_{k+j+1} = 0$, then j must be even. For if $k = 2i-1$ is odd and j is odd then the tree T would contain a cycle $w_0 w_i w_{i+1} \cdots w_{i+(j+1)/2} w_0$. If $k = 2i$ is even and j is odd, then T is not connected since the path $w_{i+1} w_{i+2} \cdots w_{i+(j+1)/2}$ is disconnected from the rest of T.

3 Combinatorial Proof of $\tau(A_n) = f_{2n+1}$

The Fibonacci number f_n counts the ways to tile a $1 \times n$ rectangle using 1×1 squares and 1×2 dominoes [1]. Alternatively, f_n counts the matchings of P_n, the path graph on n vertices, whose vertices are consecutively denoted $p_1, \ldots, p_n$. Let M be an arbitrary matching of P_{2n+1}. We construct a subgraph T_M of A_n as follows:

1. For $1 \le i \le n$, T_M has an edge from from a_0 to a_i if and only if

vertex p_{2i} has degree 0 in M. (For $i = 1$ or n, then this refers to the red edge.)

2. For $0 \le i \le n-1$, T_m has an edge from a_i to a_{i+1} if and only if p_{2i+1} has degree 1 in M. (For $i = 0$, this refers to the blue edge.)

3. T_M has a blue edge from a_0 to a_n if and only if p_{2n+1} has degree 1 in M.

Notice that these rules make it impossible for T_M to contain two edges from a_0 to a_1 or two edges from a_0 to a_n. The bijection is illustrated in Figure 5

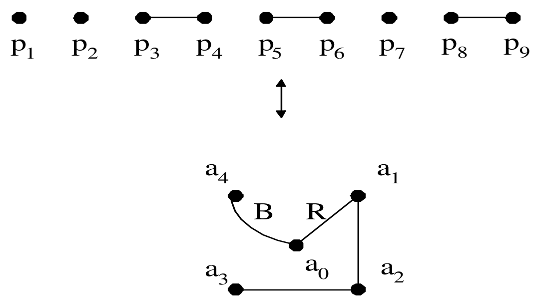

Figure 5: An example of the bijection for $n = 4$.

Like before, we prove that T_M is a spanning tree of A_n. Suppose that M has a and b vertices of degree 0 and 1 respectively; thus $a + b = 2n + 1$. Reasoning as before, M has $b/2$ odd vertices of degree 1 and $(a-1)/2$ even vertices of degree 0. Thus, T_M has $(a-1)/2 + b/2 = n$ edges. Suppose for the sake of contradiction, that T_M has a cycle C. Then C, denoted by $a_0 a_i a_{i+1} \cdots a_k a_0$, must use two edges adjacent to a_0. Thus p_{2i} and p_{2k} have degree 0 in M and hence some vertex p_{2j+1} must also have degree 0 where p_{2j+1} is strictly between p_{2i} and p_{2k} on C. But since p_{2j+1} has degree 0, there is no edge in T_M from a_j to a_{j+1}, a contradiction. Hence no cycle C exists on T_M and so T_M is a tree.

The process is also reversible since a spanning tree T of A_n completely determines the degree $d_k \in \{0, 1\}$ of each vertex p_k in a subgraph of P_{2n+1}. Again, not all d_k are equal to 1, since T would contain the cycle $a_0 a_1 \cdots a_n a_0$. To prove that P_{2n+1} has a unique matching that satisfies this degree sequence, suppose that for some k, j, $d_k = 0$, $d_{k+1} = d_{k+2} = \cdots = d_{k+j} = 1$, and $d_{k+j+1} = 0$. As before, if $k = 2i$ is even and j is odd, then the tree T contains the cycle $a_0 a_i a_{i+1} \cdots a_{i+(j+1)/2} a_0$. If $k = 2i - 1$ is odd and j is odd, then T is not connected since the path $a_i a_{i+1} \cdots a_{i+(j-1)/2}$ is disconnected from the rest of T. Thus j must be even, and the matching generating T is unique.

References

[1] A. T. Benjamin and J. J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America (2003).

[2] T. Koshy, *Fibonacci and Lucas Numbers With Applications*, Wiley and Sons (2001).

[3] B. R. Myers, Number of Spanning Trees in a Wheel, *IEEE Transactions on Circuit Theory*, CT-18 (1971) 280-282.

[4] K. R. Rebman, The Sequence: 1 5 16 45 121 320 ... in Combinatorics, *The Fibonacci Quarterly*, 13:1 (1975) 51-55.

[5] J. Sedlacek, On the Skeletons of a Graph or Digraph, *Proceedings of the Calgary International Conference of Combinatorial Structures and Their Applications*, Gordon and Breach (1970) 387-391.

[6] D. West, *Introduction to Graph Theory, Second Edition*, Prentice Hall (2002).

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, benjamin@hmc.edu, cyerger@hmc.edu