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Mathematical Models and Pedagogy of Marxist Political Economy

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Synopsis

How can we teach people about the economics of labor and exploitation in mathematics courses? We define a mathematical model for describing the relationships embodied by commodities and labor. We then use this model to illustrate the exploitative nature of profit and the tendency for catastrophic chain-reactions that lead to market crashes. Lastly, we discuss applications to pedagogy in mathematics courses using a simplified version of the model.

1. Introduction

A typical economics problem found in a math course concerns a business which produces a given product. The cost for the business to produce $x$ units of that product given by is a function $c(x)$, and the product is sold at a unit price given by a function $q(x)$ in terms of the number of units sold. The business’s revenue is then $r(x) = x \cdot q(x)$ and its profit is $p(x) = r(x) - c(x) = x \cdot q(x) - c(x)$. Often students are asked to determine how many units must be produced to yield a profit by solving $p(x) > 0$, or they are asked to find values for $x$ which maximize profit.

Such a problem is not particularly relevant to most students’ lives, as most people are workers rather than business owners. The few problems which deal directly with a worker’s compensation often concern salespeople earning commission, and finance problems involving investments earning interest are equally disconnected from the typical student who may not have disposable income to invest after graduating into our current economy.
Of course there is nothing wrong with modeling or assigning problems such as these, but how can mathematics educators supplement these concepts with additional content more focused on the struggles of the typical worker? Here we enter the realm of critical mathematics pedagogy, which is deeply influenced by the work of Paulo Freire and in particular his seminal 1970 work on critical pedagogy, *Pedagogy of the Oppressed* [3].

Interested readers are strongly encouraged to read Freire themselves, but for the purposes of this article we summarize Freire’s critical pedagogy as follows: Knowledge is neither static nor independent from people, but rather it is continuously created and re-created as people reflect on and act upon the world. Consequently no knowledge is “finished”, and furthermore this assertion implies there is no meaningful distinction between subjective and objective knowledge, so it is necessary to critically analyze who may benefit when something is presented as a “neutral” or “objective” fact. Finally, reflection which is not accompanied by action upon the world (*praxis*) produces nothing more than empty, useless knowledge, which is why Freire emphasizes the paramount importance of education which promotes critical consciousness and can lead to concrete actions for social change.

Critical mathematics pedagogy was first developed in the 1980s by Marilyn Frankenstein as a means of applying Freire’s theories to adult mathematics education [2], but “critical mathematics pedagogy is most often framed as teaching mathematics for social justice” [6]. A number of frameworks for teaching mathematics for social justice have been developed by various researchers, but for this article we focus on the Eric Gutstein’s framework [4, pages 23–31] which considers two dialectically related goals: social justice pedagogical goals and mathematics pedagogical goals. Social justice pedagogical goals include using mathematics to “read the world” (such as using mathematics to understand relations of power, inequality, and discrimination), using mathematics to “write the world” (using mathematics to enact change in the world), and developing positive cultural and societal identities (grounding mathematics education in students’ real lives). Mathematical pedagogical goals include reading the mathematical word (developing concrete mathematical skills and the means to apply them), succeeding academically in the traditional sense, and changing one’s orientation to mathematics (viewing mathematics not as a static thing to memorize but as a continually-evolving, powerful set of analytical tools for understanding the world).
Much has been written on critical mathematics pedagogy as applied to statistical education, in particular how statistics can be used to illuminate social issues and how statistical literacy can help people avoid being misled by statistics in the media. Others have studied how math anxiety serves to keep oppressed people from acquiring the mathematical skills necessary for this sort of critical thinking. Despite the Marxist origins of critical pedagogy, however, little (if any) work has been done on using Marxist political economy in teaching mathematics for social justice. In this article we present some basic concepts from Marxist political economy which can form a basis for teaching economic and financial justice in mathematics, as well as a novel mathematical model for describing economic scenarios from this perspective. Finally, we present some proposed activities suitable for a number of courses which use a simplified version of this model to allow students to investigate these concepts without detracting from more usual economics applications found in math courses.

Acknowledgements

I would like to thank Wong Tian An for encouraging me to write this paper and David F. Ruccio and [omitted] for their helpful feedback, as well as everyone who took the time to look at my course activities in order to help me reduce the necessary mathematical background. Finally, I would especially like to thank the anonymous Starbucks Workers United union organizer who provided the data in §4.2.

2. Production, labor, and exchange

Here, we present some basic concepts from Marxist political economy. For more on Marxist political economy, see [1, 5]. The content of this section along with the conclusions made in §4 are merely a restatement of the basic principles of Marxist economics as presented in these works, which themselves are based on the work of Marx and Engels.

For tens of thousands of years, humans have performed labor to produce goods from raw materials. We labored to make spears, we labored to hunt and kill animals, and then we labored to prepare and cook the meat to make food, to prepare the animal hides and pelts into clothing, and to carve bone into combs and other tools. We labored to weave grasses into baskets so that we could more efficiently labor at gathering the products of nature and thereby turn those products into goods.
Here an important distinction must be made that the products of nature are not considered goods until labor of some manner is performed to obtain them. This is because all goods have a use-value, that is the manner in which that good satisfies some human want or need. An animal in the wild satisfies no wants or needs until it is killed or captured for use by humans, and wild fruit growing in a bush satisfies no wants or needs until it is picked.

It was during this period of simple communism that in addition to producing goods for use by oneself, one’s family, or one’s tribe that humans began producing goods for exchange, also known as commodities. Now the question arises: what constitutes a “fair” exchange, i.e., why does some quantity of one commodity exchange for some quantity of another commodity?

For example, after killing some animals one might prepare the hides for use and then trade some of them in the village for more arrows so that they could hunt some more animals. This process recognized that the two villagers were trading finished commodities which represented roughly the same amount of labor-time used in their production. We know this is the case because in such a scenario the first person wouldn’t want to trade their hides for some feathers, flint, and sticks which would require them to take time out of their other labor to make these raw materials into arrows. To the hunter who trades hides, the arrows have use-value while the raw materials do not. In most texts on Marxist political economy this process is described as the simple commodity exchange $c \rightarrow c'$: commodities ($c$) are exchanged for other commodities ($c'$), and in order for this exchange to take place $c'$ must have some use-value for the person with $c$ and vice-versa.

But what if the commodity one wants isn’t available at the time when they go to the market, or the person with that commodity doesn’t want what the first person has, i.e., that person has no use-value for the commodity the first person has? Instead of simply exchanging one commodity for another, another commodity was used as an exchange medium. Early exchange mediums were often livestock: one exchanges part of their wheat harvest for some goats which could later be exchanged for a new iron plow. The problem, however, with using livestock as the exchange medium is that livestock can’t be divided into smaller units (a whole cow that can produce milk and reproduce to create more cows is worth more than two half cows which can only be used for meat and leather) and they don’t retain their exchange-value over a long period of time.
The solution to this problem came with the invention of money. Early money as an exchange medium was typically precious metals which could be divided as necessary, and the high value of gold and silver comes from the considerable amount of labor and skill needed to locate the ore, mine the ore, extract the metal, and refine it to a certain level of purity. In other societies with less advanced metallurgical techniques, various rare gems and stones played a similar role. Now the process could be described as the monetary exchange \( c \rightarrow m \rightarrow c' \): commodities are sold for money \( m \) to purchase other commodities. In most modern economies, precious metals are no longer used for everyday exchange, but token or fiat money currencies in the form of (non-precious) metal coins, paper bills, and electronic account balances play the same role as an exchange medium between commodities.

3. A mathematical model for production, labor, and exchange

We now describe a mathematical model that we will then use to describe economic scenarios that is inspired by the context provided in the previous section.

3.1. Quantified commodities

**Definition 1** (Unital commodity). A unital commodity is a commodity denominated in an appropriate unit for that commodity. Let \( C_1 \) denote the set of all unital commodities including currencies and (abstract) labor-power.\(^1\)

For example, “eggs”, “dozen eggs”, “pints beer”, “kegs beer”, “lbs rice”, and “kg rice” are all considered distinct unital commodities. The units of (abstract) labor-power will generally be given as hours and the units of currencies will be the currency itself.

**Definition 2** (Quantified commodity). A quantified commodity \( q \cdot c \) or simply \( qc \) consists of a unital commodity \( c \in C_1 \) and a quantity \( q \in \mathbb{R} \).

In most scenarios the quantity of a commodity will be non-negative, but we allow for negative values in the case of monetary deficits, in particular. For example, we could quantify 2·(pints beer) and 5·(USD) or simply 2 pints beer and 5 USD.

\(^1\) See §3.3 for the distinction between labor and labor-power.
Definition 3 (Quantity space). The quantity space of a unital commodity \( c \in C_1 \) consists of all possible quantities of \( c \) and will be denoted \( \mathbb{R}c = \{qc : q \in \mathbb{R}\} \). Given a unital commodity \( c \in C_1 \) we define the operations of addition and scalar multiplication on the quantity space \( \mathbb{R}c \) so that \( pc + qc = (p + q)c \) and \( a \cdot qc = (aq)c \) for any \( a, p, q \in \mathbb{R} \).

For example,

\[
\text{2 bushels corn} + \text{5 bushels corn} = \text{7 bushels corn},
\]

\[
3 \cdot (\text{2 bushels corn}) = \text{6 bushels corn}.
\]

In this manner, \( \mathbb{R}c \) forms a 1-dimensional vector space with basis \( \{c\} \).

We can use quantification to express equivalences between quantified commodities of the same type expressed in different units.

Definition 4 (Unital equivalence). If \( c_1, c_2 \in C_1 \) are commodities of same type denominated in different units and \( q_1c_1 \) and \( q_2c_2 \) express the same real-world quantity of that commodity type, we denote this unital equivalence by \( q_1c_1 \sim q_2c_2 \).

For example, 1 dozen eggs \( \sim \) 12 eggs.

Definition 5 (Commodity space, commodity vector). The commodity space \( C \) as the set of all (finite) formal sums of quantified commodities, i.e., \( C = \bigoplus_{c \in C_1} \mathbb{R}c \) is a vector space with basis \( \{c\}_{c \in C_1} \). Elements of \( C \) are called commodity vectors, and an arbitrary commodity vector

\[
\bar{v} = q_1c_1 + q_2c_2 + \cdots + q_nc_n \in C
\]

should be thought of as a set consisting of the commodities \( c_1, c_2, \ldots, c_n \) with the given quantities \( q_1, q_2, \ldots, q_n \).

For example,

\[
\text{2 loaves bread} + \text{1 dozen eggs} + \text{6 pints beer}
\]

could be thought of as a “grocery bag” containing two loaves of bread, one dozen eggs, and a six-pack of beer in pint cans. Because \( C \) is a vector space, it also has the operations of addition and scalar multiplication on commodity vectors which satisfy all the usual vector space axioms.
Definition 6 (Reduced commodity space, reduced commodity vector). The reduced commodity space $\bar{C}$ as the set of all (finite) formal sums of quantified commodities up to unital equivalence, i.e., $\bar{C} = \left( \bigoplus_{c \in C_1} \mathbb{R}c \right) / \sim$ is a vector space with basis $\{c\}_{c \in C_1}$ modulo unital equivalences. Because we have factored out unital equivalence, if $q_1c_1 \sim q_2c_2$ is a unital equivalence then we can simply say that $q_1c_1 = q_2c_2$ in $\bar{C}$. Elements of $\bar{C}$ are called reduced commodity vectors.

For example,

$1$ dozen eggs + $6$ pints beer = $12$ eggs + $96$ oz beer

as reduced commodity vectors in $\bar{C}$.

Definition 7 (Quantity map). For each unital commodity $c \in C_1$, we define the quantity map $q_c : C \rightarrow \mathbb{R}$ as the linear projection of $C$ onto $\mathbb{R}$ via $\mathbb{R}c$, i.e., $q_c(qc) = q$ and $q_c(qc') = 0$ if $c' \neq c$. We similarly define quantity maps $q_c : \bar{C} \rightarrow \mathbb{R}$ for reduced commodity vectors.

For example,

\begin{align*}
q_{\text{loaves bread}}(2 \text{ loaves bread} + 1 \text{ dozen eggs} + 6 \text{ pints beer}) &= 2, \\
q_{\text{dozen eggs}}(2 \text{ loaves bread} + 1 \text{ dozen eggs} + 6 \text{ pints beer}) &= 1, \\
q_{\text{pints beer}}(2 \text{ loaves bread} + 1 \text{ dozen eggs} + 6 \text{ pints beer}) &= 6, \\
q_{\text{lbs rice}}(2 \text{ loaves bread} + 1 \text{ dozen eggs} + 6 \text{ pints beer}) &= 0.
\end{align*}

Considered as a reduced commodity vector, we also have

\begin{align*}
q_{\text{eggs}}(2 \text{ loaves bread} + 1 \text{ dozen eggs} + 6 \text{ pints beer}) &= 12.
\end{align*}

Definition 8 (Exchange-value of a quantified commodity, base currency). For each unital commodity $c \in C_1$ we denote the exchange-value of $c$ by $|c| \in \mathbb{R}$. If $qc \in \mathbb{R}c$ is a quantified commodity then we define its exchange-value to be $|qc| = q|c|$. In all situations being considered we will express all exchange-values in terms of a chosen base currency and denote the monetary value $m$ in base currency $b$ as the quantified commodity $[m] = mb$ so that $m = ||m|| = |mb| = m|b|$.

For example, if we choose US dollars as the base currency then $[1] = 1$ USD so that

\begin{align*}
m = ||m|| = |m \text{ USD}| = m|1 \text{ USD}|.
\end{align*}
and the exchange rates to British Pounds, Euros, and Chinese Yuan could be expressed as

\[ 1 = |[1]| = |1 \text{ USD}| = |.94 \text{ GBP}| = |1.04 \text{ EUR}| = |7.18 \text{ CNY}|. \]

Similarly, if again we choose USD as the base currency then we could express

\[ |1 \text{ barrel Brent crude oil}| = 85.06, \]

i.e., one barrel of Brent crude oil has an exchange-value of 85.06 USD.

**Definition 9** *(Exchange-value of a commodity vector)*. *More generally if \( \bar{v} = q_1 c_1 + q_2 c_2 + \cdots + q_n c_n \in \mathcal{C} \) is a commodity vector we define its exchange-value to be*

\[
|\bar{v}| = |q_1 c_1 + q_2 c_2 + \cdots + q_n c_n| = |q_1 c_1| + |q_2 c_2| + \cdots + |q_n c_n| = q_1 |c_1| + q_2 |c_2| + \cdots + q_n |c_n|,
\]

*i.e., we define exchange-value as a linear map \(| \cdot | : \mathcal{C} \to \mathbb{R}|.*

For example, with base currency USD we have that

\[
|2 \text{ barrels Brent crude oil} + 3.4 \text{ oz t gold}| = 2 \cdot |1 \text{ barrel Brent crude oil}| + 3.4 \cdot |1 \text{ oz t gold}| = 2 \cdot 85.06 + 3.4 \cdot 1630.50 = 5713.82,
\]

i.e., the total exchange-value of two barrels of Brent crude oil and 3.4 troy ounces of gold is 5713.82 USD.

It is more difficult to establish the exchange-values of reduced commodity vectors, because although we can say 2 socks = 1 pair socks in \( \mathcal{C} \), we wouldn’t think of a single sock as having half the exchange-value of a pair of socks or really any exchange-value at all because socks are not typically sold individually.\(^3\)

\(^2\) All currency exchange rates and commodity prices in this subsection are based on data from September 27, 2022.

\(^3\) My spouse’s grandmother once told a story about growing up poor in Mexico and finding a single boot which she sold to a man with one leg. For the man with one leg, a single boot had more use-value than a pair of boots.
3.2. Economic entities and exchange

**Definition 10** (Economic entity, entity-interval). An economic entity is anything which can possess or facilitate the exchange of commodities including people, businesses, banks, and markets. Although the products of nature are not considered commodities until labor is performed to transform them into commodities, nature will also be considered an economic entity possessing potential commodities because products of nature contain use-values. An economic entity is described by a map $E : \mathbb{R} \to C$ and the quantified commodity vector $E(t)$ represents the commodities possessed by the economic entity $E$ at time $t$ and the quantity of the commodity type $c \in C$ possessed by $E$ at time $t$ will be denoted as $E_c(t) = q_c \circ E(t) = q_c(E(t))$. If $E$ is an economic entity and $[r, s]$ is a time interval then the entity-interval $E[r, s]$ will be used to denote the entity $E$ during the time interval $[r, s]$. This is to emphasize in the diagrams below when $E$ is engaged in various actions.

For example, if $E$ is a person who currently possesses at time $t = r$ a “grocery bag” containing two loaves of bread, one dozen eggs, and a six-pack of beer in pint cans, we can say that

$$E(r) = 2 \text{ loaves bread} + 1 \text{ dozen eggs} + 6 \text{ pints beer}$$

and thus

$$E_{\text{loaves bread}}(r) = q_{\text{loaves bread}}(E(r))$$

$$= q_{\text{loaves bread}}(2 \text{ loaves bread} + 1 \text{ dozen eggs} + 6 \text{ pints beer})$$

$$= 2.$$

It is often unnecessary to describe all of the commodities possessed by a given economic entity at any time, so we concern ourselves only with those commodities actually involved in the scenario being considered (e.g., it’s usually not necessary to consider a person’s clothing and other possessions when talking about labor or commodity exchanges they’re engaged in).

**Definition 11** (Hold, hold arrow). If $r < s < r' < s'$, an economic entity $E$ holds during the time interval $[s, r']$ if $E(t)$ is constant on $[s, r']$. We depict this by a hold arrow

$$E[r, s] \rightsquigarrow E[r', s']$$

The left side $E[r, s]$ denotes $E$ during some earlier time interval $[r, s]$ and the right side $E[r', s']$ denotes $E$ during some later time interval $[r', s']$. 
Definition 12 (Consumption, consumption arrow). An economic entity $E$ consumes a commodity vector $\vec{v} = q_1c_1 + \cdots + q_nc_n \in C$ during the time interval $[r, s]$ if for each $i$

1. $E_{c_i}(r) = E_{c_i}(s) + q_i \geq q_i \geq 0$, and
2. $E_{c_i}(t)$ is non-increasing on $[r, s]$.

In other words, $E$ can only consume $\vec{v}$ if it already possesses a quantity of each component $c_i$ which is at least $q_i$. We depict this by a consumption arrow

\[ E[r, s] \xrightarrow{\vec{v}} E[r, s] \]

Definition 13 (Transfer, transfer arrow). An economic entity $E$ transfers a commodity vector $\vec{v} = q_1c_1 + \cdots + q_nc_n \in C$ to another economic entity $F$ during the time interval $[r, s]$ if for each $i$

1. $E_{c_i}(r) = E_{c_i}(s) + q_i \geq q_i \geq 0$,
2. $E_{c_i}(t)$ is non-increasing on $[r, s]$, and
3. $F_{c_i}(t) = F_{c_i}(r) + E_{c_i}(r) - E_{c_i}(t)$ on $[r, s]$.

Consequently, $F_{c_i}(s) = F_{c_i}(r) + q_i$ and $F_{c_i}(t)$ is non-decreasing on $[r, s]$ for each $i$. In other words, $E$ can only transfer $\vec{v}$ to $F$ if $E$ already possesses a quantity of each component $c_i$ which is at least $q_i$. We depict this by a transfer arrow

\[ E[r, s] \xrightarrow{\vec{v}} F[r, s] \]

Nature is not considered to be capable of transferring its potential commodities to other economic entities but they can be obtained and commodified via labor, which is discussed in the following subsection.

Definition 14 (Exchange). An economic entity $E$ exchanges a commodity vector $\vec{u}$ with $F$ for a commodity vector $\vec{v}$ during the time interval $[r, s]$ if $E$ transfers $\vec{u}$ to $F$ and $F$ transfers $\vec{v}$ to $E$ during that time interval. We depict this by

\[ E[r, s] \xrightarrow{\vec{u}} F[r, s] \xleftarrow{\vec{v}} \]

We now put all of this together, and use these diagrams to illustrate simple economic scenarios.
Example 15 (Simple commodity exchange \( c \rightarrow c' \)). A person \( A \) possessing a commodity vector \( \bar{u} \) exchanges it with a person \( B \) for a commodity vector \( \bar{v} \). Assuming the exchange takes place during the time interval \([r, s]\) we could depict this as

\[
\begin{array}{c}
A[r, s] \xrightarrow{\bar{u}} B[r, s] \\
\end{array}
\]

Example 16 (Monetary exchange \( c \rightarrow m \rightarrow c' \)). A person \( A \) possessing a commodity vector \( \bar{u} \) sells that \( \bar{u} \) for its exchange-value \( |\bar{u}| = m \) to another person \( B \) (in some currency). Assuming the exchange takes place during the time interval \([r, s]\) we could depict this as

\[
\begin{array}{c}
A[r, s] \xrightarrow{\bar{u}} B[r, s] \\
\end{array}
\]

Then \( A \) holds onto this money until exchanging it to a third person \( C \) for the commodity vector \( \bar{v} \) with equal exchange-value \( |\bar{v}| = m \) during the time interval \([r', s']\). The whole scenario can then be depicted as

\[
\begin{array}{c}
A[r, s] \xrightarrow{\bar{u}} B[r, s] \\
\downarrow & m \\
\downarrow & m \\
A[r', s'] \xrightarrow{\bar{v}} C[r', s'] \\
\end{array}
\]

3.3. Labor and productions processes

Here it is important to emphasize the distinction between labor and labor-power. Labor-power is a commodity which can be bought and sold. Labor, however, is not a commodity but does have value which can be determined due to the fact that it is the necessary component in all production processes. We use \( L \) to denote the labor-power commodity and \( L' \) to denote labor used in labor processes.
Definition 17 (Purchase of labor power). The purchase by an economic entity $E$ of $h$ hours of labor-power $L$ from a person $A$ for commodity vector $\vec{v}$ (usually money) is considered as an exchange with only the transfer of $\vec{v}$ to $A$ for the agreement that $h = s - r$ hours of labor will be performed in some labor process during the time interval $[r, s]$, since $E$ cannot possess $A$’s labor-power. We depict this by

\[
E[r, s] \xrightarrow{\vec{v}} A[r, s]
\]

For example, if an economic entity $E$ hires a worker $A$ for 8 hours of labor at 15 USD per hour during the time interval $[0, 8]$, we would depict this as

\[
E[0, 8] \xrightarrow{[120]} A[0, 8]
\]

Definition 18 (Labor, production process, labor arrow, value of labor). A person $A$ performs $h = s - r$ hours of labor $L'$ in the production process $P$ which consumes the commodity vector $\vec{u} = p_1 c_1 + \cdots + p_m c_m \in \mathcal{C}$ from entity $E$ to produce the commodity vector $\vec{v} = q_1 d_1 + \cdots + q_n d_n \in \mathcal{C}$ for the entity $F$ during the time interval $[r, s]$ if for each $i$

1. $E_{c_i}(r) = E_{c_i}(s) + q_i \geq q_i \geq 0$,
2. $E_{c_i}(t)$ is non-increasing on $[r, s]$,
3. $F_{d_i}(s) = F_{d_i}(r) + p_i \geq p_i \geq 0$, and
4. $F_{d_i}(t)$ is non-decreasing on $[r, s]$.

We depict this by

\[
A[r, s] \xrightarrow{\vec{v}} E[r, s] \xrightarrow{\vec{u}} P[r, s] \xrightarrow{\vec{0}} F[r, s]
\]
Even though the production process $P$ can’t possess commodities, we think of the arrows $E \rightarrow P$ and $P \rightarrow F$ as transfer arrows occurring during the time interval $[r, s]$, and similarly $A \implies P$ is called a labor arrow. Even though labor is not a commodity, we will still denote its value as $|hL'| \in \mathbb{R}$. Determining the value of labor is the main purpose of the following subsection.

The commodity vector $\bar{u}$ consumed in the production process $P$ consists of the materials and means of production used in the production process. While tools, buildings, and other means of production are typically not wholly consumed during a single production process they are not presumed to be indestructible, and hence a fraction of their quantity is consumed during the production process. For example, if it is known that a given piece of equipment $e$ used in some production process can be used 100 times without any sort of maintenance, then each use consumes $(1/100)e$. Maintenance, on the other hand, is itself a production process which uses labor to restore equipment or some other commodity to its previous quantity.

It is possible for the worker themself to contribute the materials and tools and produce commodities for themself, i.e., $A = E = F$. We depict this by

$$
\begin{align*}
A[r, s] & \xrightarrow{\bar{\theta}} \left( \bar{u}, hL' \right) hL' \\
E[r, s] & \xleftarrow{\bar{\theta}} \bar{u} \\
F[r, s] & \xrightarrow{\bar{\theta}} \bar{u} \\
\end{align*}
$$

It is possible for nature to contribute materials to a production process even though it is not possible for nature to transfer commodities, in which case it is still necessary for nature to have the appropriate quantities available in order to produce the new commodities. We depict this by

$$
\begin{align*}
A[r, s] & \xrightarrow{hL'} \text{Nature} \\
E[r, s] & \xleftarrow{\bar{u}} \bar{u} \\
F[r, s] & \xrightarrow{\bar{\theta}} \bar{u} \\
\end{align*}
$$

where the same conditions as above are satisfied.
It is possible that there are multiple people performing labor in a single production process, multiple entities contributing materials and means of production, or multiple entities receiving the newly produced commodities. We depict this general case by

\[
A_1[r, s] \cdots A_m[r, s] \rightarrowNature[r, s] \rightarrow E_1[r, s] \cdots E_n[r, s] 
\]

\[
\downarrow h_1 L' \quad \downarrow \quad \downarrow u_1 \quad \downarrow \quad \downarrow u_n
\]

\[
P[r, s] \rightarrow F_1[r, s] \cdots F_k[r, s]
\]

where the same conditions above are satisfied for every entity and commodity involved.

3.4. Axioms of exchange and production

If labor is not a commodity, how do we determine its value? We assert the following two axioms for enabling our calculations using economic diagrams:

**Definition 19 (Axiom of exchange).** Commodity exchange for equal exchange-values.

That is, if

\[
E[r, s] \rightarrow \bar{u} F[r, s] \leftarrow \bar{v}
\]

is an exchange, then \(|\bar{u}| = |\bar{v}|\).

The axiom also applies to the purchase of labor-power. If

\[
E[r, s] \rightarrow \bar{v} A[r, s] \leftarrow h L
\]

is a purchase of labor-power then \(|hL| = |\bar{v}|\) or \(|hL| = |[m]| = m\) in the case where \(\bar{v} = [m]\) is money, i.e., wages paid are equal to the exchange-value of the labor-power purchased.
Definition 20 (Axiom of production). Commodities produced in production processes have an exchange-value equal to the sum of the exchange-values of the commodities used in production plus the value of labor, and consequently new values are only created by labor.

That is, if

\[
\begin{align*}
A[r, s] & \xrightarrow{hL'} E[r, s] \\
\downarrow & \\
P[r, s] & \xrightarrow{\bar{u}} F[r, s]
\end{align*}
\]

is a production process, then \(|\bar{v}| = |hL'| + |\bar{u}|.|\\n
In the case where products of nature are used in production, those products are considered to have no initial exchange-value, but they obtain exchange-value as they are transformed via labor into commodities, i.e., if

\[
\begin{align*}
A[r, s] & \xrightarrow{hL'} \text{Nature} \xrightarrow{\bar{u}} E[r, s] \\
\downarrow & \\
P[r, s] & \xrightarrow{\bar{v}} F[r, s]
\end{align*}
\]

is a production process, then \(|\bar{v}| = |hL'| + |\bar{u}|.|\\n
In the general case, if

\[
\begin{align*}
A_1[r, s] & \cdots A_m[r, s] \xrightarrow{h_{1}L'} \cdots \xrightarrow{h_mL'} \text{Nature} \xrightarrow{\bar{u}_1} \cdots \xrightarrow{\bar{u}_n} E_1[r, s] & \cdots E_n[r, s] \\
\downarrow & \\
P[r, s] & \xrightarrow{\bar{v}_1} \cdots \xrightarrow{\bar{v}_k} F_1[r, s] & \cdots F_k[r, s]
\end{align*}
\]

is a production process, then the sum of the exchange-values of all commodity vectors used in production plus the total value of labor is equal to the sum of the exchange-values of all commodity vectors produced:

\[
|\bar{v}_1| + \cdots + |\bar{v}_k| = |h_{1}L'| + \cdots + |h_mL'| + |\bar{u}_1| + \cdots + |\bar{u}_n|.
\]
With this we can finally define the following:

**Definition 21** (Economic scenario diagram). An economic scenario diagram $\Gamma$ is a finite directed graph with labeled edges where the vertices are entity-intervals $E[r, s]$ and production processes $P[r, s]$ during the specified time intervals, the edges are hold, consumption, transfer, and labor arrows labeled by commodity vectors such that the axiom of exchange (Definition 19) holds for all exchanges and the axiom of production (Definition 20) holds for all production processes.

### 4. Exploitation and the cycle of capitalist production

In this section time intervals are omitted from all diagrams, but we consider time to be proceeding as noted on the left side. Equations obtained from the axioms of exchange and production at each step are given on the right side.

As previously noted, the monetary exchange $c \rightarrow m \rightarrow c'$ can be described by a person $A$ exchanging a commodity vector $\bar{u}$ with $B$ for $[m]$, followed by $A$ exchanging $[m]$ with $C$ for a commodity vector $\bar{v}$:

\[
\begin{align*}
  t = r & & A \xrightarrow{\bar{u}} B & & |\bar{u}| = m \\
  t = s & & A \xrightarrow{\bar{v}} C & & |\bar{v}| = m
\end{align*}
\]

By the axiom of exchange (Definition 19), we have that $|\bar{u}| = m = |\bar{v}|$, so no value is created or lost in this process.

#### 4.1. The Cycle of Capitalist Production

In Marxist political economy, the *cycle of capitalist production* is typically described as $m \rightarrow c \rightarrow m'$: the initial money ($m$) is used to buy commodities ($c$) which are sold for more money ($m'$), with the capitalist’s profit or *surplus value* being the difference $p = m' - m$ between the final money ($m'$) and the initial money ($m$). This cycle completely describes *merchant capital*, where commodities are bought at one market and sold for a profit at another market where those commodities yield higher exchange-values.
In the modern capitalist economy there is really only one global market and additionally merchant capital does not produce any new values, so a more detailed treatment of this process is needed to describe capitalist production. This is typically described as $m \rightarrow c \rightarrow P \rightarrow c' \rightarrow m'$: the initial money is used to buy commodities for the production process ($P$) of new commodities ($c'$) which are sold for more money, earning a profit $p = m' - m$.

In either case, this is an oversimplification of the process which simultaneously ignores the dialectical nature of money, labor, and commodities as representing relationships between people, other economic entities, and the material world. A more detailed model is illustrated below:

Let’s unpack what’s going on here.

First, the capitalist invests an amount of money $m$ in the business by transferring the commodity $[m]$ to it. The money is split up as $m = c + v$ into constant capital $c$ used to purchase materials and the means of production and variable capital $v$ used to purchase labor-power, i.e., variable capital is spent on wages. The business exchanges $[c]$ to the market for a commodity vector $\bar{u}$ consisting of the materials and means of production used in production, and it exchanges $[v]$ to the workers to buy $h$ hours of their labor-power $L$. 
By the axiom of exchange (Definition 19), we have that \( c = |[c]| = |\bar{u}| \) and \( v = |[v]| = |hL| \). The amount spent on the initial commodities is called constant capital because the business has no choice but to pay the market price for the commodities used in production, whereas the business can choose to spend more or less variable capital, e.g., by raising or lowering wages.

Then the initial commodities \( \bar{u} \) and \( h \) hours of labor \( L' \) are brought together into production to produce a commodity vector \( \bar{w} \) consisting of the finished products. If products of nature are involved in production (e.g., mining, oil extraction, etc.) they are not considered to have any initial exchange-value and hence do not make up any components of \( \bar{u} \). By the axiom of production we have that \( |\bar{w}| = |hL'| + |\bar{u}| \). Then the business exchanges \( \bar{w} \) to the market for revenue \([m']\), and by the axiom of exchange we have that \( m' = |[m']| = |\bar{w}| \).

Finally, the business transfers this revenue \([m']\) to the capitalist. The capitalist’s profit for this cycle is then \( p = m' - m \), and factoring through the equations obtained from the diagram and axioms we find that

\[
p = m' - m \\
= |\bar{w}| - (c + v) \\
= (|hL| + |\bar{u}|) - (|\bar{u}| + |hL|) \\
= |hL'| - |hL|,
\]

so \( p = |hL'| - |hL| \) or profit = value of labor – wages, i.e., all profit is equal to the value of unpaid labor. We similarly find that \( p/h = |L'| - |L| \) or hourly profit = hourly value of labor – hourly wages.

The capitalist’s primary motivation is to accumulate more and more profit. Unlike the monetary exchange \( c \rightarrow m \rightarrow c' \) which begins and ends with commodities which are used to satisfy some want, using money only as the exchange medium, capitalist commodity production \( m \rightarrow c \rightarrow m' \) begins and ends with money. For the capitalist, there are no wants satisfied by money itself except as a means to acquire more money, and the primary means of acquiring more money is through the exploitation of the labor of workers. By keeping wages down and coaxing more labor out of workers, the capitalist increases their profits, and this exploitation is imperative for the capitalist if they wish to successfully outperform their competitors in the market.
4.2. Rates of Surplus Value and Profit

What else can be observed from this simple model of capitalist production? First, the revenue $m' = |\bar{w}|$ obtained from the finished products satisfies $m' = c + v + p$, i.e., revenue is the sum of the constant capital, variable capital, and surplus value. From this we can define the rate of surplus value $\sigma = p/v$ and rate of profit $\pi = p/(c + v)$. These are, respectively, the ratio between profit and wages and the ratio between profit and initial capital spent. From these we can find, respectively, the proportion $p/(p + v) = \sigma/(1 + \sigma)$ of new values made up by profit and the proportion $p/m' = \pi/(1 + \pi)$ of revenue made up by profit.

For example, consider the following data obtained from an anonymous Starbucks Workers United (SBWU) union organizer on a single eight-hour shift at a suburban Starbucks store\footnote{The data consisted of register totals for each half-hour between 5:30am and 1:30pm and the employee schedule for that shift.}: with 2–5 employees working at any single time, a total of $h = 31.5$ hours of labor at $|L| = 12.50$ USD/hour was performed and the store spent a total of $v = |hL| = 393.75$ USD on variable capital. Total revenue earned was $m' = 2294.48$ USD with from $n = 332$ customers, which an average of revenue of $m'/n = 6.91$ USD per customer. While we don’t have data on the amount of constant capital spent by the store, we can estimate the average constant capital per customer to be $x = c/n$.

Then the profit earned as a function of average constant capital per customer $x$ is $p(x) = m' - c - v = 1900.73 - 332x$. With this we can consider the rate of surplus value $\sigma(x) = p/v$ and the rate of profit $\pi(x) = p/(c + v)$ as functions of $x$; see Figure 1.

Considering that average revenue per customer is $m'/n = 6.91$ USD, we find that even with a conservative estimate of $x = 4$ USD, the rate of profit is $\pi(4) = p/(c + v) = 0.33$. Or in other words, the proportion of revenue made up by profit is $p/m' = \pi/(1 + \pi) = 25\%$. It is likely that the average constant capital per customer is actually less than this, meaning that even more of the revenue is taken by the owners as profit.

One of the key demands of many SBWU locals is an increase in wages to 15 USD/hour. How would such a change impact the rates of surplus value and profit? In this case we would have $v = 472.5$ and profit as a function of
Figure 1: The rate of surplus value $\sigma$ and rate of profit $\pi$ as functions of average constant capital per customer.

$x$ would be $p(x) = 1821.98 - 332x$, and even with an estimate of $x = 4$ USD, the rate of profit would be $\pi(4) = p/(c + v) = 0.29$ and the proportion of revenue made up by profit would be $p/m' = \pi/(1 + \pi) = 22\%$. In other words, if the store met the demand for increased wages, it would decrease profit’s proportion of revenue by only three percentage points.

5. Banks and Economic Crisis

How can we use and extend the model introduced in §3 to describe the functions of banks?

The simplest example of this is a loan with compound interest which is paid off at once. Suppose that a borrower takes out a loan with a principal $m$ at a rate $r$ compounded $k$ times per year for $T$ years. At the end of $T$ years, the borrower repays the bank $m' = m(1 + \frac{r}{k})^{kT}$. The bank’s profit is the interest earned $p = m' - m$ and the entire scenario can be depicted as

\[ t = 0 \hspace{1cm} \text{Borrower} \hspace{0.5cm} \xrightarrow{[m]} \hspace{0.5cm} \text{Bank} \]

\[ t = T \text{ years} \hspace{0.5cm} \text{Borrower} \hspace{0.5cm} \xrightarrow{[m']} \hspace{0.5cm} \text{Bank} \]

A slightly more complicated example is a loan with compound interest which is amortized, i.e., it is repaid with regular payments. Suppose that a borrower takes out an amortized loan with a principal $m$ at a rate $r$ compounded $k$
times per year for $T$ years. At the end of each compounding period, the borrower makes a payment of $s = \frac{m(r/k)}{1-(1+r/k)^{-kT}}$ to the bank. The bank’s profit is then $p = kTs - m$ and the entire scenario can be depicted as:

\[
\begin{align*}
t = 0 & \quad \text{Borrower} \xleftarrow{[m]} \text{Bank} \\
& \downarrow \quad \downarrow \\
 t = 1/k \text{ year} & \quad \text{Borrower} \xrightarrow{[s]} \text{Bank} \\
& \downarrow \quad \downarrow \\
 t = 2/k \text{ year} & \quad \text{Borrower} \xrightarrow{[s]} \text{Bank} \\
& \downarrow \quad \downarrow \\
 t = T \text{ years} & \quad \text{Borrower} \xrightarrow{[s]} \text{Bank}
\end{align*}
\]

5.1. Investments

It is somewhat more complicated to consider investments. The usual perspective merely reverses the roles by considering an investment as a loan borrowed from the investor. However, it is less clear where the “interest” comes from in such a scenario. Ultimately new values must be produced by labor, so the typical explanation that invested money “grows” is insufficient to explain investments.

**Definition 22** (Investment arrow, investment diagram). We will depict an investment by an economic entity $E$ in another entity $F$ during the time interval $[r, s]$ with expected return $p$ with an investment arrow

\[
E[r, s] \xrightarrow{[p]} F[r, s]
\]

As an edge in a directed graph, $E[r, s]$ is considered to be the head and $F[r, s]$ is the tail.

An investment diagram $\Gamma$ is a finite directed graph with labeled edges where the vertices are entity-intervals $E[r, s]$ and the edges are investment arrows labeled by returns.

As before, we omit time intervals for the remainder of the section. These diagrams are admittedly an oversimplification, and the mechanics of an investment could be modeled more accurately using the more general economic scenario diagrams introduced in §3 as detailed below.
For a simple example, suppose that a bank offers savings accounts to customers and that the total amount of deposits at the beginning of the year which will earn a total amount of interest $i$ at the end of the year. Where does this interest come from? This year the bank invests a portion of the total deposits in a business which employs workers who perform labor, with an expected return of $r$ for this investment by the end of the year. Then the bank’s profit is $p = r - i$ after paying interest on deposits, and we can depict these investments by

$$\text{Customers} \xrightarrow{[i]} \text{Bank} \xleftarrow{[r]} \text{Business}$$

There are many ways this arrangement could fail to produce profit for all involved or even result in a loss of money. The business could be unprofitable, in which case the bank might not be able to pay interest on the deposits, or there could be a bank run where customers try to withdraw more money than the bank is holding in liquid assets because a portion of their total assets are held as fixed assets tied up in investments.

This investment diagram can also be expressed in more detail as a general economic scenario diagram:

$$t = 0 \quad \text{Bank} \xrightarrow{[m_1]} \text{Other Investors} \xrightarrow{[v]} \text{Workers} \xrightarrow{[c]} \text{Business} \xrightarrow{[hL]} \text{Market} \quad m = m_1 + m_2 = c + v$$

$$v = |hL|, \quad c = |\bar{u}|$$

$$|\bar{w}| = |hL'| + |\bar{u}|$$

$$|\bar{w}| = m'$$

$$t = 1 \text{ year} \quad \text{Business} \xrightarrow{[m'_1]} \text{Market} \xrightarrow{[m'_2]} \text{Other Investors} \quad m' = m'_1 + m'_2$$

The customers’ deposits at the beginning of the year are $d$, and the bank holds a portion of it as liquid assets $l$ and invests $m_1$ in a business. The bank’s return on the investment is $r = m'_1 - m_1$, so its total assets are now $m'_1 + l = d + i + p$, where $i$ is interest earned by the savings accounts and $p$ is the bank’s profit. Then because $d = m_1 + l$ we have that $p = m'_1 - m_1 - i = r - i$. 
For another example, consider an insurance company. An insurance company uses statistics to estimate the amount $c$ that the company will have to pay for claims on the policies the company underwrites. Policyholders pay premiums $a$ for their policies, and the company uses a portion of the premiums to make other investments with expected return $r$ in order to cover all of the claims. The company’s profits are then $p = r + a - c$, and from this perspective an insurance policy could be seen as an investment by the insurance company in the policy holders with return $a - c$. We can depict these investments by

As in the previous example, there are many ways this arrangement can fail. The other investments may fail to provide enough returns to cover all the claims, or more claims could be filed than the insurance company is able to pay out with its liquid assets.

5.2. Economic Crisis

Consider a scenario similar to what happened during the 2007–2008 financial crisis: a large regional bank invested in the construction of houses during a housing boom, but the bank isn’t earning a profit on this investment because they haven’t sold enough homes. The bank then makes a number of risky investments by underwriting subprime mortgages to buyers whose credit isn’t stable. To ensure its ability to make a profit, the bank then bundles together these mortgages into mortgage-backed securities, which it markets to hedge funds, retirement funds, and other investors as a secure investment.

Mortgage-backed securities are an example of financial derivatives, which are assets whose values are based on or derived from some other asset such as mortgages or stocks. Financial derivatives are an example of what Marx called fictitious capital because their (often inflated) values are based on speculation on anticipated future income but not based on direct investment in actual production and labor. In particular, mortgage-backed securities are a wager on peoples’ ability to pay their mortgages on time rather than on the profitability of a business that actually produces commodities.
Then the bank takes out a credit default insurance policy on these securities which pays out to the bank in case those securities default and the bank can’t pay returns to the investors. Finally the bank has its other investments. The whole scenario can be depicted by

![Diagram of Investors, Mortgage-Backed Securities, Bank, Insurance Company, Credit Default Insurance Policy, Homeowners, Other Investments]

In fact, this diagram should be much bigger because there could be many banks and other financial institutions doing similar things in complex interconnected investments, as was the case during the financial crisis.

As in the previous examples, in order for every investor to earn a profit, the upward flow of money through the diagram can’t be disrupted. There are a number of ways that this arrangement could fall apart. Most economists claim that there is a business cycle of recessions followed by expansions, so it’s reasonable to consider that an economic recession could cause the bank’s other investments to fail. It’s also reasonable to consider that a recession could result in mass layoffs and unemployment resulting in a wave of homeowners defaulting on their mortgages. If enough of the bank’s investments fail the investors could choose to sell their mortgage-backed securities, in which case the bank would need to file an insurance claim to avoid default. If enough banks are in similar situations or if the insurance company’s own investments aren’t performing well as a result of the recession, the insurance company would be unable to pay out to policy holders. Now the hedge funds, retirement funds, and other investors have completely lost all of the money invested in the securities, the homeowners have lost their homes, and the banks and insurance companies either end up failing, being bought by larger rivals, or being bailed out by the government.

Modern finance is so interconnected that it is conceivable that such catastrophic chain reactions of defaults are likely to happen again without substantial regulation. One possible way to model the stability of such a system would be to consider economic scenario diagrams of interconnected investments as flow networks. From this perspective, defaults, labor strikes, logistics issues, and other disruptions would be “cut” edges removed from the
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diagram to study the extent of the resulting disruption to the flow of goods, labor, and capital. As in situations such as the one described above, such disruptions could even lead to catastrophic chain reactions of cascading defaults due to speculation in assets with greatly inflated values.

6. Anti-Capitalist Pedagogy in Mathematics

In this section we present some potential activities which can be used to supplement math courses with economic and financial topics and example problems. These are designed more as discussion workshop activities rather than homework assignments, and are intended primarily for courses such as college algebra, precalculus, and finite math but are also appropriate for calculus I and applied/business calculus. Ultimately we hope to expand this into a more detailed curriculum for a course such as finite math which discusses financial mathematics with an emphasis on labor.

The purpose of the first activity (§§6.1) is to supplement the usual “profit = revenue − costs” problems found in many math courses by presenting such a problem from the perspective of the workers using a simplified version of the model of capitalist production presented in §4. This activity meets mathematics pedagogical goals by providing an application which utilizes a number of techniques taught in math courses, and it meets social justice pedagogical goals by helping students understand the nature of profit and challenging the notion that labor unions make businesses unprofitable.

The purpose of the second activity (§§6.2) is to supplement discussions of compound interest. This activity meets mathematical pedagogical goals in courses where compound interest is taught, and it meets social justice goals by asking students to consider what it means for money to “grow” and under what sort of circumstances it may fail to “grow”.

6.1. Profit and Wages

A business’s total costs \( m = c + v \) can be broken up as a sum of constant capital \( c \) and variable capital \( v \). Constant capital is spent on things which the business has no choice but to pay market price (such as raw materials, equipment, fees, taxes, real estate, etc.), whereas variable capital is spent on wages which the business can choose to pay more or less, i.e., by raising or lowering wages. If the business’s revenue is \( r \), then its profit is \( p = r - m = r - c - v \).
Exercises

A local coffee shop owner employs workers who are paid an hourly wage $w$. On a given day the workers perform a total of $h$ hours of labor, the shop earns revenue $r$, and the shop spends constant capital $c$ to pay for ingredients, equipment, rent, etc.

1. How much money $v$ is spent on variable capital (wages) for the day? Express your answer in terms of $w$ and $h$.

   Answer: $v = wh$.

2. How much money $m$ is spent in total on constant and variable capital for the day? Express your answer in terms of $w$, $h$, and $c$.

   Answer: $m = c + v = c + wh$.

3. How much profit $p$ does the owner make for the day? Express your answer in terms of $w$, $h$, $c$, and $r$.

   Answer: $p = r - m = r - c - wh$.

The rate of profit is the ratio $\pi = p/(c + v) = p/m$ of profit to initial money spent, i.e., for every dollar spent on constant and variable capital $\pi$ dollars worth of profit are earned.

4. Express the coffee shop’s rate of profit $\pi$ in terms of $w$, $h$, $c$, and $r$.

   Answer: $\pi = \frac{p}{m} = \frac{r - c - wh}{c + wh}$.

Suppose that the workers perform a total of $h = 32$ hours of labor, constant capital spent is $c = $1300, and the revenue earned is $r = $2300.

5. Express profit $p(w)$ as a function of the hourly wage $w$ and graph this function.

   Answer: $p(w) = 1000 - 32w$. See Figure 2 for the graph.

6. Express the rate of profit $\pi(w)$ as a function of the hourly wage $w$ and graph this function.

   Answer: $\pi(w) = \frac{1000 - 32w}{1300 + 32w}$. See Figure 3 for the graph.
The workers currently earn an hourly wage of $w = 12/\text{hour}$, but they have recently formed a union to demand an increase in hourly wages to $w = 15/\text{hour}$.

7. If no worker works more than 8 hours per day, what is the maximum total daily wages earned by a worker (before taxes) at $12/\text{hour}$? At $15/\text{hour}$?

Answer: The maximum earned by any of the workers is $96$ at $12/\text{hour}$ and $120$ at $15/\text{hour}$.
8. How much is earned in total by all workers when wages are $12/hour? $15/hour?

Answer: Total wages are equal to variable capital \( v(w) = wh = 32w \), so \( v(12) = 384 \) and \( v(15) = 480 \).

9. What are the profits and rates of profit when wages are $12/hour? $15/hour?

Answer: \( p(12) = 616 \), \( \pi(12) = \frac{154}{421} \approx 0.37 \), \( p(15) = 520 \), and \( \pi(15) = \frac{26}{89} \approx 0.29 \).

10. The owner of the coffee shop is refusing the union’s demands, claiming that if hourly wages are increased from $12/hour to $15/hour, then the shop will need to raise prices (and hence revenue) in order to remain profitable. Based on your previous answers, is the owner’s claim true? Why or why not?

Answer: As mentioned in the previous discussion, all profits are unpaid labor. The workers are the ones performing the labor that generates revenue, not the owner. If hourly wages are increased to $15/hour, the owner will still be making more profit ($520) than the total wages earned by all workers ($480) and the rate of profit would only decrease slightly. The owner’s claim is false because the shop would still be profitable, it would just be slightly less profitable than it had been.

6.2. Compound Interest

An account or financial asset earns compound interest if it earns interest on previously earned interest. Suppose that a principal amount \( P \) is placed into a savings account which pays compound interest at a rate \( i \) at the end of each compounding period. What is the account balance \( A(n) \) after \( n \) compounding periods?

The initial balance at the time of deposit is \( A(0) = P \). After one period the interest earned is \( A(0) \cdot i = Pi \), so \( A(1) = P + Pi = P(1+i) \). After two periods the interest earned is \( A(1) \cdot i = P(1+i)i \), so \( A(2) = P(1+i) + P(1+i)i = P(1+i)^2 \).

In general, after \( n \) periods the balance or future value is \( A(n) = P(1+i)^n \). Typically compound interest is calculated in terms of an annual rate or stated rate \( r \) which is compounded \( m \) times per year for \( t \) years. Then the rate per
period is \( i = r/m \) and the number of compounding periods in \( t \) years is
\[ n = mt, \]
so we obtain
\[ A(t) = P \left( 1 + \frac{r}{m} \right)^{mt}. \]

But where does compound interest come from? In the case of a savings account, the bank typically splits the total savings deposits \( d \) from all customers into liquid assets \( l \) which are held in the bank and are readily available for withdrawal and fixed assets \( f \) which are invested, i.e., \( d = l + f \). The bank expects the value of their investments to increase from \( f \) to \( f + r \), where \( r \) is the return on the investment.

Then the total value of the bank’s assets are \( l + f + r = d + r \). Then a portion \( i \) of the returns are added to the savings accounts as interest earned and the remainder \( p \) is kept as the bank’s profit, i.e., \( r = p + i \). Thus the bank’s profit is \( p = r - i \).

Money deposited in a savings account does not “grow” on its own because the bank needs to invest the money in order to pay interest, and in order for investments in general to be profitable the investment must create new values. New values aren’t created on their own, so ultimately labor needs to be performed by workers to create those new values and generate those profits.

Exercises

1. If \$500 is deposited in a savings account with an annual interest rate of 2\% compounded monthly \((m = 12)\), what is the balance after 5 years?

\[ \text{Answer:} \ P = \$500, \ r = 0.02, \ m = 12, \ t = 5 \]
\[ \implies \ A(5) = 500 \cdot \left( 1 + \frac{0.02}{12} \right)^{12 \cdot 5} = \$552.54. \]

2. In the 1999 episode “A Fishful of Dollars” of the science fiction comedy series \textit{Futurama}, the main character Philip J. Fry learns that after being cryogenically frozen for 1000 years, his bank is somehow still in business! If his savings account had a balance of \$0.93 in the year 2000 and the account pays compound interest annually \((m = 1)\) at a rate of 2.25\%, what is the balance in the year 3000? Round your answer to the nearest million dollars.
Answer: $P = 0.93, r = 0.0225, m = 1, t = 1000$
\[\implies A(1000) = 0.93 \cdot \left(1 + \frac{0.025}{1}\right)^{1-1000} = 4,284,000,000.\]

3. In the 1988 episode “The Neutral Zone” of the science fiction series Star Trek: The Next Generation, the crew of the USS Enterprise finds three people who had been cryogenically frozen for over 300 years. One of them is a financier who is eager to learn the value of his investments. He estimates that his investments valued at $50 million in the year 2000 have grown like compound interest which compounds annually ($m = 1$) at a rate of 1.7%. What would he expect the value of investments to be in the year 2364? Round your answer to the nearest million dollars.

Answer: $P = 50,000,000, r = 0.017, m = 1, t = 364$
\[\implies A(364) = 50000000 \cdot \left(1 + \frac{0.017}{1}\right)^{1-364} = 23,110,000,000.\]

4. What are some scenarios in which a bank may not be able to pay interest on savings account?

Answer: The bank may not be able to pay interest if its investments are not profitable, and it’s possible for the bank to go out of business if its customers attempt to withdraw more money than is held in liquid assets. Additionally, it is possible that investments of any type may not survive recessions, wars, or changes of government.

References


5 In the Star Trek universe, humanity nearly annihilates itself in the 21st century during World War III. Later after making contact with the alien Vulcans, humanity establishes a world government which eliminates poverty and eventually helps to found the United Federation of Planets along with the Vulcans and other alien species. In “The Neutral Zone” the USS Enterprise’s Captain Jean-Luc Picard informs the financier that humanity no longer uses money, so his lawyer’s firm and his investments are unlikely to exist anymore!

