Journal of Humanistic Mathematics

Volume 13 | Issue 2 July 2023

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Recommended Citation

Benjamin Dickman & Julia Feinberg, "Critical Co-Investigators of Math Trails: Reflections from a Student and Teacher," *Journal of Humanistic Mathematics*, Volume 13 Issue 2 (July 2023), pages 102-125. DOI: 10.5642/jhummath.JGZG2852. Available at: https://scholarship.claremont.edu/jhm/vol13/iss2/8

Critical Co-Investigators of Math Trails: Reflections from a Student and Teacher

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Abstract

In this article, a K-12 mathematics educator and a recent (2020) high school graduate discuss curricular work related to math trails, which are based around the idea of mathematizing potential discoveries along a physical walk. The intersection of math trails with the realities of schooling amid the COVID pandemic is described, along with ways in which math trail learning has ramified beyond classroom walls. This collaboration serves not only to draw attention to the under-researched topic of math trails, but also to exhibit how students and teachers can, in the language of Freire, work together as critical co-investigators.

Keywords: math trails, problem posing education, problem posing pedagogy, student-centered learning

1. Introduction

This article represents a collaboration between a K-12 mathematics educator at an independent girls school in the Northeastern United States and a recent graduate who completed her senior capstone project on math trails in Spring 2020. The topic related to earlier coursework in a class entitled Mathematical Problem Solving & Problem Posing. Central to the goals of the course, in general, and this collaborative effort, in particular, are a commitment towards that which Freire refers to as "problem-posing education" [4, page 79]. More precisely:

> "Through dialogue, the teacher-of-the-students and the students-of-the-teacher cease to exist and a new term emerges: teacher-student with students-teachers. The teacher is no longer merely the-one-who-teaches, but one who is himself taught in dialogue with the students, who in turn while being taught also teach. They become jointly responsible for a process in which all grow. In this process, arguments based on "authority" are no longer valid; in order to function, authority must be on the side of freedom, not against it. Here, no one teaches another, nor is anyone self-taught. People teach each other, mediated by the world ...
>
> ... The problem-posing method does not dichotomize the activity of the teacher-student: she is not "cognitive" at one point and "narrative" at another. She is always "cognitive," whether preparing a project or engaging in dialogue with the students. He does not regard cognizable objects as his private property, but as the object of reflection by himself and the students. In this way, the problem-posing educator constantly re-forms his reflections in the reflection of the students. The students — no longer docile listeners — are now critical co-investigators in dialogue with the teacher. The teacher presents the material to the students for their consideration, and re-considers her earlier considerations as the students express their own. The role of the problem-posing educator is to create ... " [4, pages 80—81]

In this respect, the paper at hand serves a dual purpose by not only presenting work on math trails that has evolved over the past five academic years but also by modeling the critical co-investigation between teacher and [graduated] student that Freire features as integral to problem-posing pedagogy. (For another recent example of an approach to teacher-student collaboration in the context of high school mathematics education, see Strong and Butterfield's "Dear Math" [18].)

1.1. Brief History of Math Trails

In “Math Trails” [17], the authors begin with the definition that “A mathematics trail is a walk to discover mathematics” (page 6) and continue by listing six characteristics of math trails: math trails are for everyone; math trails are cooperative, not competitive; math trails are self-directed; math trails are voluntary; math trails are opportunistic; and math trails are temporary (pages 8-9). The history of math trails is traced back to 1985, for which the initial locations included England and Australia; in particular, Dudley Blane and colleagues created math trails based in Melbourne [2, 3]. Subsequent work includes that of Kay Toliver, whose videos and educational television series *The Eddie Files* explicitly broach math trails; more can be found in [20] where she begins a section entitled “Creating and Traveling the Math Trail” by writing:

> I teach in a community where the students do not hear a lot of positive comments about the area in which they live. Our school building is usually referred to by visitors as a “relic.” To counter these negative assessments, I try to show my students that there is a wealth of mathematics in our neighborhood. I developed a lesson called the “Math Trail” to give students an appreciation for the community as well as an opportunity to see mathematics at work.
>
> To create a Math Trail, the class must first do some research on the history of East Harlem. Then, they are instructed to plot a course, starting from the school plaque in the lobby of the building, that leads the class through the community and back to school, with stops along the way to visit several interesting sites and create math problems about various real-life situations (page 43).

Toliver continues by discussing the dichotomy of classroom mathematics (e.g., test-taking) and real-world mathematics (e.g., that which is encountered while walking along a math trail). To this end, she continues:

> I have discovered that some students who are not the best test takers have a very good understanding of how to apply the mathematical concepts involved in these Math Trail exercises. I recall one girl who pointed out two triangles on a rooftop and explained

> to me that they must be similar. When I asked her what one would have to know to prove this, she explained that the sides would have to be proportional and the angles congruent. She then went on to explain that the angles would remain the same regardless of the size of the triangle. Imagine my surprise to hear this from a girl who had failed three-quarters of the tests I had given in class! I knew then that she had really learned a great deal of mathematics and learned it well enough to apply it in practical fashion to the world around her. With other students, I have noticed the reverse. They do well on tests but have difficulty applying what they have studied. Math Trail exercises force these students to make a connection between the ideas they memorize for the tests and the real world. However, as one group of students wrote in the introduction to their book of Math Trail problems, the purpose of this experience is "to prove that the classroom is NOT the only place to learn math" (pages 43-44).

Besides Toliver's work in New York City, other early examples of math trails can be found in Boston where Carole Greenes, working at the time at Boston University, "created a historical mathematics trail in Boston centered on the Common and the Public Garden . . . walkers on Greenes' trail followed a human guide who knew the historical and mathematical aspects of the trail and who could give hints and suggestions to walkers who got stuck on a task or idea" [17]. Three decades after the first math trail, Boston would again be the site of a similar walk in the Math Challenge Tour written for the 2015 NCTM Annual Meeting & Exposition [5]. This work focuses on "math tours" and references the work of Math in the City (`https://www.mathsinthecity.com/`). More recent outdoor mathematics projects, although not necessarily presented in the form of a sequenced walk, include Public Math (`https://www.public-math.org/`) and Sidewalk Math, the latter of which is sometimes referred to as Math Walks [13, 15]. Additional math trails work has begun to incorporate technology, as well; for example, see the Instagram-based math trail as discussed by Maldonado in [6, page 154]. In our context — an elective course that focuses largely on mathematical problem posing — the topic of math trails is also a good fit; for example, see recent work on problem posing as connected to math trails and math walks [12, 23].

1.2. Context and Goals

The first author, Benjamin, became familiar with math trails after taking graduate courses with Henry Pollak, who is one of the authors of the 2004 book [17] on the topic. Subsequently, he worked at Boston University during the time period in which professors and graduate students assembled the aforementioned math tours for the 2015 NCTM conference. Since Fall 2016, Benjamin has been teaching at an independent girls day school in the Northeastern United States. For each year at this institution, he has taught a course on mathematical problem solving and posing, and first broached math trails as a topic in the 2018-19 academic year.

The second author, Julia, became familiar with math trails as a student enrolled in Mathematical Problem Solving & Problem Posing in the 2019-20 academic year. In addition to the standard coursework associated with math trails, she also completed a dual senior capstone project in Spring 2020, for which one of the two components was an investigation of "social justice math trails" with Benjamin as the faculty advisor. Additional details about this project and ideas that came out of it can be found in Sections 3-4.

Our math trail context differs from those previously referenced, and, in particular, from that of Toliver [20] for whom a central goal of this topic is to "counter [the] negative assessments" of the area, community, and school building in which she and students were located (page 43). Our school is a predominantly white institution with students who possess privileged identities across many axes, especially pertaining to race and class, even as they inhabit marginalized identities across others, specifically pertaining to gender identity at a site classified as a girls school. Thus, our curricular attempts to weave societally-relevant mathematics into our curriculum embeds us within the context described in Kokka's "Social Justice Pedagogy for Whom? Developing Privileged Students' Critical Mathematics Consciousness" [10] even as we engage in efforts to transition the mathematical learning along the five phases adapted from Countryman's work [7, page 84] from "Womanless mathematics" to "Women as central to mathematics" and beyond [9, page 3].

For the purpose of the paper at hand, our goals are decidedly modest: We describe the origin of math trails and their use in classroom teaching and learning across multiple years; discuss the intersection of math trails as a way of mathematizing one's local community with the shrunken worlds wrought by the COVID pandemic; and aim to center student-voice with one author as a recently graduated student who writes candidly about her own experience and that of peers when engaged in the work of math trails. The evolution of this curricular component is ongoing, and the reader is invited to participate by considering how math trails can be implemented, modified, and otherwise incorporated at their own learning sites.

2. Genesis of Assignment: Math Trails

In our grade 11 and 12 elective course on Mathematical Problem Solving & Problem Posing, the approach to math trails has evolved over the past several years. In this section, we delve into the initial forays and how they have varied over time.

2.1. Original Exploration: 2018-19

In Spring 2019, a group of students along with Benjamin, as their instructor, brainstormed potential ways to incorporate math trails into our course. The ultimate outcome of these brainstorming sessions was a presentation to the high school faculty in which he communicated student thinking in response to the two questions: Why for our class? Why for our community? This was paired with the sample slide found in Figure 1, and followed by additional student thinking around Moving Forward and a collection of Selected Themes that arose after discussing math trails in class and having started to create a trail that could correspond to our school. The three specific themes identified by students and Benjamin for inclusion in a prospective math trail were:

Access: for students of all stages and ages; for all adults at the school; for visitors and guests;

Connection: across grades; across mathematics courses; across subjects; and

Joy: creative; expressive; mathematical.

Among the suggestions for a future, full-unit implementation of math trails was the specific student-generated recommendation to involve passersby in the mathematical features of our school by posting QR codes that linked to the problems posed by participants. As one student wrote, "My first idea to continue creating/implementing the Math Trail at [our school] was to create QR codes that correspond to each problem at each stop.

It would be an easy, paperless/ecofriendly way for people on tours, in the hallways, etc to access each problem and participate in it. I think that it would be interesting if we had someone submit their answers like they do in The Riddler (https://fivethirtyeight.com/tag/the-riddler/) that gets released every Friday."

As to the variety of mathematical problems posed in the sample slide, one can see different types listed in Figure 1. The first question is an open-ended, no-wrong-answers approximation question. The second is of a similar nature, but asks about a feature of graduated classes from the past: specifically, that young women at our school in earlier times wore gloves in their graduation photographs. The intention behind the formulation of this problem was not only to engage in estimation or approximation, but also to provide a potential inroad for asking further questions around the history of our school, in particular, and other questions (around girls schools, fashion, women's rights movements, etc) more generally. The third question is an applied mathematics question: it asks where to place subsequent graduation photos, which was a real-world consideration at the time for the school. Finally, the fourth question can be classified as a pure mathematical problem: Although it corresponds to a real-world feature insofar as it references a staircase in the school, it is a somewhat contrived formulation of a question adapted from an earlier problem set in which the Fibonacci numbers arose (https://playwithyourmath.com/2017/07/27/7-step-up/).

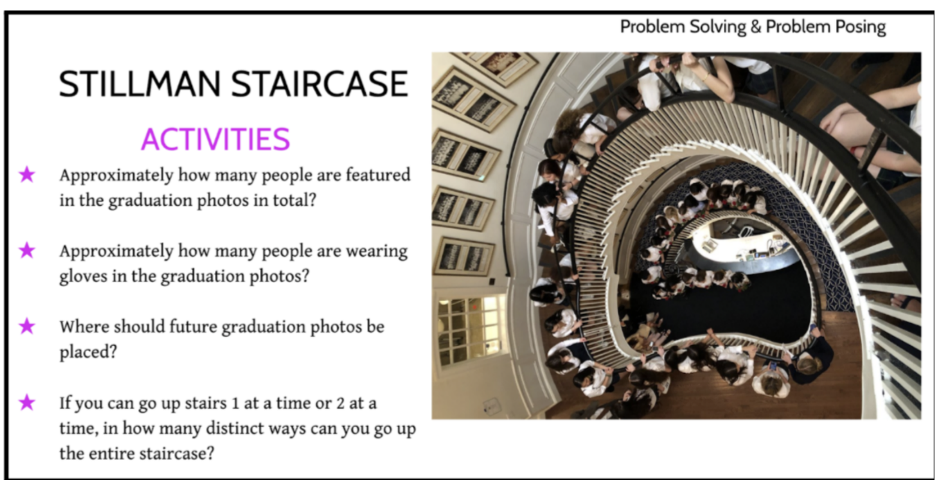

Figure 1: Sample slide from Spring 2019 Math Trail unit.

These trail items, along with others that included observations of shapes related to our school logo, windows, various light fixtures, and how to graph two-dimensional versions of these in Desmos were generally well-received by the faculty, and the stage was set to continue with a deeper dive into math trails for the following school year.

2.2. COVID Interference

In the subsequent academic year of 2019-20, the intention was to broach original math trails in the Spring after completing an earlier unit during which students read about Sidewalk Math in a blog post [16], an excerpt from the *Math Trails* book [17], and Walker's article [22] on "Cultivating Mathematics Identities In and Out of School and In Between" in the *Journal of Urban Mathematics Education*. This was the year in which Julia was a senior at the school and enrolled in one of the three sections of the Problem Solving & Problem Posing course. Although we, as a class, were able to visit a nearby park with ample foot traffic and engage in Sidewalk Math, the sudden transition to distance learning as a result of the March 2020 onset of the COVID pandemic prevented us from discussing math trails while located in a single, shared physical space.

The math trail unit from this time period is discussed from the student perspective by Julia in the subsequent section. The student-generated items included, rather than markers along a single trail to be followed, a collection of mathematized spaces and experiences that capture the particular experiences of high school seniors spending their final semester physically apart due to the acute threat of a pandemic. Asked to provide insight into the daily sights and sounds of Spring 2020, the students created and collated individual items into slide shows that serve as a reminder of what life was like for so many at the unfolding of a catastrophe that has yet to conclude. Topics included optimizing space for Zoom-based dance instruction and the acoustics of pots and spoons banged together for nightly celebrations of first responders and health care workers — these items, and their back story, are delved into by Julia. Additional examples included card games played with family members, jigsaw puzzles, book reading, and other ways to entertain oneself indoors, such as the trajectory followed by a rubber band repeatedly shot across a room with the goal of landing it inside of a cup. The math trail goal of seeing mathematics everywhere, after having successfully engaged in Sidewalk Math at a nearby park only weeks earlier, juxtaposed sharply with the work of students who, even after being encouraged to get fresh air and spend time outdoors, produced such items as "A Typ(e)ical Day in Quarantine" (Figure 2) in which the photograph is of a student's laptop, and the problems posed relate to letter combinations.

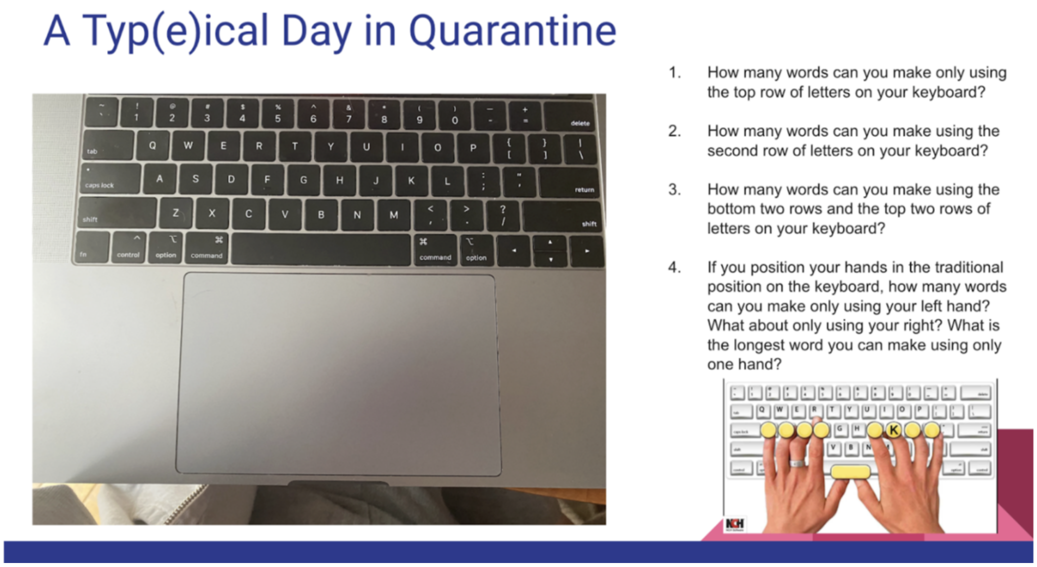

Figure 2: An example of a Spring 2020 student-created math trail item.

Although an item such as the one in Figure 2 may indicate a lack of fidelity to math trails in their original formulations, it faithfully represents the inner-worlds of resilient young people who somehow continued to complete mathematical coursework amid a daily deluge of death, disease, and the unknown. The banality of a lived experience reduced to staring at a keyboard or repeatedly firing a rubber band across a room exemplifies both the privilege and tragedy of secondary school culminating with days upon weeks spent quarantining indoors.

2.3. Virtual Teaching

In the 2020-21 academic year, Benjamin taught the Mathematical Problem Solving & Problem Posing course entirely virtually. Due to this physical separation, the assignment continued to involve mathematizing a part of one's life rather than a physical trail, but this time there was an added component in which students were asked to connect their mathematization to the real world by checking whether there was anything related through a search on congress dotgov. Moreover, the final assignment asked students to identify their House representative, as well. The prompt given as a sample and designed by Benjamin is presented in Figure 3.

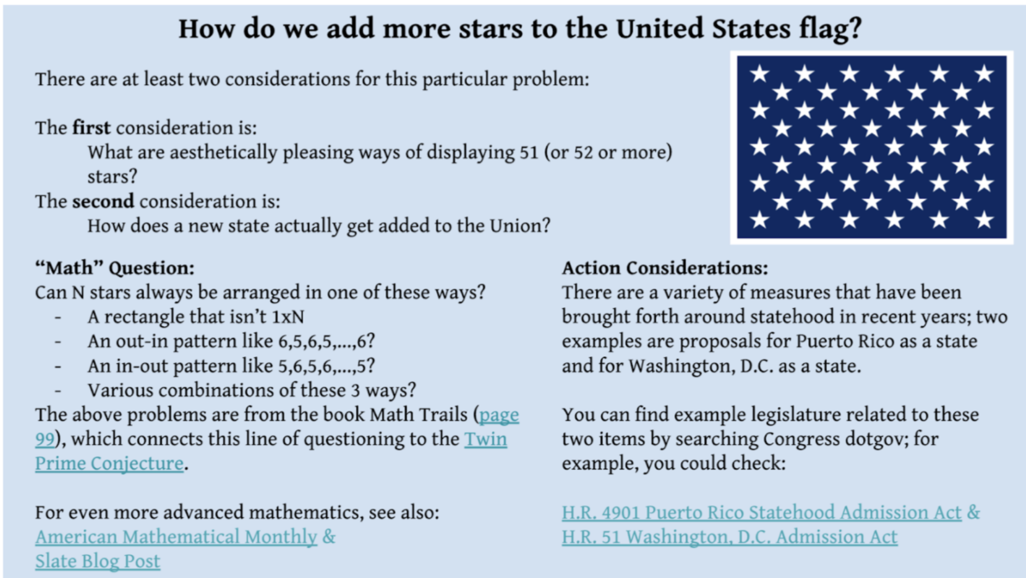

Figure 3: Benjamin's 2020-21 math trail item about star arrangements.

This item was borne out of the example of star arrangements in which the geometry of various patterns was considered [17], but the hypothetical of a state added to the union was presented without any further context. The *Monthly* article [11] and a blog post through *Slate* (`https://slate.com/human-interest/2010/06/13-stripes-and-51-stars.html`) were included as related mathematical examples in which the context of star arrangements was not extricated from the realities of current statehood considerations. The intention behind this modification of the math trail inspired assignment was to support students in connecting the abstracted mathematics with real-world events. Students were not encouraged to support or oppose any pending legislature, nor to form an opinion on past, present, or future decision-making as pertains to statehood.

Figures 4, 5, and 6 in the following pages contain parts of an individual student-generated trail item from 2020-21, in which the student presented data with a description; posed mathematical wonders; and contextualized the wonders with respect to relevant legislation. (Additional work from the same student included further reading and analyses related to wages and wage gaps at the intersection of race and gender.)

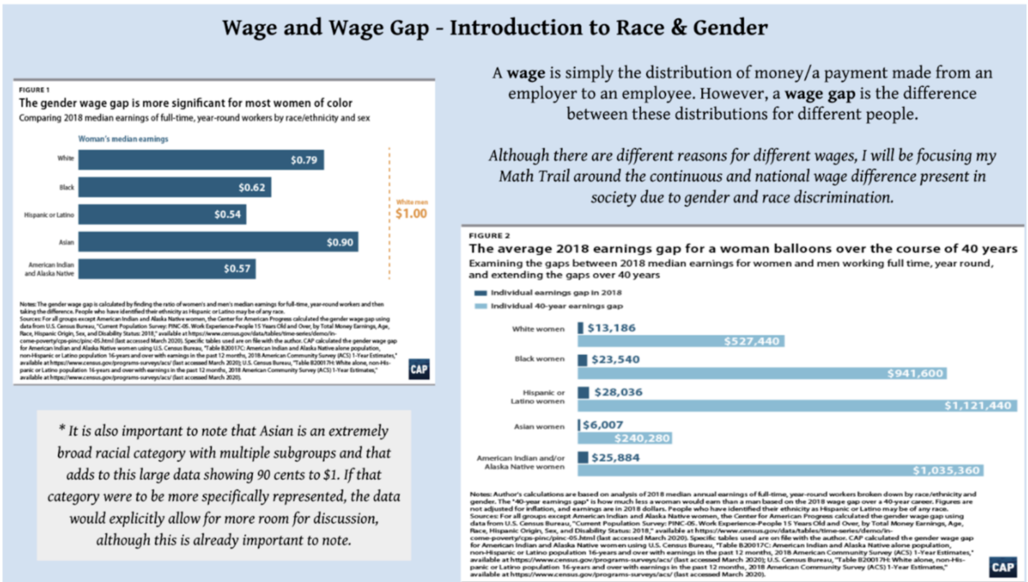

Figure 4: A student's math trail item during our 2020-21 virtual course.

Different Mathematical Approaches

Go back to Figure 1. As you can see, there is a dashed line along the right of the graph labeled: "White men $1.00." Consider the scaling and representation of the data.

The subheading of the graph says "Comparing 2018 median earnings of full-time, year-round workers by race/ethnicity and sex." However, the graph does not necessarily do this. The graph compares the "2018 median earnings of full-time, year-round" women workers by their race/ethnicity and uses white men's average of $1.00 as the benchmark. To accurately represent what its description suggests, the graph should include the median earnings of white men as a bar itself. This can raise questions for the message the figure is trying to send. Here are some possible questions:

1) Are women of color underpaid and $1.00 is the "just" median earning? Are WOC supposed to work up to that?
2) Are white men overpaid? What does it really mean to be "under" or "over" paid?
3) If white men are overpaid, how do we determine what the average salary/median earning should be?

Regardless of how you answer the questions above, also consider Figure 2 and the message it sends of the pay gap within these individual groups of color. Note also that the article repeatedly says "men" instead of a more specific term like "white, non-Hispanic men." What narrative does this support and/or normalize?

Figure 5: A student's corresponding mathematical wonders from 2020-21.

Context in Wonder - Intersection of Race & Gender:
Role & Involvement of Congress

After some research and looking through Congress.gov, I found these:

1. H.R.7 Bill: Paycheck Fairness Act; Introduced for Steps of Legislation; Passed House
2. H.R.2039 Bill: Fair Pay Act of 2019; Introduced for Steps of Legislation; Introduced to House
3. H.Con.Res.30 Concurrent Resolution; Introduced; Recognize significance of equal pay and the disparity between wages paid to men and women
    a. H.Con.Res.76 H.Con.Res.122; disparity in wages paid to Latina women in comparison to men
    b. H.Con.Res.59; disparity in wages paid to Black women in comparison to men

Figure 6: Student-identified congressional considerations for the trail item from Figures 4-5.

Although the inspiration for this assignment was the discussion of star arrangements with its lack of context in the book on math trails, the assignment continued to represent the mathematizing of individual student wonders as they sought to read and write the world with mathematics [8] and connect their thinking with easily located legislative realities.

3. Moving Beyond Classroom Walls

Despite the original wish by students in Spring 2019 for future students to mathematize a physical trail at our school, the realities of the COVID pandemic forced the next cohort of students to mathematize their immediate surroundings during a time of quarantine in Spring 2020, and the fully virtual delivery of the course in the subsequent year led to students diverging from a physical trail and segueing back into a more general form of mathematization as related to society — along with congressional connections — in Spring 2021. In this section, we discuss two ways in which the ongoing work that originated in math trail investigations has made its way out of the mathematics classrooms: first, how the modified math trail assignment that included legislative queries evolved in a student-led social justice club; second, how Julia was able to engage with "social justice math trails" as a part of her senior capstone project in Spring 2020, and continue to extend her thinking thereafter.

3.1. ACTION: Reaching out to Representatives project

Beginning in Spring 2021 and continuing through the 2021-2022 academic year, our school's social justice club, ACTION, engaged in the "Reaching out to Representatives" project. Benjamin served as one of the two faculty advisors for this club; the student whose work is included in Figures 4-6 served as one of the two student leaders for this club.

As the idea of identifying one's representative percolated from our mathematics class to the social justice club, club members formalized their approach to moving a step further. In particular, they created a Canva-based presentation, from which an introductory video still is visible in Figure 7, and included sample materials to facilitate this process for members of our school community. Club members gave this presentation to each advisory (akin to a homeroom) for students in grades 9 to 12, and presented their work through a Zoom-based full faculty and staff meeting, as well. The presentation contained scripts, tips, and video-based content that included an actual call by one student to her representative.

After the original presentations to students and faculty/staff members of the school, the project was pushed further the next fall by incorporating the QR codes initially suggested by the 2018-19 Problem Solving & Problem Posing cohort. A variety of clickbait-style flyers were created and posted throughout the school with a QR code that allowed students to access the Reaching out

Figure 7: Screenshot from the ACTION club's Canva presentation.

to Representatives project. This project included not only the Canva-based information and encouragement around identifying and calling one's House Representative, but also surveys that determined what school members were interested in, and follow-up information around committing to call one's representative to advocate for a specific legislative issue. Although no grown ups advocated for or against particular legislation as a part of the ACTION club, Benjamin was successful in encouraging his representative to adopt one of the items supported by the American Mathematical Society[1], namely, House Resolution 6836 the Robert Parris Moses Congressional Gold Medal Act,[2] which asks congress to award a posthumous gold medal to civil rights activist and Algebra Project founder Bob Moses. In particular, Benjamin brought this resolution to the attention of his representative, who agreed to join as a cosponsor, and whose press secretary subsequently agreed to visit the school and speak directly with the ACTION social justice club about her experiences and answer any questions that they had. Securing this cosponsorship is evidenced by the Representative's direct response to Benjamin on `Twitter`; a screenshot of this virtual exchange is contained in Figure 8 on the next page.

3.2. Senior Capstone Project: Social Justice Math Trails

[Here, Julia writes about her experience and thinking as a high school senior.]

In the Spring of 2020, each high school senior enrolled in Problem Solving and Posing, including me, was tasked by Benjamin with the formulation of a "math trail" item within and with regard to the physical environment and situation they were inhabiting. In the near past, the entire cohort of students enrolled in the course physically attended school and were living in Manhattan, but in the wake of COVID, the majority of the cohort had left their primary residences in NYC and relocated outside of the five boroughs. As a result, a student problem poser had to grapple with how the particulars of her immediate living environment, the specifics of her residential area, and the personal extent to which she can safely observe or access her extra-residential surroundings might manifest in the trail item she poses, differ from

[1]See `https://www.ams.org/government/getinvolved-dc#/` for this and other current action items.

[2]Information about the bill can be found at `https://www.congress.gov/bill/117th-congress/house-bill/6836`, last accessed on July 13, 2023.

Figure 8: Representative Carolyn Maloney tweets cosponsorship for HR 6836 in Spring 2022.

the corresponding particulars of a peer attempting to solve the trail item, and interfere with the peer's ability to engage with the trail item organically, meaningfully, or completely.

Grappling with the possibility of mismatch between the environmental particulars of problem posers and solvers proved challenging for the cohort and gave way to conversations between me and three of my peers in which each disclosed a concern specific to their living environment and demonstrated that developing a student-forged math trail can engage a student's critical consciousness [4], critical mathematics consciousness [10], and natural but repressed and discredited impulse to mathematize for human flourishing [19].

Exploring her townhouse in search of a site for a trail item, Peer 1 discovered a propensity to perceive the mathematical potential of two things:

(1) aesthetically beautiful and geometrically complex art, architecture, and interior design, and

(2) the prices of already purchased items or items in need of purchase.

Peer 1 noted that sites around her house of aesthetic beauty and geometric complexity could be made accessible to peers by a photograph, but she felt as though presenting these sites as two-dimensional, static photographs would cause the sites to incur arbitrariness, which led her to consider the possibility that the sites themselves are arbitrary for any non-resident of her house. When speculating about peer problem solvers' experiences of problems emerging from the prices of purchases specific to her household, Peer 1 felt certain that any such trail item would rival textbook word problems' degree of separation from the real and realistic economic considerations of problem solvers in their day-to-day lives as well as uncomfortable at the prospect of portraying the ways in which her household exercises its socioeconomic privilege.

Peer 1's discomfort emerged out of her worry that portraying her socioeconomic privilege would place her on the receiving end of judgment by her peers, but upon voicing this discomfort and its origins to me and positioning a potential peer problem solver as an interlocutor rather than a potential critic, she developed and expressed concern for how portraying her socioeconomic privilege through a word problem, a medium generically troped as heeding normative frameworks or indicating divergence, could encourage a peer problem solver to apply a scarcity mindset to their own socioeconomic situation.

Student-forged math trails incentivize mutualism and empathy between student problem posers and student problem solvers not only because each student fulfills both roles but also because students will bear witness to the problem solving experiences that their trail items produce. Once Peer 1 ceased to conceive of her trail item as an authored artifact and began to view it as a dynamic script awaiting a peer problem solver's enactment and interpretation, she activated her critical consciousness. She exchanged her fear that a peer's class consciousness might lead to critique of her for fearing that she might cause a peer undesirable shame by propagating a narrative that could stoke desire to emulate her socioeconomic privilege.

Like Peer 1, Peer 2 worried that the situation she had identified as mathematically intriguing — the act of banging pots and spoons outside her window each night at 7:00 p.m. to cheer on first responders and healthcare workers — would seem arbitrary to the majority of her peers who were living outside of NYC and in areas where pot and spoon banging did not achieve pertinence through the celebratory custom. She felt comfortable assuming that the entire cohort of students enrolled in Problem Solving and Posing would have access to pots and spoons, but she felt equally confident that her peers who did not partake in this celebratory custom would not want to mathematically investigate their pots and spoons by banging them and using a decibel measurement application to identify which combination produces the loudest noise. Such a trail item might lead these peers to ponder the physics of acoustics with a flittering eye towards the physicality of pots and spoons, but it will not effect a problem solving experience that notably exceeds the real-worldliness of most high school physics problems and labs. But unlike Peer 1's discarded math trail items that dealt with purchase prices evidencing a degree of socioeconomic privilege unshared by most of her peers and purchased items extraneous to the lives of all of her peers, Peer 2's pots and spoons trail item represented a living context unshared by most of her peers because they had exercised their greater degree of access to socioeconomic privilege to flee the trail item's geographic context and the preponderance of COVID cases and deaths therein.

Peer 2 expressed to me her concern that her math trail item would insufficiently engage the majority of her peers and prompted me to defend the item. I had remained in NYC where I feared for the health of my immediate family, which included my sister, an ER doctor, and greatly benefited from the 7:00 p.m. opportunity to participate in a communal expression of gratitude, promotion of hope, and recollection of its vibrancy. My experience of the Spring of 2020 caused me to recognize the pots and spoons trail item's potential to enrich the playfulness and acoustic effectiveness of the celebratory custom for students who remained in NYC, braving greater health risks and reductions to their freedoms. Moreover, my experience of the Spring of 2020 facilitated my awareness of the injustice that results from suppressing mathematical questions, considerations, or problems perceived as pertinent to only a minority of students. I told Peer 2 that I believed discarding the trail item would erase from the math trail the experiences of students who lacked an alternative to remaining in NYC and conceal joys unique to remaining in NYC that countered the scarcity narrative that prevailed within the cohort.

Formulating this belief required me to explicitly engage my critical mathematics consciousness and its ongoing processes "involv[ing] interrelated development of sociopolitical understanding, critical civic empathy, and taking action" [10, page 782] expressed and engendered by mathematics. I understood that the privileged backgrounds of the students in the cohort prefigured the basis of their sociopolitical understanding of the communities they populate and communities to which they are extrinsic, which enabled me to anticipate the propensity of the cohort's trail items to cater to the upper quartiles of privilege within an already privileged cohort as well as the predisposition of the cohort to normalize such a propensity. Questioning whether one should apply regression-toward-the-mean logic to a cohort of eighteen high school students of privileged but varied backgrounds moved me to exercise critical civic empathy for peers vulnerable to characterization as negligible deviants from the mean, which culminated in my defense of Peer 2's trail item. Peer 2 agreed with my defense and decided to proceed with the trail item. Further, Peer 2 proceeded with the trail item with the intention of forwarding a counternarrative to the scarcity narratives surrounding NYC in the Spring of 2020 and produce varied problem solving experiences that variably involve play, mathematical and physical truths, societal truths, desire for justice in the face of COVID, and apprehending injustice. Thus, the explicit engagement of my critical mathematics consciousness impelled Peer 2's critical mathematics consciousness. Participation in a student-forged math trail compelled Peer 2 and me to consider jointly our roles as problem posers and problem solvers, which resulted in the authorization of a math trail item prioritizing problem solvers with whom we shared situational similarity and the threat of neglect by peers' trail items.

Peer 3 shared with Peer 2 a desire to use mathematics as a tool for adapting to her COVID context. The inaccessibility of a dance studio and the insufficient space of her NYC apartment (as then configured) for dancing prevented Peer 3 from practicing her passion, which exacerbated the challenging and unpleasant emotions ushered in by the Spring of 2020. Peer 3 had a psychological need to resume dancing, and she wanted to address this need through her math trail problem involving a hand-drawn floor plan of her bedroom and its furniture accompanied by a prompt to produce a furniture arrangement that optimizes the amount of space she has for dancing without rendering her bedroom dysfunctional in its capacity as a bedroom. Due to its many considerations of functionality that evade algorithmic instruction, its fundamental abstraction from the physically involved and experimental realities of furniture rearrangement, and its hyper-specific particulars, Peer 3's trail item makes minimally manifest the algebraic, geometric, and spatial-dynamic realities of the problem scenario and instead encourages problem solvers to repeatedly and arbitrarily reconfigure the furniture and compute the area of the largest contiguous stretch of floor within the arbitrarily derived floorplans. Peer 3 determined that her trail item achieved minimal accessibility for peer problem solvers, but she had no desire to supply her trail item with imprecise but utilitarian constraints nor did she wish to limit her trail item to a list of rules for rearranging a space that peer problem solvers could apply to a physically accessible room. She wanted to maintain her trail item as it was, despite its issues of untranslatability, because the real-world problem, Peer 3's inability to carry on dancing and reaping the psychological benefits, demanded her attention, and, in intertwining a mathematical problem with her real-world problem, answered the call of the real-world problem. Peer 3 recognized the theoretical irreconcilability of a math trail item mathematically cogent and realistically germane solely for its poser, and I confirmed this irreconcilability while seconding Peer 3's conviction that she should not discard or radically alter the trail item to the detriment of her real-life flourishing.

COVID drove students in the cohort to necessarily dissonant contexts and environments that substandardly positioned Peer 3's impulse to mathematize for her own flourishing for the purposes of producing a trail item attendant to the flourishing of peer problem solvers. Nevertheless, Peer 3 indulged her impulse and made me recognizant of the vitality of the human instinct to mathematize for human flourishing, which led me to suspect the culpability of mathematics education and curricula in the repression and discreditation of students' sensitivity to the presence of mathematical considerations in real-world problems without explicitly mathematical presentations, their willingness to make these considerations, and their drive to derive real-world solutions from mathematical ones. Overall, through these three conversations with my peers, I witnessed the capacity of student-forged math trails for engaging students' critical consciousnesses, critical mathematics consciousnesses, and mathematizing tendencies. To me, this proved their value as a method of exit from the hegemony of decontextualized, abstract mathematics and as an instrument for redistributing the power to mathematize.

Traditional math trails have six basic characteristics. They are (1) for everyone, (2) cooperative, not competitive, (3) self-directed, (4) voluntary, (5) opportunistic, (6) and temporary; see [17, pages 8–10], where a paradigm of math trails is outlined that focuses on ensuring the agency of trail walkers. Such a paradigm, however, does not describe the student-forged math trails I witnessed and participated in. In my senior capstone project, I set forth six tenets for a paradigm of student-forged math trails (SFMT) that prioritize the agency and experience of student trail forgers:

I. SFMTs encourage students to claim the freedom to redefine what is worth mathematizing and empower students to generate problems that counter-narrativize and interject;

II. SFMTs optimize the relevance of students' pre-existing social, cultural, political, and experiential knowledge;

III. SFMTs work towards developing students' understanding that math does not operate neutrally and emerges from sociopolitical, cultural, and deeply human practices;

IV. SFMTs center students and their lived encounters with mathematics in the classroom;

V. SFMTs create an opportunity to engage multiple senses in the practice of mathematics, thereby minimizing abstractions and repositioning math as existing within students' worlds;

VI. SFMTs welcome students to restructure a hierarchy of mathematical knowledge based on relevancy and applicability.

These tenets intentionally exclude "for everyone" and associated calls for collectivism because all contextualized math problems reflect biases toward their structural components (i.e., mathematical structure, semantic structure, context, and format (compare with [21]) and their resultant processes of mathematization and narrativization. Moreover, "for everyone" risks marginalizing non-dominant contexts in the name of inclusivity, with the result of exclusion. In addition, these preliminary tenets of student-forged math trails disregard "cooperative, not competitive" because "cooperative" connotes cohesion between individuals and their problem solving or problem posing cohort, which one cannot assume. Furthermore, "cooperative" implicitly advocates for sacrificing the self to the collective, which hazards silencing deviant voices and ways of thinking.

4. Closing Thoughts

[Here Julia briefly discusses the evolution of her thinking around math trails, which she initially encountered in Spring 2020 and which formed the focus of her senior capstone project, in the three years that have since elapsed.]

In my senior capstone project, I defended the six tenets I proposed using the practical and pedagogical literature of social justice mathematics. Having mounted my defense, I stood by the value of the tenets but questioned the value of their prescription. I returned to "The Kay Toliver Mathematics Program" and realized that, in their most liberated constructions, math trails are a form of storytelling; math trails reflect how their forgers navigate the world and map the complex ways in which students' lives braid together. Thus, a singular central tenet of student-forged math trails emerged: narrative. Shifting students' communicative goal from solution retrieval to construction of greater mathematical meaning engages metacognition, which facilitates access to latent stores of knowledge [14]. If narrative becomes students' communicative goal when mathematizing, students can access latent solutions for re-humanizing math for themselves as well as the student trail walkers and forgers who follow.

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