Journal of Humanistic Mathematics

Volume 14 | Issue 1 January 2024

Sharing Four Biscuits Between Three People: An Illustrative Example of How Mathematics is Intertwined with Human Values

Lovisa Sumpter
Stockholm University

David Sumpter
Uppsala University

Recommended Citation

Lovisa Sumpter & David Sumpter, "Sharing Four Biscuits Between Three People: An Illustrative Example of How Mathematics is Intertwined with Human Values," *Journal of Humanistic Mathematics*, Volume 14 Issue 1 (January 2024), pages 74-93. DOI: 10.5642/jhummath.KJEO2817. Available at: https://scholarship.claremont.edu/jhm/vol14/iss1/6

JHM is an open access bi-annual journal sponsored by the Claremont Center for the Mathematical Sciences and published by the Claremont Colleges Library | ISSN 2159-8118 | http://scholarship.claremont.edu/jhm/

Sharing Four Biscuits Between Three People: An Illustrative Example of How Mathematics is Intertwined with Human Values

Cover Page Footnote

Lovisa Sumpter’s part of the research was funded by the Swedish Institute for Educational Research (Skolforskningsinstitutet), project 2021-00068.

Sharing Four Biscuits Between Three People: An Illustrative Example of How Mathematics is Intertwined with Human Values

Lovisa Sumpter

Stockholm University, Stockholm, SWEDEN
lovisa.sumpter@su.se

David Sumpter

Uppsala University, Uppsala, SWEDEN
david.sumpter@it.uu.se

Abstract

Despite convincing arguments by mathematicians, philosophers, sociologists and machine learning practitioners to the contrary, there remains a widespread notion amongst many members of the general public (and some practitioners) that mathematics is neutral, that it is free from human values. One reason why this notion persists is that we lack clear-cut examples that demonstrate how mathematics and values are intertwined. In this paper, we offer one such example. In particular, we show that when sharing four biscuits between three people, several possible mathematical and ethical frameworks can be used. We demonstrate that different solutions—hiding one biscuit, arbitrarily sharing the extra biscuit, randomizing allocation, dividing the extra biscuit into three parts, and successively dividing it into smaller and smaller parts—involve different mathematical methods and evoke different human values. We discuss the construction of quantum biscuit splitting devices and the use of machine learning to divide biscuits. We argue that the multitude of different mathematically-correct solutions to this problem (each with its own ethical justification) might influence the values held by practicing mathematicians. The example we propose here has been used in teaching to help students understand why mathematics cannot be cleanly separated from human values.

1. Introduction

There is a widespread notion—among many members of the general public, but also among some practitioners—that mathematics can offer objective, fair, or neutral solutions to problems, and that mathematics is free from human values [19, 20]. For example, if we were asked to share three biscuits between three people, for many people, all else being equal, the fair answer to this problem is to give one biscuit to each person. This is also the mathematical answer of the division $3 \div 3 = 1$.

Notions of objectivity and neutrality are often extended to statistics, mathematical modeling and artificial intelligence (AI). For example, age-estimation based on neural networks has been suggested as an "objective" way of judging the outcome of plastic surgery [14]; machine learning has been suggested as a way of "overcoming human bias" in decision-making in patient care [9]; and sentencing in criminal cases is viewed as suffering from "excessive subjectivity, inconsistency, bias and potential stereotyping" in comparison to algorithmic approaches [8].

Such thinking is misguided for several reasons. Even in apparently straightforward questions of sharing, many cultures do not view strict mathematical division as fair. For example, when third-grade students in Liberia were asked to share \$45 between three children, they typically assigned \$20, \$15 and \$10, with the most money going to the oldest child [33]. Similarly, when children are asked to share biscuits between soft toys, they use a variety of notions of "fairness" in their reasoning [18, 26, 35]. On the societal scale, inheritance from parents to children in many Western countries creates inequality in society [24], but is usually considered fair. Indeed, what is considered fair can then be based on a variety of factors, especially when we include the recipients' wishes [34].

In the context of machine learning and AI, several recent studies have emphasized the impossibility of fairness [1]. This is captured in the title of the Dwork article "Fairness through awareness" [17]. These authors show that in classification problems (e.g., criminal sentencing or university admissions), a decision in different demographic groups (e.g., different ethnicities or genders) cannot simultaneously respect both statistical parity (where the demographics of those groups that are sentenced or recruited are the same as those of the underlying population) and the principle that every individual should be treated equally. Fairness is itself value-laden: by solving the problem on the basis of one fairness criterion, we are being unfair to another.

Arguments as to why mathematics is not objective or value-free come from a range of different disciplinary approaches throughout academic inquiry. Philosophers, for example, have pointed out that objectivity cannot be demarcated from subjectivity because the sense data on which we base our decisions necessarily comes from a subject who has his or her own way of looking at the world [28]. Similarly, we cannot even clearly separate empirical questions from mathematical questions [31]. In the context of public policy, Douglas argues that by claiming to adopt value-free objectivity in scientific work, we are only masking or hiding the decisions we have made about values [15, 16]. She concludes that scientists should acknowledge that values must "play in scientific reasoning, while not allowing values to supplant good reasoning" [16]. Statisticians Gelman and Henning encourage fellow researchers to use values such as transparency, consensus, impartiality, and correspondence to observable reality instead of objectivity and awareness of multiple perspectives and context dependence instead of subjectivity [22]. These all become explicit "virtues" (to use Gelman and Hennig's own term), which a practicing statistician should, they believe, follow. As part of a course taught at Cambridge University, Chiodo and Bursill-Hall [10] provide case studies—ranging from financial services and cryptography to targeted advertising and applications of AI—highlighting the ethical dilemmas arising from the mathematics learned by students during their undergraduate degrees.

Despite sociological, engineering, mathematical, and philosophical arguments against the value-free objectivity of mathematics, the notion still persists in some places. For example, an article by financial mathematician John Armstrong in the *Spectator*, accompanied by a letter co-signed by several eminent mathematicians, criticized "the sinister attempts to 'decolonise' mathematics", by objecting to the Quality Assurance Agency for Higher Education proposal to teach students that some "mathematicians had connections to the slave trade, racism or Nazism" [3]. Armstrong argues that mathematics should not be diluted with lessons on "diversity, sustainable education, and entrepreneurship." His message is that mathematics and social values are separate and should not be mixed.

Within scientific research, Armstrong's viewpoint is (thankfully) isolated. However, there is a tendency of those working in mathematics, statistics, and machine learning to ignore the role of values in their subject areas. To tease out these hidden values, Birhane and others [6] analyzed a hundred highly cited papers from two of the most important machine learning conferences. They found that the articles emphasize performance, generalization, quantitative evidence, efficiency, novelty, and very seldom (in only 1% of articles) mention negative potentials of machine learning. Chiodo and Bursill-Hall highlight that the modern economy works in ways that hide the ethical context of decisions and that encourage disengagement from moral reasoning [10]. Alayont [2] notes that the American Mathematical Society (AMS) ethical guidelines lack depth compared to that of, for example, actuaries and engineers.

One persuasive argument sometimes put forward to defend a lack of engagement in values and ethics is that mathematics is neutral once the problem is fully specified. In the fields of machine learning and economics, reference is made to an objective function (the meaning of the word "objective" refers to meeting a specified objective, not explicitly to objectivity). In the context of some of the examples provided at the start of this introduction, the objective function argument says that these problems were not properly specified, because we did not specify what we considered fair. For example, if among Liberian school children it is considered fair that the oldest child receives the largest share, then a \$20, \$15, and \$10 sharing solution becomes a correct solution to that objective function. Similarly, Dwork and others [17] propose that, for problems concerning university admissions, setting a clear objective function of "fair affirmative action" can allow us to circumnavigate the problems they identify from complementary fairness goals. This type of approach has led to the suggestion that bias, once identified, can be removed from data sets and from models (see, for example, [7]). Under this argument, mathematics thus recovers its neutrality once the problem is properly specified.

There are problems with these (somewhat weaker) claims of neutrality too. In the context of using data in machine learning, Birhane [4] criticizes how the dominant view (embodied in the chosen objective function) is taken to be a neutral "view from nowhere," so that questions of privilege and oppression become issues with which data scientists and mathematicians need not concern themselves. Indeed, defining an objective function for college admissions or criminal sentencing involves adopting a view that says that categorizing people using statistics is morally acceptable (see [1] for more discussion of this point). The specification of the objective function can itself be thought of as an act of disguising a value-judgement—i.e., that human beings should be evaluated using numbers—in a cloud of apparent objectivity.

Green [23] criticizes how the movement to do "social good" in computer science proceeds with small, technology-centric reforms, without reflecting on what doing "good" in society actually entails. Even with an objective defined, what is the "good" of optimizing criminal sentencing or college admissions in the USA, given the racial injustices and social inequalities upon which these systems have been built?

Ernest [19] argues forcefully that the very act of learning mathematics shapes our thinking in an (apparently) ethics-free and amoral way, which then supports instrumentalism. The impression that mathematics is ethics-free is then exploited in warfare, corporate activity, exploitation of humans and mismanagement of the environment. For Ernest [19] this approach results in the treatment of persons as objects rather than as moral beings. Whitaker and Guest [38] give a concrete example of such instrumentalism in the form of what they refer to as, #bropenscience. They ask why, for example, reproducibility of experimental results in psychology (a scientific trend which arose from statistical observations of problems with p-values when testing hypotheses) "is placed as central [to open science], while sexual assault that closes off academia to too many women is not?"

We are sympathetic to the position of Birhane, Ernest, Guest and others. But we also believe that such arguments might be aided by illustrative examples of how readily human values and mathematics become intertwined. Ernest's argument, for example, links together capitalism, ethical values, mathematical education, corporate management and many more aspects of society in a way which makes it hard to pinpoint where precisely mathematics (as opposed to human thought in general) is problematic [19, 20]. When Douglas [16] points out that claims of value-free objectivity are really hiding the values expressed, she does not tell us where and how the values are hidden. Similarly, the warnings in Abebe and colleagues [1], that we should not forget that there are many ways of viewing problems, do not explicitly show where the limits of the computational viewpoint lie. And calls to decolonize mathematics and computer science (see, for example, [5]) might be seen as too vague and general, without concrete examples of how mathematical modeling choices inform our values and visa-versa.

It is exactly this type of concrete example, of how our mathematical view is related to values, which we aim to provide in this article.

Some previous work addresses similar problems. Most notably, one potential framework is encompassed in what are sometimes called *wicked problems* [25]. These are problems which are loosely formulated, it is unclear when a solution has been reached, and a lot depends on the viewpoint of those presenting them [11]. The term is particularly important in explaining why, for example, sustainable development problems cannot be solved by a single approach [30]. Devlin [13] recognizes that mathematics can contribute to solutions of such problems but rarely yields the solution on its own. He goes on to describe steering a bike, which involves a counterintuitive swing to the right in order to turn left, as an example [13]. However, there has not been an extensive investigation of what might be defined as wicked problems in mathematics.

One well-known dilemma that does relate values and mathematics comes in the form of Foot's [21] trolley problem. Jarvis Thompson [37] describes the dilemma as follows:

> Suppose you are the driver of a trolley. The trolley rounds a bend, and there come into view ahead five track workmen, who have been repairing the track. The track goes through a bit of a valley at that point, and the sides are steep, so you must stop the trolley if you are to avoid running the five men down. You step on the brakes, but alas they don't work. Now you suddenly see a spur of track leading off to the right. You can turn the trolley onto it, and thus save the five men on the straight track ahead. Unfortunately, Mrs. Foot has arranged that there is one track workman on that spur of track. He can no more get off the track in time than the five can, so you will kill him if you turn the trolley onto him. Is it morally permissible for you to turn the trolley? (page 1395)

Originally proposed to investigate moral dilemmas related to abortion, the problem succeeds in abstracting away from the specific issue, while still emphasizing the fact that a problem cannot just be solved by looking at numbers or assigning values to lives. Diverting the train is an active choice to kill the single workman, while letting the trolley roll is a passive acceptance of the fate of the five people already on the track. Thompson proposed variations of the original problem which bring out this dilemma even more clearly.

Would you, for example, push a person on to the track in order to stop the train (assuming that such a measure was guaranteed to work)?

There remain two limitations of the trolley problem when it comes to illustrating how values and mathematical thinking are intertwined. Firstly, the drastic nature of the outcomes draws attention very strongly to values and ethics. Secondly, the trolley problem does not emphasize mathematical modeling choices per se. The model is relatively clear—actively causing one person to die versus passively failing to save five—and there are few mathematical decisions to be made before providing an answer (or, as Foot and Thompson may have intended, accepting that there is no answer).

In this article, we discuss an everyday dilemma which, we will argue, illustrates how choices about how to model a system cannot be separated from social values. The example we choose derives from six sharing scenarios used in experiments by Sumpter and Hedefalk [35]. In the first scenario, children (ages 5-8 years old) were asked to share twelve "biscuits" (made of paper) among three recipients (soft toys). In the second scenario, the children were asked to share four biscuits amongst three soft toys with no further information about the soft toys. In later scenarios, designed to introduce wider forms of mathematical and ethical reasoning, background stories are provided about the soft toys making the sharing more complex. The soft toys have different needs such as being hungry, sad, or not having their own biscuit factory. The results of these studies showed that the children used ethical reasoning as a support for their mathematical reasoning, but also vice versa [18, 26, 35]. They were able to model the situation and adapt their reasoning with respect to how the context was interpreted. For example, in some cases bigger soft toys were given a larger share, but not if a smaller toy was in a greater need and that need had, according to the children, more weight than the difference in size. In some cases, fairness was viewed by the children as giving the exact same amount, independent of need, and they used ethical reasoning to argue for why division was the correct mathematical operation to apply.

We focus on this second scenario of sharing four biscuits amongst three recipients with no further information about the recipients. Our choice is made because the problem does not have an immediately obvious solution (like $12 \div 3 = 4$ as in the first scenario above), but also has not introduced any explicit ethical concerns (such as differing hunger levels).

Sharing four biscuits amongst three recipients provides an example where we might expect mathematical properties of the problem alone (and not values or ethics) to fully determine the solution.

In this article, we argue that even within this simple sharing problem, ethics and mathematics are intertwined. We first give six solutions to the biscuit problem based on five mathematical specializations (quotients, partitions, probability, geometry and the infinitesimal) and point to how each solution connects to particular social values. We then use an allegory to look at how discussions between mathematicians who specialize in one of these five areas will appear value-laden both to each other and to outsiders. We then zoom out and ask how more advanced mathematical methods, including quantum mechanics and machine learning, might influence our view of the problem.

2. Solutions to the biscuit problem

How can we share four square-shaped biscuits fairly between three people? We build on our earlier work [36] comparing divisible and indivisible solutions to this problem, to propose six distinct solutions; each uses a different mathematical framework and each gives rise to different value judgements. We give a human name to each solution in order to help represent that solution in discussions in later sections of the paper. Think of the name of the solution as the name of the person proposing the solution.

Quotient solution (Codie)

Words: Take one biscuit away (possibly 'eat it yourself') and share the other three.
Mathematics: 4 divided by 3 is 1 remainder 1. Only distribute the quotient to the people.
Values: Simplify the problem. Avoid conflict around the extra biscuit. Equality is more important than resource efficiency.

Arbitrary solution (Arby)

Words: Give an extra biscuit to one of the people, chosen according to an arbitrary criterion (such as how "cute the person is"), while sharing the others equally.
Mathematics: 4 is partitioned into 2+1+1.
Values: In situations where equality is not straightforward, it is acceptable to base decisions on whims. Right to choose more important than absolute equality.

Random solution (Randy)

Words: The extra biscuit could be distributed at random. For example, we roll a die: if it lands on 1 or 2, the first person gets it; if it lands on 3 or 4, the second person gets it; if it lands on 5 or 6, the third person gets it.
Mathematics: Probability that the extra biscuit goes to any given individual is 1 in 3. Expected number of biscuits is 4/3.
Values: Everyone has the same probability of receiving the extra biscuit. Equality in opportunity for the extra biscuit.

Gambling solution (Gabriel)

Words: All biscuits are distributed at random in the way outlined above, either individually or as a winner-takes-all outcome of a game.
Mathematics: The expected number of biscuits per person remains 4/3.
Values: A way of distributing the biscuits that involves gambling and gaming. Equality in opportunity for all available resources.

Geometric solution (Jenny)

Words: We cut the 'extra' biscuit into three equal-sized rectangles.
Mathematics: A square can be cut into three equally shaped rectangles. It is impossible to make the division so that the three smaller biscuits are square.
Values: It is the taste and not the shape of the biscuit that matters. Equality in outcome.

Infinitesimal solution (Ingrid)

Words: On the first round of sharing everyone gets one biscuit. We then cut the extra biscuit into four squares, of which everyone gets one. Then we cut the remaining quarter biscuit into four smaller bits, share out three bits. We continue cutting smaller and smaller squares.
Mathematics: The cut of the biscuit gives a quarter biscuit to each person. The sequence of cuts of the biscuit gives a share:

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.$$

Values: This method produces absolutely equal amounts of biscuit in the same shape, down to the last crumb, for all three people. All biscuit parts are squares (of decreasing size). Equality in outcome, while preserving the tradition of square biscuits.

3. When each person knows only one solution method

We now proceed by using an allegory. Imagine a world where each of six friends (Codie, Arby, Randy, Gabriel, Jenny and Ingrid) only understand the mathematical aspects of their own solution and see the others' arguments in terms only of value judgements. We now follow some of the potential discussions between them.

Ingrid has learnt about the geometric series, but Jenny does not understand it. What Jenny hears, when Ingrid tries to explain her method of cutting smaller and smaller squares, is that Ingrid is unnecessarily obsessed with squares. Because Jenny does not understand the geometric series, she feels a lingering doubt as to whether everyone will get the same amount. What will happen to the bits that are left over after we have made a lot of cuts?

For her part, Ingrid does not understand how Jenny accepts the use of rectangles in a problem that is about square objects. Just like Jenny does not understand geometric series, Ingrid does not understand that the area of a rectangle can be the same as the area of a square. So, for Ingrid, the doubt lies in the use of rectangles in a problem involving squares.

When Randy listens to Ingrid and Jenny, arguing about cutting up biscuits, he is confused. He understands neither area nor geometric series, and explains that all their problems can be solved by the throw of a dice. Ingrid and Jenny are shocked. To them, Randy appears to value equality of opportunity over equality of outcome! The two of them could at least agree on the importance of outcomes. Randy understands the distinction between opportunity and outcome (they all understand value-based arguments) but explains that since the expected value of the dice throw is one third, it guarantees fairness, unlike their methods, which guarantee that the last biscuit is destroyed according to some ritual that he has no insight into.

Gabriel sees Randy's solution as arbitrary. There are many different ways in which randomness can be used to share the biscuits. For example, one person could take all the biscuits, or each of them in turn could be distributed according to the roll of a die. Each of them will expect to get 4/3 biscuits at the end, so the process is fair, Gabriel claims.

Arby admits, like Gabriel, that some parts of this problem are simply arbitrary. He knows the 4 can be partitioned as $4+0, 3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. But the only partition suitable for three people is $2+1+1$. This is why Gabriel's method does not make mathematical sense, because it can produce $4+0$ or $3+1$. So, he agrees with Randy that one biscuit has to be assigned to someone, but the use of a dice is incomprehensible to him. He believes we should trust ourselves to decide who gets the last biscuit and be able to explain our choices. Gabriel, Randy, Ingrid and Jenny are shocked by Arby's solution. Unlike each of theirs, it does not follow a principle of any kind.

Codie looks at the five arguing friends. She knows that the quotient when we divide 4 by 3 is 1. She shares out the three biscuits, takes the extra one and eats it. Problem solved!

4. Increased mutual understanding

Even relaxing our assumption that the modelers cannot understand each other's approach, we can still envisage conflicts between them. For example, Ingrid might know how Jenny's geometrical approach works, but she may still feel that her experience with more advanced calculus techniques makes her solution more rigorous. Having an understanding of convergence of series is viewed as having a more advanced grasp of mathematics. This type of issue often comes up in discussions between practicing mathematical modelers: they prefer their own "advanced" method and can (often unconsciously) diminish the methods used by others as resulting from their subjective values: Ingrid thinks that Jenny accepts rectangles because she does not fully understand her method involving squares.

The debate between equality of outcome and equality of opportunity is another example of how such differences in mathematical background play out. Financial mathematicians and many of those working in machine learning are familiar with using probability theory to answer questions in finance and technology. For them Randy's solution, and even that proposed by Gabriel, might well appear fair. These solutions are also convenient and reflect an underlying reality about the world: much of what happens in finance and in life is random. It is not too much of a stretch to see how Randy and

Gabriel might see an economic system in which randomness plays a large role in determining an individual's success as fair.

Such opportunity-based thinking sits uncomfortably with both Ingrid and Jenny (despite their disagreements about exactly how the cutting should be performed). Their training leads them to emphasize deterministic solutions, which they know exist. Thinking probabilistically, for them, involves introducing new mathematical techniques that are neither needed nor effective in improving the situation. When everyone can get the same, it seems unnecessary to complicate things with randomness. To Ingrid and Jenny, the solution based on equality of outcome reflects how the world should be. This is something akin to the mathematics used in the accounting profession or in taxation systems. Fairness should be fulfilled down to the last cent or penny.

Arby can be viewed as a pure mathematician, who shows how 4 can be partitioned into $4, 3 + 1, 2 + 1 + 1$ and $2 + 2$, and then argues that only one of these partitions is fair, while pointing out that any further conclusion about this problem is impossible. It is reasonably common to hear pure mathematicians take the view that their job is only to provide an outline of the structure of the problem. After this is done, the division of the last biscuit is not their responsibility. To Arby, arguments concerning equality of opportunity and outcome simply do not belong in the world of mathematics.

Codie is a pragmatist. She wants to move forward and not listen to the other five argue. To Codie, arguments of this type are avoided by taking practical steps in advance of them occurring.

These discussions show the difficulty of separating values and mathematics. Differences arise between our modelers, not because they do not understand different ways of doing mathematics, but from a connection between the background and experiences of a mathematician and the types of values they appear to espouse. Some arguments will appear more intuitive to some people than to others, and they will tend to rely on those arguments. As a result, they will appear to adopt (and in some cases really adopt) the values associated with their favorite argument. Human values and mathematical thinking become intertwined in our approach to problems.

5. The all-knowing mathematician

The argument we have made in the sections up to now links mathematical arguments to values and backgrounds of the protagonists, then shows how conflicts arise because the protagonists fail to fully realize that others are also using a mathematical framework for their reasoning. An objection to our allegory-based approach is to first point out that all six modelers are carrying out valid mathematical operations, the ones we outlined earlier. Whatever we might think of the various arguments put forward by the modelers, when contrasted with ways of sharing the biscuits that are not supported by reasoning of any kind, these all have mathematical validity. We could then take the view that, provided the six protagonists all understand the relative merits of and logic behind their different ways of sharing the biscuit, they can collectively make a rational, informed decision.

This view can be characterized as follows:

All-knowing (Norman)

Words: Listens carefully to all the arguments of the others and negotiates a collective decision. Arguments without a mathematical foundation are discarded.

Mathematics: Understands all of the arguments above and helps explain the reasoning to the six modelers.

Values: All views are listened to and an informed process is followed to come to a solution.

Norman's approach is very appealing, and, in practice, this will usually lead to acceptable solutions. In educational settings, Norman might be the teacher, explaining what is mathematically valid and what is not, and then demonstrating how values and mathematics have become entangled.

Norman's approach does, nonetheless, also encode values. The most important of these is that Norman is required to understand all arguments in coming to a decision about the last biscuit. The value-implication is that decision-making should be led by experts. Also, in order for mutual agreement to be met, Norman should be able to help the others see the pros and cons of their ways of arguing. Norman is in a position of knowledge, authority and power.

The difficulty in this approach is that trusting Norman implies that only those who understand the inner workings of all mathematical methods should be allowed to share out biscuits! This is a strong requirement. Insisting that everyone involved in decision-making should be able to calculate the expectation of a random variable and convergence of a geometric series might be considered anti-democratic. It could be seen as a tyranny of experts.

A further difficulty arises if we question Norman's competence. How can we know that he knows all the math for a particular problem? Consider, for example, Katty who is represented by the following solution.

Schrödinger's solution (Katty)

Words: Katty builds a quantum computing device that takes in a biscuit and outputs three boxes. These boxes contain a quantum superposition of states, corresponding to a one-in-three probability that they contain a biscuit. After the device is fed one biscuit, if one of the three people wanting biscuits opens their box and finds one there, it is guaranteed that there is not a biscuit in the other boxes. If the first person finds no biscuit in their box, the second and the third still share a quantum superposition of states, now corresponding to a one-in-two probability that their box contains a biscuit.

Mathematics: Katty presents a solution based on quantum mechanics, which proves that the machine works.

Values: Not only is the assignment of biscuits random, but it also allows the box owners to both have a biscuit and not have a biscuit (as a quantum superposition of states) at the same time!

Norman would have to understand quantum mechanics in order to have a complete treatment of the biscuit problem. This is possible, of course, but we can see how it becomes increasingly more difficult to be all-knowing. What other, unknown solutions might be suggested? Can we only have quantum physicists as arbitrators of biscuit sharing problems?

There are other ways in which the problem can be solved. In line with a current trend in society, one further proposal comes in the following form.

Machine learning solution (Matt)

Words: Matt builds a machine learning model (a neural network, for example) which takes in information about the decision makers and the recipient and outputs a suggested division of cookies.

Mathematics: The neural network or other model is trained on previous decisions and makes decisions based on minimizing an objective function (distance between predictions and previous decisions).

> Values: Past decisions provide a guide for future decisions. The human element is removed from the loop by learning preferences.

Matt sees his approach as simply reflecting what has happened in the past. It is (for him) a neutral solution, taken from the available data. But for the other eight friends proposing their own solutions, Matt's solution lacks any form of logic. Norman, in particular, is provoked by it. What if the people making previous decisions were prejudiced in some way? What is the underlying logic to Matt's black box? Matt's solution highlights many of the dilemmas which arise when we apply machine learning to solve social problems [29]. Machine learning methods fail to account for historical biases and can reinforce prejudice [6].

6. Discussion

We now have nine mathematical approaches to the seemingly innocuous problem of sharing four biscuits. Each has its strengths and weaknesses, and each is linked to social values.

Our example can aid discussions around ethics and human values at all levels, from pre-school to university level. Indeed, our earlier work has shown that the children who worked with the cases have used several of the solutions proposed in the present paper, albeit in reduced versions. For instance, it is not uncommon for young children to divide the biscuits into four, just as in the infinitesimal solution, until they reach a small piece that is not possible to share anymore (e.g., [26]). Or they start by saying that they need to get all information in place before they can discuss how to share the biscuits (e.g., [18]). Sumpter and Hedefalk [35] show that children are aware of the consequences of various strategy choices. Eriksson and colleagues [18] go on to show that they understand that "fair" can have different meanings. These results show that young children are able to use mathematical reasoning and ethical reasoning together to make sense of the world.

The way we have presented the biscuit problem here shows a connection to many different aspects of mathematical modeling learnt during an undergraduate education. The nine solutions span the university curriculum: from partitions and infinitesimals, through probability and statistics, to quantum mechanics. The biscuit problem could be used within individual courses. For example, students could, as part of an integration of ethics across the mathematics curriculum along the lines suggested in [2], be asked to find the corresponding biscuit problem for each math course they study. Alternatively, it could form a part of an overall course on applications of mathematics, statistics and machine learning, once the technical skills are already learnt. In this case, students can first be asked to propose their own mathematical solution to the biscuit problem, before discussing it with others. They can then think about whether there are any values associated with these choices and discuss these together.

We see the biscuit problem as a way to help students discuss values more explicitly and examine the principles on which they make moral statements [32]. At university level, talking about the biscuit problem can be a complement to an ethical analysis of textbook problems. For example, Shulman [32] looks at examples ranging from a German second world war textbook which asks students to find how to kill as many people as possible with poisoned gas to one taken from a standard American calculus text about minimizing the financial cost of laying an oil pipeline, without considering the environmental impact of the chosen solution. It can also complement real-world ethical dilemmas presented in the course designed by Chiodo and Bursill-Hall [10]. In an ethics course for mathematics undergraduates, the biscuit problem can be used to illustrate that ethical dilemmas do not simply arise because the application areas are complex or because the application of models has not been well thought-out. Values are implicit in our choice of a model even in the most straightforward of applications.

This last point relates to claims that mathematics is neutral once the problem is fully specified, which we describe in the introduction. The problem of dividing a biscuit is fully specified or, at the very least, any argument that the biscuit problem in not fully specified leaves open the question as to whether any mathematical modeling question is ever fully specified. The teacher, using the biscuit problem as an example, can demonstrate the limitations of the "fully specified" or "objective function" claims, by asking students to consider (given the nine potential solutions to the biscuit problem) the pitfalls of claiming that more complex problems can be solved once they are fully specified. Specifying a mathematical model for a complex system is part of the modeling problem, and so the modeling problem cannot be solved without bringing our values into play.

We hope that the biscuit problem will help those, like Armstrong [3], who claim that mathematics and social values should be kept separate, see that they are themselves misguided. And we hope it will help us all understand the subtle and varied ways in which values are a central part of how we do mathematics.

References

[1] R. Abebe, S. Barocas, J. Kleinberg, K. Levy, M. Raghavan, and D.G. Robinson, "Roles for computing in social change," pages 252-260 in *Proceedings of the 2020 Conference on Fairness, Accountability, and Transparency (Barcelona, 2020)* (Association for Computing Machinery, New York, 2020) .doi:10.1145/3351095.3372871

[2] F. Alayont, "A Case for Ethics in the Mathematics Major Curriculum," *Journal of Humanistic Mathematics*, Volume **12** Issue 2 (2022), pages 160-177. doi:10.5642/jhummath.CXSI3022

[3] J. Armstrong, "The Sinister Attempts to Decolonise Mathematics," *The Spectator*, 13 November 2022; available at `https://www.spectator.co.uk/article/the-sinister-attempts-to-decolonise-mathematics/`.

[4] A. Birhane, "Algorithmic injustice: a relational ethics approach," *Patterns*, Volume **2** Issue 2 (2021). doi:10.1016/j.patter.2021.100205

[5] A. Birhane, and O. Guest, "Towards decolonising computational sciences," arXiv preprint, 2020; `arXiv:2009.14258`.

[6] A. Birhane, P. Kalluri, D. Card, W. Agnew, R. Dotan, and M. Bao, "The values encoded in machine learning research," pages 173-184 in *2022 ACM Conference on Fairness, Accountability, and Transparency (Seoul, 2022)* (Association for Computing Machinery, New York, 2022). doi:10.1145/3351095.3372871

[7] T. Bolukbasi, K.W. Chang, J.Y. Zou, V. Saligrama, and A.T. Kalai, "Man is to computer programmer as woman is to homemaker? debiasing word embeddings," *Advances in Neural Information Processing Systems*, Volume **29**: Proceedings of the 30$^{\text{th}}$ International Conference on Neural Information Processing Systems (December 2016), pages 4356–4364.

[8] T. Brennan, W. Dieterich, and B. Ehret, "Evaluating the predictive validity of the COMPAS risk and needs assessment system," *Criminal Justice and Behavior*, Volume **36** Issue 1, pages 21-40.

[9] E. Celtikci, "A systematic review on machine learning in neurosurgery: The future of decision-making in patient care," *Turkish Neurosurgery*, Volume **28** Issue 2 (2018), pages 167–173.

[10] M. Chiodo, and P. Bursill-Hall, "Teaching ethics in mathematics," *European Mathematical Society Magazine*, Number 114 (2019), pages 38-41. doi:10.4171/NEWS/114/10

[11] R. Coyne, "Wicked problems revisited," *Design Studies*, Volume **26** Issue 1 (2005), pages 5-17.

[12] K. Devlin, "Do math and chess make you a better problem solver? Teaching math for life in a wicked world," blog post at *Math Values: Devlin's Angle*, November 1, 2019; available at `https://www.mathvalues.org/masterblog/do-math-and-chess-make-you-a-better-problem-solver-teaching-math-for-life-in-a-wicked-world`.

[13] K. Devlin, "The 'wicked problems' problem, recycled," blog post at *Math Values: Devlin's Angle*, December 2, 2021; available at `https://www.mathvalues.org/masterblog/the-wicked-problems-problem-recycled`.

[14] R. Dorfman, I. Chang, S. Saadat, and J. Roostaeian, "Making the subjective objective: Machine learning and rhinoplasty," *Aesthetic Surgery Journal*, Volume **4**0 Issue 5 (2020), pages 493-498.

[15] H. Douglas, "Inductive risk and values in science," *Philosophy of Science*, Volume **67** Issue 4 (2000), pages 559-579.

[16] H. Douglas, "The irreducible complexity of objectivity," *Synthese*, Volume 138 Issue 3 (2004), pages 453-473.

[17] C. Dwork, M. Hardt, T. Pitassi, O. Reingold, and R. Zemel, "Fairness through awareness," pages 214-226 in *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, (Association for Computing Machinery, New York, 2012). doi:/10.1145/2090236.2090255

[18] H. Eriksson, M. Hedefalk, and L. Sumpter, "The tension between division and fair share," in *Teaching Mathematics as to be Meaningful – Foregrounding Play and Children's Perspectives: Results from the POEM5 Conference*, H. Palmér, C. Björklund, E. Reikerås and J. Elofsson, eds., Springer, 2023.

[19] P. Ernest, "The ethics of mathematics: Is mathematics harmful?" pages 187-216 in *The Philosophy of Mathematics Education Today* edited by P. Ernest (ICME-13 Monographs, Springer, 2018).

[20] P. Ernest, "The dark side of mathematics: Damaging effects of the overvaluation of mathematics," pages 29-42 in *Debates in Mathematics Education* edited by Gwen Ineson and Hilary Povey (Routledge, 2020).

[21] P. Foot, "The problem of abortion and the doctrine of the double effect," *Oxford Review*, Number **5** (1967), pages 5–15.

[22] A. Gelman and C. Hennig, "Beyond subjective and objective in statistics," *Journal of the Royal Statistical Society: Series A (Statistics in Society)*, Volume **180** Issue 4 (2017), pages 967-1033.

[23] B. Green, "'Good' isn't good enough," in *Proceedings of the AI for Social Good workshop at NeurIPS*, Volume **17** (2019).

[24] A. Hager and H. Hilbig, "Do inheritance customs affect political and social inequality?" *American Journal of Political Science*, Volume **63** Issue 4 (2019), pages 758-773.

[25] B. W. Head, "Wicked problems in public policy," *Public Policy*, Volume **3** Issue 2 (2008), pages 101-118.

[26] M. Hedefalk, H. Eriksson, P. Markkanen, and L. Sumpter, "Five year olds in between sharing and division," *Philosophy of Mathematics Education Journal*, Volume **39** (2022).

[27] J. Kleinberg, J. Ludwig, S. Mullainathan, and A. Rambachan, "Algorithmic fairness," in *AEA Papers and Proceedings*, Volume **108** (2018), pages 22-27. doi:10.1257/pandp.20181018

[28] T. Nagel, "What is it like to be a bat?" *The Philosophical Review*, Volume **83** Issue 4 (1974), pages 435-450.

[29] Cathy O'Neill, *Weapons of Math Destruction: How Big Data Increases Inequality and Threatens Democracy*, Penguin Random House, 2017.

[30] J. Pryshlakivsky and C. Searcy, "Sustainable development as a wicked problem," pages 109-128 in *Managing and engineering in complex situations* edited by Samuel F. Kovacic and Andres Sousa-Poza (Springer, Dordrecht, 2013).

[31] W. Quine, "Two dogmas of empiricism," pages 109-128 in *Can Theories be Refuted? Essays on the Duhem-Quine Thesis* edited by Sandra G. Harding (Springer, Dordrecht, 1976).

[32] B. Shulman, "Is there enough poison gas to kill the city?: The teaching of ethics in mathematics classes," *The College Mathematics Journal*, Volume **33** Issue 2 (2002), pages 118-125.

[33] B.S. Stemn, "Rethinking mathematics teaching in Liberia: Realistic mathematics education," *Childhood Education*, Volume **93** Issue 5 (2017), pages 388-393.

[34] W. Suksompong, "Constraints in fair division," *ACM SIGecom Exchanges*, Volume **19** Issue 2 (2021), pages 46-61.

[35] L. Sumpter and M. Hedefalk, "När dela lika är olika" ("When Equals are Different" in Swedish), *Nordisk Barnehageforskning*, Volume **20** Issue 2 (2023), pages 110–129. doi:10.23865/nbf.v20.256.

[36] L. Sumpter, and D.J.T. Sumpter, "Ethics as part of mathematical reasoning", *Prometeica-Revista de Filosofía y Ciencias*, Volume **27** (2023), pages 649–657.

[37] J.J. Thomson, "The Trolley Problem," *The Yale Law Journal*, Volume **94** Number 6 (1985), pages 1395–1415.

[38] K. Whitaker, and O. Guest, "#bropenscience is broken science: Kirstie Whitaker and Olivia Guest ask how open 'open science' really is," *The Psychologist*, November 2020, pages 34-37; available at `https://www.bps.org.uk/psychologist/bropenscience-broken-science`.