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EXISTENCE OF POSITIVE SOLUTIONS FOR A SUPERLINEAR ELLIPTIC SYSTEM WITH NEUMANN BOUNDARY CONDITION

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Abstract. We prove the existence of a positive solution for a class of nonlinear elliptic systems with Neumann boundary conditions. The proof combines extensive use of a priori estimates for elliptic problems with Neumann boundary condition and Krasnoselskii's compression-expansion theorem.

1. Introduction

The purpose of this paper is to prove that the system

$$\begin{aligned} -\Delta u + \alpha u &= \beta v + f_1(x,u,v) \quad \text{in } \Omega \\ -\Delta v + \delta v &= \gamma u + f_2(x,u,v) \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

has a nontrivial positive solution. In (1.1) Δ denotes the Laplacian operator, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ are real parameters. We also assume that $f_1(x,u,v), f_2(x,u,v)$ are measurable in x, differentiable in (u,v), and bounded on bounded sets. Our main result reads as follows.

Theorem 1.1. *If there exist $b \in (1, \min\{2, (N+1)/(N-1)\})$, $m > 0$, and $M > 0$ such that*

$$m(u+v)^b \le f_i(x,u,v) \le M(u+v)^b \quad \text{for } i = 1,2, u,v \ge 0, \tag{1.2}$$

and $\beta\gamma < \alpha\delta$, then the problem (1.1) *has a positive solution.*

The main tool in our proofs is Krasnoselskii's compression-expansion theorem (see Theorem 1.2 below) which we state for sake of completeness. For a proof of this theorem the reader is referred to [12, Theorem 13.D]. To apply Theorem 1.2 to Theorem 1.1, in Section 3 we use of a priori estimates for elliptic equation with Neumann boundary conditions, see [11].

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Theorem 1.2. *Let X be a real ordered Banach space with positive cone K. If $\Upsilon : K \to K$ is a compact operator and there exist real numbers $0 < R < \overline{R}$ such that*

$$\Upsilon(x) \nleq x, \text{ for } x \in K, \ \|x\| = R,$$
$$\Upsilon(x) \ngeq x, \text{ for } x \in K, \ \|x\| = \overline{R}.$$

then Υ has a fixed point with $\|x\| \in (R, \overline{R})$.

There is rich literature on systems like (1.1) in the presence of *variational structure* and Dirichlet boundary condition, see [2, 3, 4, 6, 7, 8]. Costa and Magalhaes [3] study system (1.1) for nonlinearities with subcritical growth. The reader may consult [2] for applications of the Mountain Pass Lemma to the study of fourth order systems. In [8], (1.1) is studied for Lipschitzian nonlinearities and $\alpha = \delta = \lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω. For a survey on the study of elliptic systems the reader is referred to [4].

Throughout this paper we denote by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ and by $\|cdot\|_{k,p}$ the norm in the Sobolev space $W^{k,p}(\Omega)$ (see [1]).

2. Linear Analysis

In this section we study the linear problem

$$\begin{aligned} -\Delta u + \alpha u - \beta v &= P_1(x) \quad \text{in } \Omega \\ -\Delta v - \gamma u + \delta v &= P_2(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $P_1(x) \geq 0$, $P_2(x) \geq 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\delta > 0$.

Lemma 2.1. *For each $P_1, v \in L^2(\Omega)$, then the equation*

$$\begin{aligned} -\Delta u + \alpha u &= P_1(x) + \beta v \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{2.2}$$

has a unique solution. Moreover, there exists $c > 0$, independent of (P_1, v), such that

$$\|u\|_{1,2} \leq c\|P_1 + \beta v\|_2, \tag{2.3}$$

Proof. Let H be the Sobolev space $H^1(\Omega)$, and $B : H \times H \to \mathbb{R}$ defined by $B[u, v] = \int_\Omega \nabla u \nabla v + \alpha uv$. Since $\alpha > 0$, $B[u, u] \geq \min\{1, \alpha\}\|u\|^2$. By the Lax-Milgram theorem (see [5]) there exists $u \in H$ such that

$$B[u, z] = \int_\Omega \nabla u \nabla z + \alpha \int_\Omega uz = \int_\Omega z(x)(P_1(x) + \beta v(x))dx. \tag{2.4}$$

Hence u is a weak solution to (2.2). Taking $z = u$ and $c^{-1} = \min\{1, \alpha\}$ the lemma is proved. □

Lemma 2.2. *Let P_1, v, and u be as in Lemma 2.1. If $v \geq 0$ then $u \geq 0$.*

Proof. Suppose u is not positive. Let $A = \{x \in \Omega, u(x) < 0\}$, and $z = u\chi_A$. By the definition of weak solution

$$\int_\Omega z(P_1 + \beta v) = \int_\Omega \nabla u \nabla z + \alpha \int_\Omega uz = \int_A \nabla u \nabla u + \alpha(\int_A u^2). \tag{2.5}$$

This is a contradiction since $\int_A \nabla u \nabla u + \alpha(\int_A u^2) > 0$, while $\int_A z(P_1 + \beta v) < 0$. This proves the lemma. $\square$

Lemma 2.3. *For each $v \in L^2$, let $u(v) \equiv u \in H^1(\Omega)$ be the solution to (2.2) given by Lemma 2.1. If $w \in H^1(\Omega)$ is the weak solution to*

$$\begin{gathered} -\Delta w + \delta w = P_2(x) + \gamma u(v) \quad \text{in } \Omega \\ \frac{\partial w}{\partial n} = 0 \quad \text{in } \partial\Omega, \end{gathered} \tag{2.6}$$

then

$$\|w\|_2 \le \frac{1}{\alpha}\|P_2\|_2 + \frac{\delta}{\alpha\gamma}\|P_1\|_2 + \frac{\beta\gamma}{\delta\alpha}\|v\|_2. \tag{2.7}$$

Proof. Multiplying (2.6) by w and using the Cauchy-Schwartz inequality we have

$$\begin{aligned} \int_\Omega \nabla w \nabla w + \delta \int_\Omega w^2 &= \int_\Omega P_2(x) \cdot w + \gamma u(v) \cdot w \\ &\le \|P_2\|_2 \cdot \|w\|_2 + \gamma \|u(v)\|_2 \cdot \|w\|_2 \\ &\le (\|P_2\|_2 + \delta\|u(v)\|_2) \cdot \|w\|_2. \end{aligned} \tag{2.8}$$

Hence

$$\|w\|_2 \le \frac{1}{\delta}\|P_2\|_2 + \frac{\gamma}{\delta}\|u(v)\|_2. \tag{2.9}$$

Similarly,

$$\|u\|_2 \le \frac{1}{\alpha}\|P_1\|_2 + \frac{\beta}{\alpha}\|v\|_2. \tag{2.10}$$

Replacing (2.9) in (2.10),

$$\begin{aligned} \|w\|_2 &\le \frac{1}{\gamma}\|P_2\|_2 + \frac{\gamma}{\delta}\|u(v)\|_2 \\ &\le \frac{1}{\gamma}\|P_2\|_2 + \frac{\delta}{\gamma}(\frac{1}{\alpha}\|P_1\|_2 + \frac{\beta}{\alpha}\|v\|_2) \\ &\le \frac{1}{\alpha}\|P_2\|_2 + \frac{\delta}{\alpha\gamma}\|P_1\|_2 + \frac{\beta\gamma}{\delta\alpha}\|v\|_2, \end{aligned} \tag{2.11}$$

which proves the lemma. $\square$

Theorem 2.4. *Given $(P_1, P_2) \in L^2(\Omega) \times L^2(\Omega)$, there exists a unique pair $(u, v) \in H \times H$ satisfying (2.1). In addition, (u, v) depends continuously on (P_1, P_2).*

Proof. Let $v_1, v_2 \in L^2(\Omega)$. Let $u(v_1)$ and $u(v_2)$ be given by Lemma 2.1 and w_1, w_2 as given by Lemma 2.3. Hence

$$\begin{aligned} &\int_\Omega |\nabla(w_1 - w_2)|^2 + \delta \int_\Omega |(w_1 - w_2)|^2 \\ &= \gamma \int_\Omega u(v_1) - u(v_2))(w_1 - w_2) \\ &\le \gamma(\|u(v_1) - u(v_2))\|_{L_2})\|w_1 - w_2)\|_2. \end{aligned} \tag{2.12}$$

Therefore,

$$\|w_1 - w_2\| \le \frac{\gamma}{\delta}(\|u(v_1) - u(v_2))\|_{L^2}. \tag{2.13}$$

Multiplying (2.2) by $u(v_1) - u(v_2)$ and subtracting we have

$$\begin{aligned} &\int_\Omega |\nabla(u_1 - u_2)|^2 + \alpha \int_\Omega (u(v_1) - u(v_2))^2 \\ &= \beta \int_\Omega ((v_1 - v_2)(u(v_1) - u(v_2)) \\ &\le \beta \|v_1 - v_2\|_2 \|u(v_1) - u(v_2)\|_2. \end{aligned} \tag{2.14}$$

Thus

$$\|(u(v_1) - u(v_2)\|_2 \le \frac{\beta}{\alpha} \|(v_1 - v_2)\|_2. \tag{2.15}$$

Replacing this in (2.13) yields $\|w_1 - w_2\|_2 \le \frac{\gamma\beta}{\alpha\delta} \|(v_1 - v_2)\|_2$. Hence by the contraction mapping principle there exists a unique w such that $w = v$. That is (u, v) satisfies

$$\begin{gathered} -\Delta u + \alpha u = \beta v + P_1(x) \quad \text{in } \Omega \\ -\Delta v + \delta v = \gamma u + P_2(x) \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \quad \text{on } \partial\Omega, \end{gathered} \tag{2.16}$$

By Lemma 2.1, u depends continuously on (P_1, v). Also, by Lemma 2.3, v depends continuously on (P_1, P_2). Hence (u, v) depends continuously on (P_1, P_2), which proves the theorem. □

Lemma 2.5. *Let $h_1, h_2 \in L^\infty(\Omega)$. For each $p > 1$ there exist $C_2(p) \equiv C_2 > 0$ such that if (y, z) satisfies*

$$\begin{gathered} -\Delta y + \alpha y = \beta z + h_1, \\ -\Delta z + \delta z = \gamma y + h_2, \quad \text{in } \Omega \\ \frac{\partial y}{\partial n} = \frac{\partial z}{\partial n} = 0 \quad \text{in } \partial\Omega, \end{gathered} \tag{2.17}$$

then

$$\|y\|_{2,p} + \|z\|_{2,p} \le C_2(\|h_1\|_\infty + \|h_2\|_\infty) \tag{2.18}$$

(see [5]*). In particular, by the Sobolev imbedding theorem, taking $p > N/2$ we may assume that*

$$\|y\|_\infty + \sup \frac{|y(\zeta) - y(\eta)|}{\|\zeta - \eta\|} + \|z\|_\infty + \sup \frac{|z(\zeta) - z(\eta)|}{\|\zeta - \eta\|} \le C_2(\|h_1\|_p + \|h_2\|_p). \tag{2.19}$$

Proof. Multiplying the first equation in (2.17) by y we have

$$\begin{aligned} \int_\Omega |\nabla y|^2 + \alpha \int_\Omega y^2 &= \beta \int_\Omega (yz) + \int_\omega h_1 y \\ &\le \beta \int_\Omega (yz) + \|h_1\|_\infty |\Omega|^{1/2} \|y\|_2. \end{aligned} \tag{2.20}$$

Similarly,

$$\begin{aligned} \int_\Omega |\nabla z|^2 + \delta \int_\Omega z^2 &= \gamma \int_\Omega (yz) + \int_\omega h_2 z \\ &\le \gamma \int_\Omega (yz) + \|h_2\|_\infty |\Omega|^{1/2} \|z\|_2. \end{aligned} \tag{2.21}$$

Since $\alpha > 0$ and $\alpha\delta - \beta\gamma > 0$, the quadratic form $G(s,t) = \alpha s^2 - (\beta+\gamma)st + \delta t^2$ positive definite. That is, there exists $C > 0$ such that $G(s,t) \geq C(s^2+t^2)$ for all $s,t \in \mathbb{R}$. This, (2.20), and (2.21) imply

$$C(\|y\|_2 + \|z\|_2) \leq 2|\Omega|^{1/2}(\|h_1\|_\infty + \|h_2\|_\infty). \tag{2.22}$$

By (2.20) and (2.22),

$$\begin{aligned}\bar{\alpha}\|y\|_{1,2}^2 &\leq \|y\|_2(\beta\|z\|_2 + |\Omega|^{1/2}(\|h_1\|_\infty + \|h_2\|_\infty)) \\ &\leq (\frac{2\beta}{C}+1)|\Omega|^{1/2}\|y\|_2(\|h_1\|_\infty + \|h_2\|_\infty) \\ &\equiv C_3\|y\|_2(\|h_1\|_\infty + \|h_2\|_\infty) \\ &\leq C_3\|y\|_{1,2}(\|h_1\|_\infty + \|h_2\|_\infty).\end{aligned} \tag{2.23}$$

Hence

$$\|y\|_{1,2} \leq \frac{C_3}{\bar{\alpha}}(\|h_1\|_\infty + \|h_2\|_\infty). \tag{2.24}$$

Similarly,

$$\|z\|_{1,2} \leq \frac{C_3}{\bar{\delta}}(\|h_1\|_\infty + \|h_2\|_\infty). \tag{2.25}$$

From (2.24), (2.25) and the Sobolev imbedding theorem (see [5, Theorem ??]) we see that

$$\begin{aligned}\|y\|_{2N/(N-2)} + \|z\|_{2N/(N-2)} &\leq S(1,2)(\|y\|_{1,2} + \|z\|_{1,2}) \\ &\leq S(1,2)\Big(\frac{C_3}{\bar{\alpha}} + \frac{C_3}{\bar{\delta}}\Big)(\|h_1\|_\infty + \|h_2\|_\infty) \\ &\equiv C_4(\|h_1\|_\infty + \|h_2\|_\infty).\end{aligned} \tag{2.26}$$

By regularity properties for elliptic boundary value problems there exists a positive real number C_2 such that if $-\Delta u + \tau u = f$ en Ω and $(\partial u)/(\partial\eta) = 0$ in $\partial\Omega$ $\|u\|_{2,P}$ when $p \in (1, (N/2)+1)$. This and (2.26) imply

$$\|y\|_{2,\frac{2N}{N-2}} + \|z\|_{2,\frac{2N}{N-2}} \leq C_2(C_4 + |\Omega|^{\frac{N-2}{2N}})(\|h_1\|_\infty + \|h_2\|_\infty)). \tag{2.27}$$

Iterating this argument finitely many times we see that there exist $p > N/2$ and $C_3 > 0$ such that

$$\|y\|_{2,p} + \|z\|_{2,p} \leq C_3(\|h_1\|_\infty + \|h_2\|_\infty)), \tag{2.28}$$

which proves the lemma. □

3. Proof of Theorem 1.1

Let $\rho = \max\{\alpha/m, \delta/m\}$ and $\bar{R} = 2(2M\rho|\Omega|)^{1/(2-b)}$ (see (1.2)). For $i = 1,2$, let

$$g_i(x,u,v) = \begin{cases} f_i(x,u,v) & \text{for } 0 \leq u+v \leq \bar{R}, \\ f_i(x, \bar{R}u/(u+v), \bar{R}v/(u+v)) & \text{for } u+v \geq \bar{R}. \end{cases}$$

Let X be the ordered Banach space $C(\bar{\Omega}) \times C(\bar{\Omega})$ with positive cone

$$\begin{aligned}K = \Big\{(u,v) \in X : u \geq 0, v \geq 0, \|u - \frac{1}{|\Omega|}\int_\Omega u\|_\infty \leq bM\bar{R}^{b-1}\int_\Omega u, \\ \|v - \frac{1}{|\Omega|}\int_\Omega v\|_\infty \leq bM\bar{R}^{b-1}\int_\Omega v\Big\}.\end{aligned} \tag{3.1}$$

Let (see (1.2) and Lema 2.5)

$$R \in \big(0, \min\{\bar{R}, (2C_2M)^{1-b}\}\big). \tag{3.2}$$

For $(u,v) \in K$, $\|(u,v)\|_X \geq R$, we define $\Upsilon(u,v) = (U,V)$ as the only solution to

$$\begin{gathered} -\Delta U + \alpha U = \beta V + g_1(x,u,v) \quad \text{in } \Omega \\ -\Delta V + \delta V = \gamma U + g_2(x,u,v) \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \quad \text{in } \partial\Omega. \end{gathered} \tag{3.3}$$

If $(u,v) \in K$ and $\|(u,v)\|_X \leq R$ we define

$$\Upsilon(u,v) = \|(u,v)\|_X \Upsilon((R/\|(u,v)\|_X)(u,v)), \quad \Upsilon(0,0) = (0.0). \tag{3.4}$$

Since g_1, g_2 are nonnegative continuous functions, $\Upsilon(u,v) = (U,V)$ satisfies $U \geq 0$ y $V \geq 0$ for $(u,v) \in K$ (see Lemma 2.2).

Suppose that for some $(U,V) = \Upsilon(u,v)$ we have

$$\|U - \frac{1}{|\Omega|}\int_\Omega U\|_\infty > bM\bar{R}^{b-1}\int_\Omega U, \tag{3.5}$$

with $\|(u,v)\|_X \geq R$. Hence $\|U\|_\infty \geq bMR^{b-1}\int_\Omega U$, which implies that if $\|U\|_\infty = U(x)$, $x \in \bar{\Omega}$, then there exists $y \in \bar{\Omega}$ such that $\|y - x\| \leq m_1\bar{R}^{(1-b)/n}$ and $U(y) \leq U(x)/2$, with m_1 a constant depending only on Ω. Hence

$$\frac{U(x) - U(y)}{\|x - y\|} \geq \frac{\|U\|_\infty}{2m_1\bar{R}^{(b-1)/N}}. \tag{3.6}$$

Let now $p > N$ be such that

$$\frac{N + p - b(p-1)}{(p-1)N} + \frac{b}{p} > 0. \tag{3.7}$$

This and Lemma 2.5 imply

$$\begin{aligned} \|U\|_\infty\bar{R}^{(b-1)/n} &\leq C_2\|g_1(\cdot,u,v)\|_p \\ &\leq C_2M\Big(\int_\Omega (u+v)^{bp}\Big)^{1/p} \\ &\leq C_2M\Big(\int_\Omega (u+v)^b(u+v)^{b(p-1)}\Big)^{1/p} \\ &\leq C_2M\|u+v\|_\infty^{b(p-1)/p}\Big(\int_\Omega (u+v)^b\Big)^{1/p}. \end{aligned} \tag{3.8}$$

Integrating the first equation in (3.3) on Ω,

$$\alpha\int_\Omega U \geq m\int_\Omega (u+v)^b, \tag{3.9}$$

(see (1.2)). From (3.8) and (3.9),

$$\begin{aligned} \|U\|_\infty\bar{R}^{\frac{b-1}{n}} &\leq C_2M\|u+v\|_\infty^{b(p-1)/p}\Big(\frac{\alpha}{m}\int_\Omega U\Big)^{1/p} \\ &\leq C_2M\|u+v\|_\infty^{b(p-1)/p}\Big(\frac{\alpha}{2mM}\bar{R}^{1-b}\|U\|_\infty\Big)^{1/p} \\ &\leq C_2M\Big(2M\bar{R}^{b-1}\int_\Omega (u+v)\Big)^{\frac{b(p-1)}{p}}\Big(\frac{\alpha}{2mM}\bar{R}^{1-b}\int_\Omega \|U\|_\infty\Big)^{1/p}. \end{aligned} \tag{3.10}$$

Therefore

$$
\begin{aligned}
\|U\|_\infty^{(p-1)/p} &\le m_2\bar{R}^{(b-1)(\frac{b(p-1)}{n}-\frac{1}{p}-\frac{1}{n})}\Big(\int_\Omega (u+v)\Big)^{b(p-1)/p} \\
&\le m_3\bar{R}^{(b-1)(\frac{b(p-1)}{n}-\frac{1}{p}-\frac{1}{n})}\Big(\int_\Omega (u+v)^b\Big)^{(p-1)/p} \\
&\le m_3\bar{R}^{(b-1)(\frac{b(p-1)}{n}-\frac{1}{p}-\frac{1}{n})}\Big(\frac{\alpha}{m}\int_\Omega U\Big)^{(p-1)/p} \\
&\le m_4\bar{R}^{(b-1)(\frac{b(p-1)}{n}-\frac{1}{p}-\frac{1}{n})}\Big(\bar{R}^{1-b}\|U\|_\infty\Big)^{(p-1)/p}.
\end{aligned} \tag{3.11}
$$

Since m_2, m_3, m_4 are independent of U,

$$
1 \le m_4\bar{R}^{(b-1)(\frac{b(p-1)}{n}-\frac{1}{p}-\frac{1}{n}-\frac{p-1}{p})}. \tag{3.12}
$$

By (1.2), there exists $p > N$ such that

$$
(b-1)\Big(\frac{b(p-1)}{n}-\frac{1}{p}-\frac{1}{n}-\frac{p-1}{p}\Big) < 0. \tag{3.13}
$$

Taking $\bar{R}$ sufficiently large we have a contradiction to (3.5). Thus $\Upsilon(u,v) \in K$. For $\|(u,v)\|_X < R$ the proof follows from the definition of Υ. Thus $\Upsilon(K) \subset K$.

Let C_2 be as in 2.5 and $x \in \bar{\Omega}$ be such that $U(x) = \max\{U(y); y \in \bar{\Omega}\}$. From the definition of C_2 we conclude that if $y \in \bar{\Omega}$ and $\|y-x\| \le C_2M(\|u\|_\infty^b + \|v\|_\infty^b)$ then by the definition of g_1, g_2, if $\{u_j, v_j\}_j$ is a bounded sequence in X so are $\{g_1(x, u_j, v_j)\}_j$ and $\{g_2(x, u_j, v_j)\}_j$ in $C(\bar{\Omega})$. Since g_1, g_2 are bounded functions, due to Lemmas 2.5, $\{U_j, V_j\}_j$ is bounded in $W^{2,p}(\Omega) \times W^{2,p}(\Omega)$. Taking $p > N/2$, by the Sobolev imbedding theorem (see [5]) we see that $\{U_j, V_j\}_j$ has a converging subsequence in the space X, which proves that Υ is a compact operator.

Suppose that for some (u,v) such that $\|u\|_\infty + \|v\|_\infty = R$, $U \ge u$, $V \ge v$. By (2.18),

$$
\begin{aligned}
R = \|u\|_\infty + \|v\|_\infty &\le \|U\|_\infty + \|V\|_\infty \\
&\le 2C_2M\|u+v\|_\infty^b \\
&\le 2C_2MR^b,
\end{aligned} \tag{3.14}
$$

which contradicts the definition of R. This proves that $\Upsilon(u,v) \not\ge (u,v)$ for $\|(u,v)\|_X = R$.

Suppose that $(U,V) = \Upsilon(u,v) \le (u,v)$ for some (u,v) with $\|(u,v)\|_X = \bar{R}$. Without loss of generality we may assume that $\|u\| \ge \bar{R}/2$. Hence, by the definition of K,

$$
\int_\Omega u \ge \bar{R}\frac{1}{2(|\Omega|^{-1} + bM\bar{R}^{b-1})} \ge C_3\bar{R}^{2-b}. \tag{3.15}
$$

Integrating the first equation in (3.3) we infer that

$$
\begin{aligned}
\alpha\int_\Omega U &= \beta\int_\Omega V + \int_\Omega g_1(u,v) \\
&= \beta\int_\Omega V + m\int_\Omega (u+v)^b \\
&\ge \beta\int_\Omega V + m\int_\Omega (U+V)^b.
\end{aligned} \tag{3.16}
$$

Similarly,

$$\delta \int_\Omega V \geq \gamma \int_\Omega U + m \int_\Omega (U+V)^b.$$

By Holder inequality and the definition of ρ,

$$\int_\Omega (U+V)^b \leq \rho|\Omega|. \tag{3.17}$$

Since $(U, V) \in K$,

$$\bar{R} \leq 2\|U\|_\infty \leq 4MR^{b-1} \int_\Omega U \leq 2M\bar{R}^{b-1}\rho|\Omega|, \tag{3.18}$$

which contradicts the definition of $\bar{R}$. Thus Υ satisfies the hypotheses of Theorem 1.2. Hence Υ has a fixed point (u, v) in $\{(y, z); \|(y, z)\| \in (R\ ,\ \overline{R})$. Therefore (u, v) is a positive solution to (1.1), which proves Theorem 1.1.

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