12-1-2014

Combinatorial Proofs of Fibonomial Identities

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COMBINATORIAL PROOFS OF FIBONOMIAL IDENTITIES

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Abstract. Fibonomial coefficients are defined like binomial coefficients, with integers replaced by their respective Fibonacci numbers. For example, $\binom{10}{3}_F = \frac{F_{10}F_9F_8}{F_3F_2F_1}$. Remarkably, $\binom{n}{k}_F$ is always an integer. In 2010, Bruce Sagan and Carla Savage derived two very nice combinatorial interpretations of Fibonomial coefficients in terms of tilings created by lattice paths. We believe that these interpretations should lead to combinatorial proofs of Fibonomial identities. We provide a list of simple looking identities that are still in need of combinatorial proof.

1. Introduction

What do you get when you cross Fibonacci numbers with binomial coefficients? Fibonomial coefficients, of course! Fibonomial coefficients are defined like binomial coefficients, with integers replaced by their respective Fibonacci numbers. Specifically, for $n \geq k \geq 1$,

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}$$

For example, $\binom{10}{3}_F = \frac{F_{10}F_9F_8}{F_3F_2F_1} = \frac{55 \cdot 34 \cdot 21}{1 \cdot 1 \cdot 2} = 19{,}635$. Fibonomial coefficients resemble binomial coefficients in many ways. Analogous to the Pascal Triangle boundary conditions $\binom{n}{1} = n$ and $\binom{n}{n} = 1$, we have $\binom{n}{1}_F = F_n$ and $\binom{n}{n}_F = 1$. We also define $\binom{n}{0}_F = 1$.

Since $F_n = F_{k+(n-k)} = F_{k+1}F_{n-k} + F_k F_{n-k-1}$, Pascal's recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ has the following analog.

Identity 1. *For $n \geq 2$,*

$$\binom{n}{k}_F = F_{k+1}\binom{n-1}{k}_F + F_{n-k-1}\binom{n-1}{k-1}_F.$$

As an immediate corollary, it follows that for all $n \geq k \geq 1$, $\binom{n}{k}_F$ is an integer. Interesting integer quantities usually have combinatorial interpretations. For example, the binomial coefficient $\binom{a+b}{a}$ counts lattice paths from $(0,0)$ to (a,b) (since such a path takes $a+b$ steps, a of which are horizontal steps and the remaining b steps are vertical). As described in [1] and elsewhere, the Fibonacci number F_{n+1} counts the ways to tile a strip of length n with squares (of length 1) and dominos (of length 2). As we'll soon discuss, Fibonomial coefficients count, appropriately enough, tilings of lattice paths!

2. Combinatorial Interpretations

In 2010 [9], Bruce Sagan and Carla Savage provided two elegant counting problems that are enumerated by Fibonomial coefficients. The first problem counts restricted linear tilings and the second problem counts unrestricted bracelet tilings as described in the next two theorems.

Theorem 2. *For $a, b \geq 1$, $\binom{a+b}{a}_F$ counts the ways to draw a lattice path from $(0,0)$ to (a,b), then tile each row above the lattice path with squares and dominos, then tile each column below the lattice path with squares and dominos, with the restriction that the column tilings are not allowed to start with a square.*

Let's use the above theorem to see what $\binom{6}{3}_F = \frac{F_6F_5F_4}{F_1F_2F_3} = \frac{8\cdot5\cdot3}{1\cdot1\cdot2} = 60$ is counting. There are $\binom{6}{3} = 20$ lattice paths from $(0,0)$ to $(3,3)$ and each lattice path creates an integer partition (m_1, m_2, m_3) where $3 \geq m_1 \geq m_2 \geq m_3 \geq 0$, where m_i is the length of row i. Below the path the columns form a complementary partition (n_1, n_2, n_3) where $0 \leq n_1 \leq n_2 \leq n_3 \leq 3$. For example, the lattice path below has horizontal partition $(3,1,1)$ and vertical partition $(0,2,2)$. The first row can be tiled $F_4 = 3$ ways (namely *sss* or *sd* or *ds* where *s* denotes a square and *d* denotes a domino). The next rows each have one tiling. The columns, of length 0, 2 and 2 can only be tiled in 1 way with the empty tiling, followed by tilings *d* and *d* since the vertical tilings are not allowed to begin with a square. For another example, the lattice path associated with partition $(3,2,2)$ (with complementary vertical partition $(0,0,2)$) can be tiled 12 ways. These lattice paths are shown below.

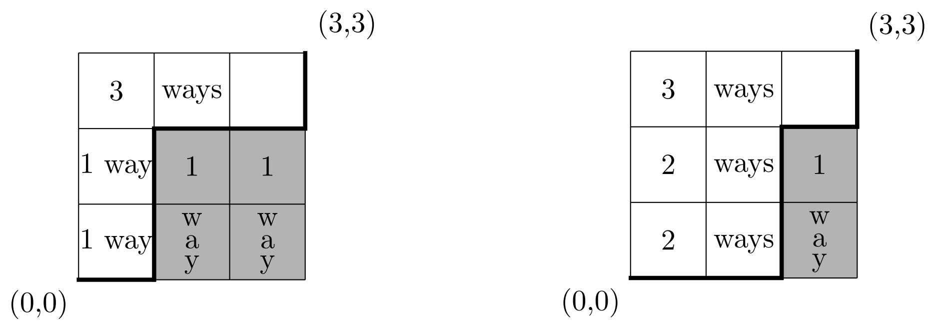

FIGURE 1. The rows of the lattice path $(3,1,1)$ can be tiled 3 ways. The columns below the lattice path, with vertical partition $(0,2,2)$ can be tiled 1 way since those tilings may not start with squares. This lattice path contributes 3 tilings to $\binom{6}{3}_F$. The lattice path $(3,2,2)$ contributes 12 tilings to $\binom{6}{3}_F$.

The lattice path associated with $(3,1,0)$ has no legal tilings since its vertical partition is $(1,2,2)$ and there are no legal tilings of the first column since it has length 1. There are 10 lattice paths that yield at least one valid tiling. Specifically, the paths associated with horizontal partitions $(3,3,3)$, $(3,2,2)$, $(3,1,1)$, $(3,0,0)$, $(2,2,2)$, $(2,1,1)$, $(2,0,0)$, $(1,1,1)$, $(1,0,0)$, $(0,0,0)$ contribute, respectively, $27+12+3+3+8+2+2+1+1+1 = 60$ tilings to $\binom{6}{3}_F$.

More generally, for the Fibonomial coefficient $\binom{a+b}{a}_F$, we sum over the $\binom{a+b}{a}$ lattice paths from $(0,0)$ to (a,b) which corresponds to an integer partition $(m_1, m_2, \ldots, m_b)$ where $a \geq m_1 \geq m_2 \cdots \geq m_b \geq 0$, and has a corresponding vertical partition $(n_1, n_2, \ldots, n_a)$ where $0 \leq n_1 \leq n_2 \cdots \leq n_a \leq b$. Recalling that $F_0 = 0$ and $F_{-1} = 1$, this lattice path contributes

$$F_{m_1+1}F_{m_2+1}\cdots F_{m_b+1}\ F_{n_1-1}F_{n_2-1}\cdots F_{n_a-1}$$

tilings to $\binom{a+b}{a}_F$.

The second combinatorial interpretation of Fibonomial coefficients utilizes circular tilings, or *bracelets*. A bracelet tiling is just like a linear tiling using squares and dominos, but bracelets also allow a domino to cover the first and last cell of the tiling. As shown in [1], for $n \geq 1$, the Lucas number L_n counts bracelet tilings of length n. For example, there are $L_3 = 4$ tilings of length 3, namely sss, sd, ds and $d's$ where d' denotes a domino that covers the first and last cell. Note that $L_2 = 3$ counts ss, d and d' where the d' tiling is a single domino that starts at cell 2 and ends on cell 1. For combinatorial convenience, we say there are $L_0 = 2$ empty tilings. The next combinatorial interpretation of Sagan and Savage has the advantage that there is no restriction on the vertical tilings.

Theorem 3. *For $a, b \geq 1$, $2^{a+b}\binom{a+b}{a}_F$ counts the ways to draw a lattice path from $(0,0)$ to (a,b), then assign a bracelet to each row above the lattice path and to each column below the lattice path.*

Specifically, the lattice path from $(0,0)$ to (a,b) that generates the partition $(m_1, m_2, \ldots, m_b)$ above the path and the partition $(n_1, n_2, \ldots, n_a)$ below the path contributes

$$L_{m_1} L_{m_2} \cdots L_{m_b} \; L_{n_1} L_{n_2} \cdots L_{n_a}$$

bracelet tilings to $2^{a+b}\binom{a+b}{a}_F$. Note that each empty bracelet contributes a factor of 2 to this product. For example, the lattice path from $(0,0)$ to $(3,3)$ with partition $(3,1,1)$ above the path and $(0,2,2)$ below the path contributes $L_3 L_1 L_1 L_0 L_2 L_2 = 72$ bracelet tilings enumerated by $2^6\binom{6}{3}_F = 64 \times 60 = 3840$.

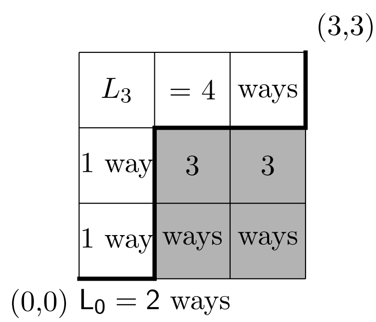

FIGURE 2. The rows above the lattice path can be tiled with bracelets in 4 ways and the columns below the path can be tiled with bracelets in $L_0 L_2 L_2 = 2 \times 3 \times 3 = 18$ ways. This contributes 72 bracelet tilings to $2^6\binom{6}{3}_F = 3840$.

In their paper, Sagan and Savage extend their interpretation to handle Lucas sequences, defined by $U_0 = 0$, $U_1 = 1$ and for $n \geq 2$, $U_n = aU_{n-1} + bU_{n-2}$. Here U_{n+1} enumerates the total weight of all tilings of length n where the weight of a tiling with i squares and j dominos is $a^i b^j$. (Alternatively, if a and b are positive integers, U_{n+1} counts colored tilings of length n where there are a colors for squares and b colors for dominos.) Likewise the number of weighted bracelets of length n is given by $V_n = aV_{n-1} + bV_{n-2}$ with initial conditions $V_0 = 2$ and $V_1 = a$ (so the empty bracelet has a weight of 2). This leads to a combinatorial interpretation of *Lucasnomial coefficients* $\binom{n}{k}_U$, defined like the Fibonomial coefficients. For example,

$$\binom{10}{3}_U = \frac{U_{10} U_9 U_8}{U_1 U_2 U_3}.$$

Both of the previous combinatorial interpretations work exactly as before, using weighted (or colored) tilings of lattice paths.

3. Combinatorial Proofs

Now that we know what they are counting, we should be able to provide combinatorial proofs of Fibonomial coefficient identities. For example, Identity 1 can be rewritten as follows.

Identity 4. *For $m, n \geq 1$,*

$$\binom{m+n}{m}_F = F_{m+1}\binom{m+n-1}{m}_F + F_{n-1}\binom{m+n-1}{m-1}_F.$$

Combinatorial Proof: The left side counts tilings of lattice paths from $(0,0)$ to (m,n). How many of these tiled lattice paths end with a vertical step? As shown below, in all of these lattice paths, the first row has length m and can be tiled F_{m+1} ways. The rest depends on the lattice path from $(0,0)$ to $(m,n-1)$. Summing over all possible lattice paths from $(0,0)$ to $(m,n-1)$ there are $\binom{m+n-1}{m}_F$ tiled lattice paths for the rest of the lattice. Hence the number of tiled lattice paths ending in a vertical step is $F_{m+1}\binom{m+n-1}{m}_F$.

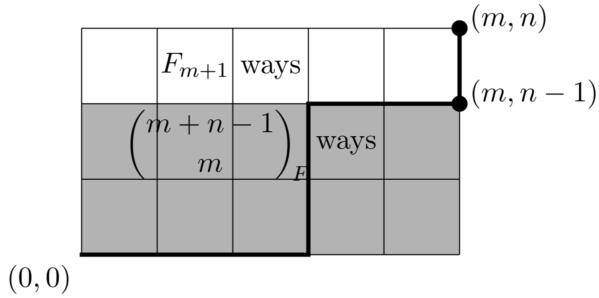

Figure 3. There are $F_{m+1}\binom{m+n-1}{m}_F$ tiled lattice paths that end with a vertical step.

How many tiled lattice paths end with a horizontal step? In all such paths, the last column has length n and can be tiled F_{n-1} ways (beginning with a domino). Summing over all lattice paths from $(0,0)$ to $(m-1,n)$ there are $\binom{m+n-1}{m-1}_F$ tiled lattice paths for the rest of the lattice. Hence the number of tiled lattice paths ending in a horizontal step, as illustrated below, is $F_{n-1}\binom{m+n-1}{m-1}_F$.

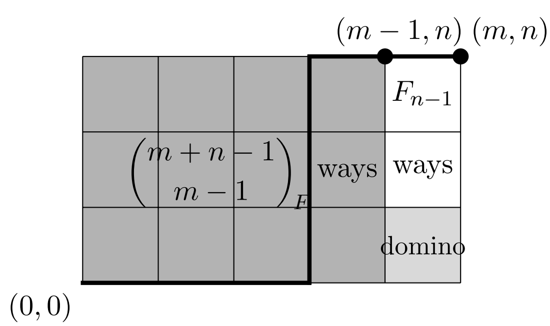

Figure 4. There are $F_{n-1}\binom{m+n-1}{m-1}_F$ tiled lattice paths that end with a horizontal step.

Combining the two previous cases, the total number of tiled lattice paths from $(0,0)$ to (m,n) is $F_{m+1}\binom{m+n-1}{m}_F + F_{n-1}\binom{m+n-1}{m-1}_F$. $\square$

Replacing linear tilings with bracelets and removing the initial domino restriction for vertical tilings, we can apply the same logic as before to get

$$2^{m+n}\binom{m+n}{m}_F = 2^{m+n-1}L_m\binom{m+n-1}{m}_F + 2^{m+n-1}L_n\binom{m+n-1}{m-1}_F.$$

Dividing both sides by 2^{m+n-1} gives us

Identity 5. *For $m, n \geq 1$,*

$$2\binom{m+n}{m}_F = L_m\binom{m+n-1}{m}_F + L_n\binom{m+n-1}{m-1}_F.$$

In full disclosure, Identities 4 and 5 are used by Sagan and Savage to prove their combinatorial interpretations, so it is no surprise that these identities would have easy combinatorial proofs. The same is true for the weighted (or colorized) version of these identities for Lucasnomial coefficients.

Identity 6. *For $m, n \geq 1$,*

$$\binom{m+n}{m}_U = U_{m+1}\binom{m+n-1}{m}_U + U_{n-1}\binom{m+n-1}{m-1}_U.$$

Identity 7. *For $m, n \geq 1$,*

$$2\binom{m+n}{m}_U = V_m\binom{m+n-1}{m}_U + V_n\binom{m+n-1}{m-1}_U.$$

By considering the number of vertical steps that a lattice path ends with, Reiland [8] proved

Identity 8. *For $m, n \geq 1$,*

$$\binom{m+n}{m}_F = \sum_{j=0}^{n} F_{m+1}^{j} F_{n-j-1}\binom{m-1+n-j}{m-1}_F$$

Combinatorial Proof: We count the tiled lattice paths from $(0,0)$ to (m,n) by considering the number j of vertical steps at the end of the path, where $0 \leq j \leq n$. Such a tiling begins with j full rows, which can be tiled F_{m+1}^{j} ways. Since the lattice path must have a horizontal step from $(m-1, n-j)$ to $(m, n-j)$, the last column will have height $n-j$ and can be tiled (without starting with a square) in F_{n-j-1} ways. The rest of the tiling consists of a tiled lattice path from $(0,0)$ to $(m-1, n-j)$ which can be created in $\binom{m-1+n-j}{m-1}_F$ ways. (Note that when $j = n-1$, the summand is 0, since $F_0 = 0$, as is appropriate since the last column can't have height 1 without starting with a square; also, when $j = n$, $F_{-1} = 1$, so the summand simplifies to F_{m+1}^{n}, as required.) All together, the number of tilings is $\sum_{j=0}^{n} F_{m+1}^{j} F_{n-j-1}\binom{m-1+n-j}{m-1}_F$, as desired.

□

By the exact same logic, using bracelet tilings, we get

Identity 9. *For $m, n \geq 1$,*

$$2^{m+n}\binom{m+n}{m}_F = \sum_{j=0}^{n} L_m^{j} L_{n-j} 2^{m+n-1-j}\binom{m-1+n-j}{m-1}_F$$

Replacing F with U and replacing L with V, the last two identities are appropriately colorized as well.

4. Open Problems

What follows is a list of Fibonomial identities that are still in need of combinatorial proof. Some of these identities have extremely simple algebraic proofs (and some hold for more general sequences than Fibonomial sequences) so one would expect them to have elementary combinatorial proofs as well.

Many simple identities appear in Fibonacci Quarterly articles by Gould [4, 5].

$$\binom{n}{k}_F \binom{k}{j}_F = \binom{n}{j}_F \binom{n-j}{k-j}_F$$

$$\binom{n}{k}_F = \sum_{j=k}^{n} \frac{F_j - F_{j-k}}{F_k} \binom{j-1}{k-1}_F$$

$$F_k \binom{n}{k}_F = F_n \binom{n-1}{k-1}_F = F_{n-k+1} \binom{n}{k-1}_F$$

Here is another basic identity for generalized binomial coefficients, first noted by Fontené [3] and further developed by Trojovský [10]

$$\binom{n}{k}_F - \binom{n-1}{k}_F = \binom{n-1}{k-1}_F \frac{F_n - F_k}{F_{n-k}}.$$

Here are some alternating sum identities, provided by Lind [7] and Cooper and Kennedy [2], respectively, that might be amenable to sign-reversing involutions:

$$\sum_{j=0}^{k+1} (-1)^{j(j+1)/2} \binom{k+1}{j}_F \binom{n-1}{k}_F = 0.$$

$$\sum_{j=0}^{k} (-1)^{j(j+1)/2} \binom{k}{j}_F F_{n-j}^{k-1} = 0.$$

Here are some special cases of very intriguing formulas that appear in a recent paper by Kilic, Akkus and Ohtsuka [6].

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k}_F = \prod_{k=0}^{n} L_{2k}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k}_F = (-1)^n \prod_{k=1}^{2n} L_{2k-1}$$

We have just scratched the surface here. There are countless others!

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AMS Classification Numbers: 05A19, 11B39

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