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Magical Miscellany

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What is a Math Fun Fact, you ask? A Math Fun Fact is any mathematical tidbit that can be presented or grasped quickly, is surprising or captivating, can be generally enjoyed by friends of mathematics, and is hopefully fun! Of course, part of the fun is thinking about why the Fun Fact is true—so we won't spoil the fun. Though, we may give you some hints and references.

However, since there are infinitely many Math Fun Facts (prove this), we can only bring you a few each time... here are a few whose conclusions might be considered "magical".

Magic "Squares," Indeed!

Perhaps you've seen the magic square

8	1	6
3	5	7
4	9	2

which has the property that all rows, columns, and diagonals sum to 15. Well, it has another "magic" and "square" property! If you read the rows as *numbers*, forwards and backwards, and square them, then:

$$816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 294^2.$$

In fact, something similar is true for the columns! Reading them forwards and backwards, and squaring them, one finds that:

$$834^2 + 159^2 + 672^2 = 438^2 + 951^2 + 276^2.$$

But wait, there's more! We'll leave it to you, O gentle reader, to try each of the "diagonals" which wrap diagonally around the square.

This amazing property is true of *any* 3×3 magic square (though if the entries contain more than one digit, you will have to carry the extra places). The proof of this fact, as you might guess, uses properties of matrix algebra. What can be said about numbers in other bases? What about $n \times n$ magic squares? You may wish to think about such questions before consulting Benjamin and Yasuda, "Magic Squares Indeed!" (*American Mathematical Monthly*, 106 (1999), pp. 152–156).

Kaprekar's Magical Constant

Take any four digit number (whose digits are not all identical), and do the following:

1. Rearrange the string of digits to form the largest and smallest 4-digit numbers possible.
2. Take these two numbers and subtract the smaller number from the larger.
3. Use the number you obtain and repeat the above process.

What happens if you repeat the above process over and over? Let's see. Suppose we choose the number 3141.

$$4311 - 1134 = 3177.$$
$$7731 - 1377 = 6354.$$
$$6543 - 3456 = 3087.$$
$$8730 - 0378 = 8352.$$
$$8532 - 2358 = 6174.$$
$$7641 - 1467 = 6174\ldots$$

The process eventually hits 6174 and then stays there!

But the more amazing thing is this: every four digit number whose digits are not all the same will eventually hit 6174, in at most 7 steps, and then stay there! The number 6174 is known as *Kaprekar's constant*, named after D.R. Kaprekar, who discovered this fact in 1955.

If you encounter a number with fewer than 4 digits, it must be treated as though it has 4 digits, using leading zeroes. Example: if you start with 3222 and subtract 2333, then the difference is 0999. The next step would then consider the difference $9990 - 0999 = 8991$, and so on.

With a little thought, we can see that any starting number must give a sequence that eventually cycles—since each number uniquely determines the next number in the sequence and there are only finitely many possibilities, the sequence must return to a number it hit before, leading to a cycle. But there might be many such cycles; the surprise is that for length 4 strings in base 10, there happens to be 1 non-trivial cycle, and that cycle has just one number in it, namely 6174.

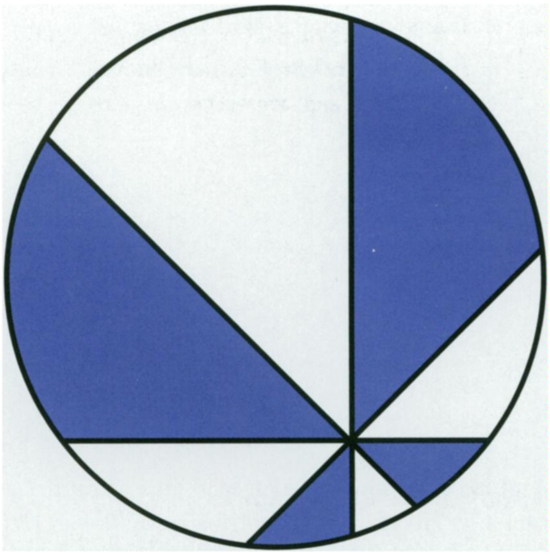

Figure 1. As long as the angles are equal at the intersection point, the blue slices and the white slices contain the same total area.

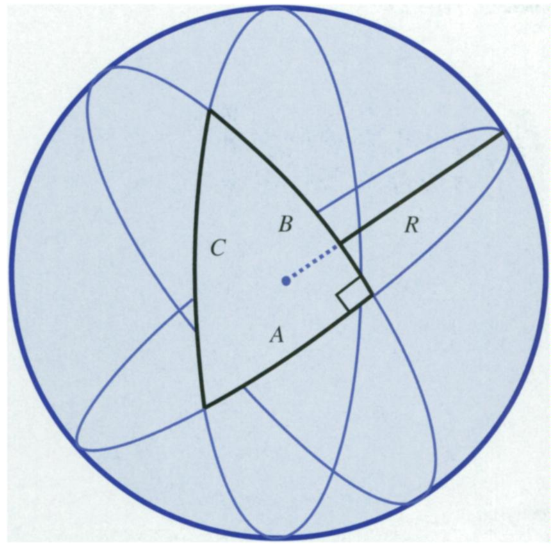

Figure 2. A spherical right triangle. Edges *A*, *B*, and *C* are arcs of great circles on the sphere of radius *R*.

You might investigate what happens for strings of other lengths or in other bases. Then check out Eldridge and Sagong, "The determination of Kaprekar convergence and loop convergence of all three digit numbers" (*American Mathematical Monthly*, 95 (1988), pp. 105–112).

Pizza Slices!

Take a pizza and pick an arbitrary point in it. Suppose you cut the pizza into 8 slices by cutting at 45 degree angles through that point, and color the alternate pieces blue and white. See Figure 1.

Surprising fact: the total area of the blue slices and the total area of the white slices will always be the same!

In fact, this theorem is true if the number of slices is any multiple of 4, except for 4, and the slices are cut by using equal angles through a fixed arbitrary point in the pizza. Alternatively, if instead of equal angles, you use equal-length arcs on the circumference and slice from a fixed arbitrary point in the pizza, the conclusion still hold if the number of slices is even and greater than 2.

These facts can be proved using calculus and polar coordinates. The book *Which Way Did the Bicycle Go*? by Konhauser, Velleman, and Wagon gives background and generalizations.

A Spherical Pythagorean Theorem

Did you know there is a version of the Pythagorean Theorem for right triangles on spheres?

First, let's define precisely what we mean by a spherical triangle. A *great circle* on a sphere is any circle whose center coincides with the center of the sphere. A *spherical triangle* is any 3-sided region enclosed by sides that are arcs of great circles. If one of the corner angles is a right angle, the triangle is a *spherical right triangle*. See Figure 2.

In such a triangle, let C denote the length of the side opposite the right angle. Let A and B denote the lengths of the other two sides. Let R denote the radius of the sphere. Then the following particularly nice formula holds:

$$\cos(C/R) = \cos(A/R) \cdot \cos(B/R).$$

It is fun to verify that the formula holds in some simple examples; for instance, try a triangle with *two* right angles, formed by great circles at the equator and two longitudes. Note that the other angle could be anything. Is that reflected in the formula above?

This formula is called the *Spherical Pythagorean Theorem* because the regular Pythagorean theorem can be obtained as a special case—just expand the cosines using their Taylor series, manipulate the resulting expression, and you will find

$$C^2 = A^2 + B^2$$

as R goes to infinity! This should make sense, since as R goes to infinity, spherical geometry becomes more and more like regular planar geometry!

By the way, there is a "hyperbolic geometry" version, too. Can you guess what it says? See Velian, "The 2500-year-old Pythagorean Theorem" (*Mathematics Magazine*, 73 (2000), pp. 259–272).

More Math Fun

These Fun Facts, and many others, can be found at the Math Fun Facts website: `http://www.math.hmc.edu/funfacts`. ■