4-1-2013

A Borsuk-Ulam Equivalent that Directly Implies Sperner's Lemma

Kathryn L. Nyman
Willamette University

Francis Su
Harvey Mudd College

Recommended Citation

Nyman, Kathryn and Francis Edward Su. "A Borsuk-Ulam equivalent that implies Sperner's Lemma." American Mathematical Monthly, 120 (2013), 346–354.

A Borsuk–Ulam Equivalent that Directly Implies Sperner's Lemma

Kathryn L. Nyman and Francis Edward Su

Abstract. We show that Fan's 1952 lemma on labelled triangulations of the n-sphere with $n + 1$ labels is equivalent to the Borsuk–Ulam theorem. Moreover, unlike other Borsuk–Ulam equivalents, we show that this lemma directly implies Sperner's Lemma, so this proof may be regarded as a combinatorial version of the fact that the Borsuk–Ulam theorem implies the Brouwer fixed-point theorem, or that the Lusternik–Schnirelmann–Borsuk theorem implies the KKM lemma.

1. INTRODUCTION. The Brouwer fixed-point theorem, the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma, and Sperner's lemma are known to be equivalent. Equally powerful, they form a triumvirate of theorems whose interconnections have been exploited with great success in fixed point algorithms [15, 17] as well as in game theory [1]. Similarly, the Borsuk–Ulam theorem, the Lusternik–Schnirelmann–Borsuk (LSB) theorem, and Tucker's lemma are another triumvirate of equivalent results. In each of these triples, the first is a topological result, the second is a set-covering result, and the third is a combinatorial result.

Moreover, these triples are related to each other. Since the Borsuk–Ulam theorem implies the Brouwer fixed-point theorem, any theorem in the second triple must imply any theorem in the first. It is an interesting question to find *direct* proofs of each implication. For instance, a topological construction shows how a Brouwer fixed point follows from Borsuk–Ulam antipodes [13], and with set-coverings, the LSB theorem can be used to directly prove the KKM lemma [11]. But in the combinatorial domain, we are unaware of a direct proof that Tucker's lemma implies Sperner's lemma.

In this article, we show that another combinatorial lemma, Fan's $N + 1$ Lemma, may be a more natural combinatorial analogue to the Borsuk–Ulam theorem, and therefore more worthy to sit in the Borsuk–Ulam triumvirate than Tucker's lemma. In particular, in Section 3 we show that Fan's $N + 1$ Lemma is equivalent to the Borsuk–Ulam theorem, and in Section 4 we exhibit a direct proof that it implies Sperner's lemma (see Figure 1).

2. BACKGROUND. We first review these theorems. Let Σ^n be a polyhedral version of the n-sphere, the set of all points in $\mathbb{R}^{n+1}$ of distance 1 from the origin in the L_1 norm:

$$\Sigma^n = \{(x_1, \ldots, x_{n+1}) : \sum |x_i| = 1\}.$$

In $\mathbb{R}^3$, Σ^2 is just the boundary of the octahedron. As with the octahedron, note that Σ^n is naturally subdivided into orthants; we will study labelled triangulations of Σ^n that refine the orthant subdivision. A *triangulation* is a subdivision by simplices that either meet face-to-face or not at all. Each simplex is the affine hull of its *vertices*; these are the *vertices of the triangulation*. A triangulation of Σ^n is *symmetric* if, when σ is a simplex of the triangulation, then $-\sigma$ is a simplex as well.
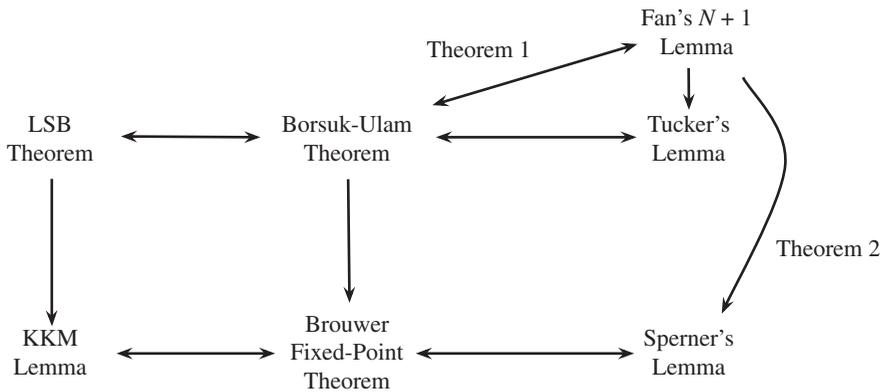
Figure 1. Connections between the topological, set-covering, and combinatorial results.

Define an *m-labelling* to be a function ℓ that assigns to each vertex v one of $2m$ possible integers: $\{\pm 1, \pm 2, \ldots, \pm m\}$. A symmetric triangulation of Σ^n has an *anti-symmetric* labelling if $\ell(-v) = -\ell(v)$ for all vertices v. A labelling has a *complementary edge* if some adjacent pair of vertices has labels that sum to zero, e.g., $\{+i, -i\}$.

Call a simplex *alternating* if its vertex labels are distinct in magnitude and alternate signs, when arranged in order of increasing value. So the labels have the form

$$\{k_1, -k_2, k_3, \ldots\} \quad \text{or} \quad \{-k_1, k_2, -k_3, \ldots\}$$

when $1 \le k_1 < k_2 < k_3 < \cdots$. The first kind is called *positive alternating* and the second is *negative alternating*, based on the sign of k_1. For instance, a triangle labelled $\{-1, +3, -7\}$ would be negative alternating, and an edge labelled $\{+2, -3\}$ would be positive alternating.

Fan's $N + 1$ Lemma. *Let T be a symmetric triangulation of Σ^n with an $(n + 1)$-labelling that is anti-symmetric and has no complementary edge. Then T has a positive alternating n-simplex.*

Thus, if the boundary of an octahedron (e.g., see Figure 7) has a triangulation anti-symmetrically labelled by $\{\pm 1, \pm 2, \pm 3\}$ and no complementary edges, then it must have a $\{+1, -2, +3\}$ triangle.

We call this Fan's $N + 1$ Lemma because Fan's original lemma [**4**] is more general; it says that for any m-labelling with the same hypotheses, there are an odd number of positive alternating n-simplices and an equal number of negative alternating n-simplices. And as [**9**] shows, the result holds for more general triangulations of S^n with a constructive proof. When $m = n + 1$, an m-labelling has only one kind of positive alternating simplex—namely, the simplex with labels of every magnitude: $\{1, -2, +3, \ldots, (-1)^n(n + 1)\}$.

Note that if an anti-symmetric m-labelling has no complementary edge, then $m \ge n + 1$, because alternating simplices must have $n + 1$ different label values (apart from sign). Since an n-labelling is an $(n + 1)$-labelling with one label missing, then as noted by Fan [**4**], the contrapositive of Fan's $N + 1$ Lemma yields Tucker's lemma as a corollary.

Tucker's Lemma. *Let T be a symmetric triangulation of Σ^n with an n-labelling that is anti-symmetric. Then T has a complementary edge. See Figure 2.*
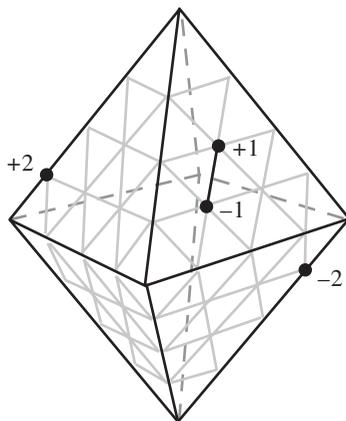
Figure 2. A complementary edge is guaranteed by Tucker's lemma when the polyhedral 2-sphere has a symmetric triangulation with an anti-symmetric 2-labelling.

Tucker's lemma [**6, 16**] was originally proposed as a combinatorial equivalent of the Borsuk–Ulam theorem [**2**], though it has found other applications as well (e.g., [**10**]).

Borsuk–Ulam Theorem. *Let $h : S^n \to \mathbb{R}^n$ be a continuous function such that $h(-x) = -h(x)$ for all $x \in S^n$. Then there exists $w \in S^n$ such that $h(w) = 0$.*

A set covering result due to Lusternik–Schnirelman–Borsuk [**2, 7**] is also equivalent to the Borsuk–Ulam theorem.

LSB Theorem. *Let $C_1, \ldots, C_{n+1}$ be a collection of closed sets that cover S^n. Then at least one of the sets must contain a pair of antipodal points.*

These theorems (Fan, Tucker, Borsuk–Ulam, LSB) concern topological or polyhedral n-spheres. The next three theorems concern topological and polyhedral n-balls.

Let B^n denote an n-ball, the set of all points within unit distance of the origin in $\mathbb{R}^n$. A polyhedral version of an n-ball is an n-simplex, which is more naturally described by its embedding in $\mathbb{R}^{n+1}$:

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) : x_i \geq 0, \sum x_i = 1\}.$$

It is homeomorphic to an n-ball. For any $v = (v_1, \ldots, v_{n+1}) \in \Delta^n$, let

$$Z(v) = \{i : v_i \neq 0\}$$

be the set of indices of coordinates of v that are nonzero. Thus in Δ^2, $Z((0, 1, 0)) = \{2\}$ and $Z((.3, 0, .7)) = \{1, 3\}$. Suppose T is a triangulation of Δ^n. A *Sperner-labelling* ℓ assigns to each vertex v a label from $\{1, \ldots, n + 1\}$ such that

$$\ell(v) \in Z(v). \tag{1}$$

This forces each main vertex of Δ^n to have a different label (the index of its one nonzero coordinate), and any vertex on a face of Δ^n can only be labelled by one of the main vertices that span that face. Call an n-simplex in the triangulation *fully-labelled* if its vertices have distinct labels (and therefore all labels $\{1, \ldots, n + 1\}$).
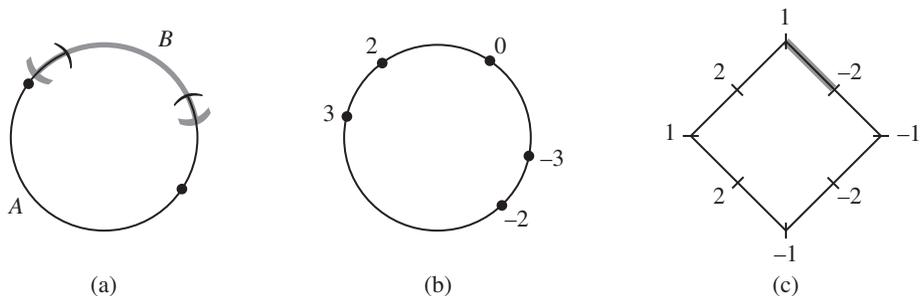
Figure 3. (a) LSB: a pair of antipodal points contained in one of two closed sets that cover S^1. (b) Borsuk–Ulam: Given a continuous, anti-symmetric function from S^1 to $\mathbb{R}$, there is a point mapped to 0. (c) Fan's $N + 1$: an antisymmetric 2-labelling of Σ^1 with no complementary edge must have a positive alternating edge (shaded).

Sperner's Lemma. *Any Sperner-labelled triangulation of Δ^n must have a fully-labelled n-simplex.*

In fact, there are an odd number of such simplices [12]. An exposition and applications may be found in [14]. Sperner's lemma provides the simplest route to proving this famous theorem of Brouwer [3].

Brouwer Fixed-Point Theorem. *For any continuous function $f : B^n \rightarrow B^n$, there exists a point $x \in B^n$ such that $f(x) = x$.*

Knaster–Kuratowski–Mazurkiewicz [5] provided the original link between the Brouwer theorem and Sperner's lemma.

KKM Lemma. *Let $C_1, \ldots, C_n$ be a collection of closed sets that cover Δ^n such that for each $I \subseteq [n + 1]$, the face spanned by the set $\{e_i | i \in I\}$ is covered by $\{C_i | i \in I\}$. Then $\cap_{i=1}^{n} C_i$ is nonempty.*
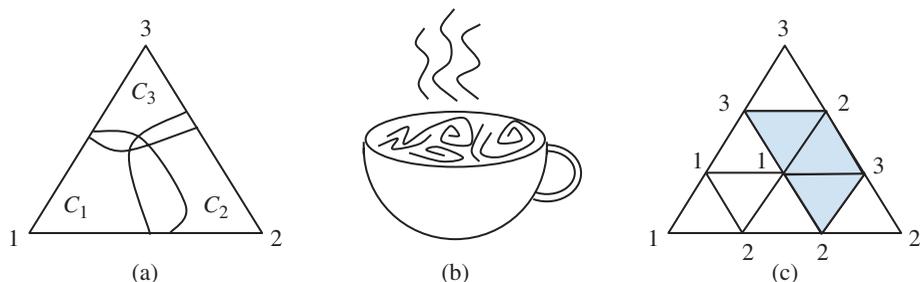
Figure 4. (a) KKM: these sets have a non-empty intersection. (b) Brouwer: the stirred coffee has a point that is in the same place as before the stirring. (c) Sperner: there's an odd number of 123-triangles.

3. EQUIVALENCE OF FAN'S $N + 1$ LEMMA AND THE BORSUK–ULAM THEOREM. As discussed earlier, Fan's general lemma with m-labellings [4] implies the Borsuk–Ulam Theorem through Tucker's lemma. Here we show that Fan's $N + 1$ Lemma is equivalent to the Borsuk–Ulam theorem.

Theorem 1. *Fan's $N + 1$ Lemma is equivalent to the Borsuk–Ulam Theorem.*

Proof. We first show that the Borsuk–Ulam Theorem implies Fan's $N + 1$ Lemma. Let T be a symmetric triangulation of Σ^n with an anti-symmetric $(n + 1)$-labelling L, in which there are no complementary edges. Let $w_i \in \mathbb{R}^{n+1}$ be the point with ith coordinate n and other coordinates -1:

$$w_i = (-1, \ldots, -1, n, -1, \ldots - 1).$$

Let $W_+ = \{w_1, \ldots, w_{n+1}\}$ and $W_- = \{-w_1, \ldots, -w_{n+1}\}$. The set $W = W_+ \cup W_-$ comprises $2n + 2$ points that lie on the n-dimensional hyperplane: $H = \{(x_1, \ldots, x_{n+1}) : \sum_{i=1}^{n+1} x_i = 0\}$.
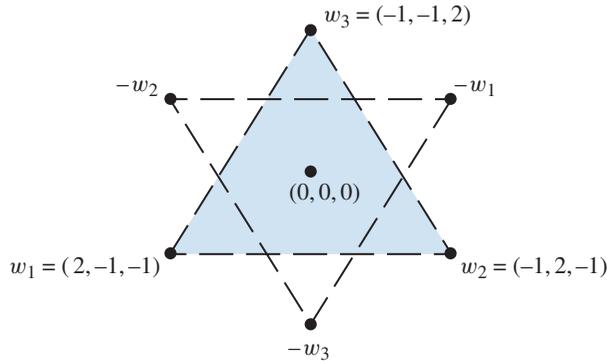

Figure 5. For $n = 2$, the points w_1, w_2, w_3 and $-w_1, -w_2, -w_3$ in the hyperplane H. The shaded region indicates the image under h of a positive alternating 2-simplex, which maps to a simplex containing all the positive w_i (and the origin).

Define a continuous map $h : \Sigma^n \to H$ as follows. For each $v \in T$, let

$$h(v) = \begin{cases} w_{L(v)} & \text{if } L(v) \text{ is odd} \\ -w_{L(v)} & \text{if } L(v) \text{ is even,} \end{cases} \tag{2}$$

where $w_{-i} = -w_i$ in case $L(v) < 0$. Extend h linearly to each simplex of T. Since L is an anti-symmetric labelling, we see $h(-x) = -h(x)$ for all $x \in \Sigma^n$. Therefore, by Borsuk–Ulam there is a $z \in \Sigma^n$ such that $h(z) = 0$.

Thus z is in some n-simplex σ such that $h(\sigma)$ contains the origin. The images of the vertices of σ form a set $K = \{h(v) : v \in \sigma, v \in T\}$, a subset of W of size $n + 1$ or smaller (if there are repeated labels). Since there are no complementary edges in T, the set K contains no pair $\{w_j, -w_j\}$. Then $K = \{w_j\}_{j \in B} \cup \{-w_j\}_{j \in B'}$, where B and B' are disjoint subsets of $\{1, \ldots, n + 1\}$.

Now consider the sum of vectors in K:

$$\hat{v} = \sum_{j \in B} w_j - \sum_{j \in B'} w_j.$$

Note that the dot products $w_i \cdot w_i = n(n + 1)$ for all $i \in [n + 1]$, and $w_i \cdot w_j = -(n + 1)$ for all $j \neq i$. So, for $i \in B$, the dot product

$$w_i \cdot \hat{v} = n(n + 1) - (|B| - 1)(n + 1) + |B'|(n + 1)$$
$$= (n + 1)(n + 1 - |B| + |B'|),$$

© THE MATHEMATICAL ASSOCIATION OF AMERICA [Monthly 120

which is positive unless $|B| = n + 1$ and $|B'| = 0$, i.e., $K = W_+$. And for $i \in B'$,

$$-w_i \cdot \hat{v} = |B|(n + 1) + n(n + 1) - (|B'| - 1)(n + 1)$$
$$= (n + 1)(|B| - |B'| + n + 1),$$

which is positive unless $|B'| = n + 1$ and $|B| = 0$, i.e., $K = W_-$. Since the convex hull of K contains the origin, it cannot be the case that all vectors in K have a positive dot product with $\hat{v}$. So either $K = W_+$ or $K = W_-$ (and indeed, in these cases, K's convex hull contains the origin).

If $K = W_+$, then (2) shows the original simplex σ has labels $\{1, -2, \ldots, (-1)^n(n + 1)\}$. If $K = W_-$, then (2) and anti-symmetry of L shows that $-\sigma$ has these labels. In either case we find a positive alternating simplex, as desired.

Now we show Fan's $N + 1$ Lemma implies the Borsuk–Ulam Theorem. Let $h : \Sigma^n \to \mathbb{R}^n$ be a continuous function such that $h(-x) = -h(x)$ for all $x \in \Sigma^n$. Assume, by way of contradiction, that there is no point $z \in \Sigma^n$ such that $h(z) = 0$. If $h(x) = (x'_1, \ldots, x'_n)$, let $\hat{h} : \Sigma^n \to \mathbb{R}^{n+1}$ be the function defined by $\hat{h}(x) = (x'_1, \ldots, x'_n, -\sum_{i=1}^n x'_i)$. So $\hat{h}$ maps Σ^n to the hyperplane H and preserves continuity and anti-symmetry. Furthermore, there is no point z such that $\hat{h}(z) = 0$.

Let T be a symmetric triangulation of Σ^n, and let the set W be as above. We wish to construct a labelling L on the vertices of T that is anti-symmetric.

For $v \in T$, define $L(v)$ to be the index i such that w_i is closest to $\hat{h}(v)$ in $\mathbb{R}^{n+1}$. Note that $i \in \{\pm 1, \ldots, \pm(n + 1)\}$. In the case of ties, choose the index with the smallest absolute value. This is well-defined because $\hat{h}(v)$ is never 0, and no nonzero point can be equidistant from w_i and $w_{-i} = -w_i$. That L is anti-symmetric follows from noting that $\hat{h}$ is anti-symmetric, so $\hat{h}(v)$ is closest to w_i if and only if $\hat{h}(-v)$ is closest to w_{-i}.

Therefore, by Fan's $N + 1$ Lemma, there exists either a complementary edge $(+i, -i)$, for some i, or an alternating simplex with labels $\{1, -2, \ldots, (-1)^n(n + 1)\}$. By taking finer and finer triangulations, and by the compactness of the Σ^n, there exists a convergent subsequence of shrinking positive alternating simplices or a convergent subsequence of shorter complementary edges involving the same index i. This gives a limit point which, by the continuity of $\hat{h}$, is either equidistant from both w_i and $-w_i$, or is equidistant from all points in $\{w_1, -w_2, w_3, \ldots, (-1)^n w_{n+1}\}$. But the only point with this property is 0. Thus, the limit point z must satisfy $\hat{h}(z) = 0$ and therefore, $h(z) = 0$. ∎

4. FAN'S $N + 1$ LEMMA IMPLIES SPERNER'S LEMMA.

Now we establish how Fan's $N + 1$ Lemma will indeed prove Sperner's lemma by a direct construction, so it is the "right" combinatorial result to sit in the Borsuk–Ulam triumvirate. Prescott [**8**] established this implication in dimension two by a different method.

Theorem 2. *Fan's $N + 1$ Lemma implies Sperner's lemma.*

Proof. Consider a triangulation S of Δ^n with a Sperner-labelling ℓ. We first extend S to a triangulation T of Σ^n by reflecting copies of S to the other orthants of Σ^n. Let $G = \{\pm 1\}^{n+1}$ denote the group of symmetries of Σ^n generated by reflections that flip the sign of selected coordinates; then the action of $g = (g_1, \ldots, g_{n+1}) \in G$ on $v = (v_1, \ldots, v_{n+1}) \in \Sigma^n$ produces $gv = (g_1 v_1, \ldots, g_{n+1} v_{n+1}) \in \Sigma^n$. So g reflects v in all coordinates i for which $g_i = -1$. Note that $g = (1, 1, \ldots, 1)$ is the identity in G. The idea of this construction is illustrated in Figure 6.
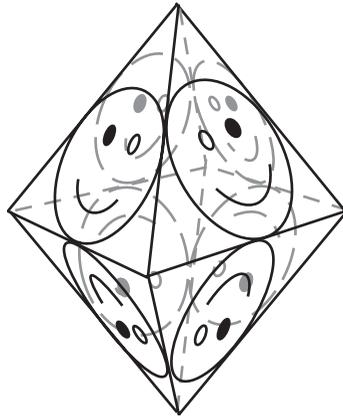
Figure 6. The actions of G on Σ^n, as shown by their effects on Mr. Smiley.

Similarly, if σ is a simplex in S spanned by a set of vertices V, we define $g\sigma$ to be the simplex spanned by the vertices in $gV = \{gv : v \in V\}$. Let T be the collection of simplices $\{g\sigma : \sigma \in S \text{ and } g \in G\}$. Then T is a triangulation of Σ^n, since the reflection method ensures that simplices of T meet face-to-face along reflected facets of S.

Now we extend the labelling ℓ on vertices of S to a labelling L on vertices of T by reflection but with possible sign modifications. Define

$$L(gv) = g_{\ell(v)} \cdot (-1)^{\ell(v)+1} \cdot \ell(v) \tag{3}$$

for each $v \in S$. Notice that $L(gv)$ and $\ell(v)$ have the same label value (but possibly different signs). When $g = (1, 1, \ldots, 1)$, this defines L on S and the factor $(-1)^{\ell(v)+1}$ turns fully-labelled simplices into positive alternating simplices. When g is non-trivial, L defines a labelling of vertices on reflected copies of S (see Figure 7).
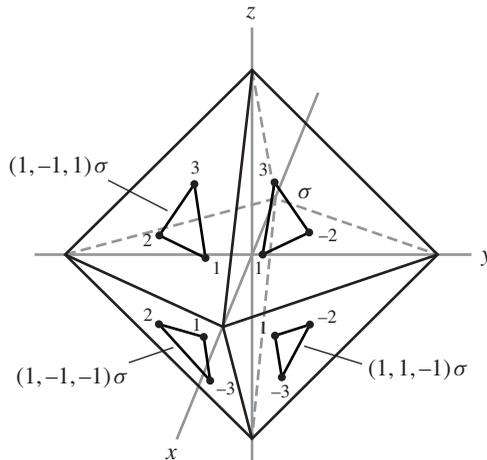

Figure 7. A positive alternating simplex σ in T arising from a fully-labelled simplex with labels $\{1, 2, 3\}$ in S, and reflected simplices $g\sigma$ for $g = (1, 1, -1)$, $(1, -1, 1)$, and $(1, -1, -1)$ with their L-labellings indicated.

We might worry that L is not well-defined where orthants meet. However, orthants meet where $gv = \hat{g}\hat{v}$, for some $g, \hat{g} \in G$ and some $v, \hat{v} \in S$. But then $g_i v_i = \hat{g}_i \hat{v}_i$

for each i, which implies $v_i = \hat{v}_i$ since $g_i, \hat{g}_i = \pm 1$. Then $g_i = \hat{g}_i$ when $v_i \neq 0$, i.e., when $i \in Z(v)$. But $\ell(v) \in Z(v)$ by (1), so that $g_{\ell(v)} = \hat{g}_{\ell(v)}$. It follows from (3) that $L(gv) = L(\hat{g}v)$, so L is well-defined.

Now we show that L satisfies the conditions of Fan's $N + 1$ Lemma. Antipodal labels sum to zero by construction: The point antipodal to v is $-v = \bar{g}v$, where $\bar{g} = (-1, -1, \ldots, -1)$, so that (3) gives $L(-v) = -L(v)$. Also, we can show that L has no complementary edges. Every edge in T is a reflected copy of some edge in S via some $g \in G$, and the Sperner-labelling ℓ of S has no complementary edges (all labels are positive). Then the rule (3) shows that for any choice of g, two vertices $v, w \in S$ will have identical ℓ-labels ($\ell(v) = \ell(w)$) if and only if their g-reflections have identical L-labels as well ($L(gv) = L(gw)$). So L has no complementary edges, because ℓ did not.

Thus Fan's $N + 1$ Lemma applies, so there exists a positive alternating n-simplex in T. Since Δ^n is the only facet of Σ^n that contains the labels $\{1, -2, 3, \ldots, (-1)^n(n + 1)\}$, there must be a fully-labeled n-simplex in S. ∎

In fact, as noted earlier, a stronger version of Fan's $N + 1$ Lemma holds, whose conclusion is that there are in fact an odd number of positive alternating n-simplices. Then the above argument would demonstrate the stronger version of Sperner's lemma, which concludes that there are an odd number of fully-labelled n-simplices in S.

REFERENCES

1. K. C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge, 1985.
2. K. Borsuk, Drei Sätze über die *n*-dimensionale euklidische Sphäre, *Fund. Math.* **20** (1933) 177–190.
3. L. E. J. Brouwer, Über Abbildungen von Mannigfaltigkeiten, *Math. Ann.* **71** (1912) 97–115, available at http://dx.doi.org/10.1007/BF01456931.
4. K. Fan, A generalization of Tucker's combinatorial lemma with topological applications. *Ann. of Math.* **56** (1952) 431–437, available at http://dx.doi.org/10.2307/1969651.
5. B. Knaster, C. Kuratowski, S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für *n*-dimensionale Simplexe, *Fund. Math.* **14** (1929) 132–137.
6. S. Lefshetz, *Introduction to Topology*, Princeton University Press, Princeton, 1949.
7. L. Lusternik, L. Schnirelman, Topological Methods in Variational Calculus, *Issledowatelskiǐ Institut Matematiki i Mechaniki pri O.M.G.U., Moscow*, 1930 [Russian].
8. T. Prescott, *Extensions of the Borsuk–Ulam Theorem*, Senior Thesis, Harvey Mudd College, 2002.
9. T. Prescott, F. E. Su, A constructive proof of Ky Fan's generalization of Tucker's lemma, *J. Combin. Theory Ser. A* **111** (2005) 257–265, available at http://dx.doi.org/10.1016/j.jcta.2004.12.005.
10. F. W. Simmons, F. E. Su, Consensus-halving via theorems of Borsuk–Ulam and Tucker, *Math. Social Sci.* **45** (2003) 15–25, available at http://dx.doi.org/10.1016/S0165-4896(02)00087-2.
11. G. Spencer, F. E. Su, The LSB theorem implies the KKM lemma, *Amer. Math. Monthly* **114** (2007) 156–159.
12. E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg* **6** (1928) 265–272, available at http://dx.doi.org/10.1007/BF02940617.
13. F. E. Su, Borsuk–Ulam implies Brouwer: a direct construction. *Amer. Math. Monthly* **104** (1997) 855–859, available at http://dx.doi.org/10.2307/2975293.
14. ———, Rental harmony: Sperner's lemma in fair division, *Amer. Math. Monthly* **106** (1999) 930–942, available at http://dx.doi.org/10.2307/2589747.
15. M. J. Todd, *The Computation of Fixed Points and Applications*, Lecture Notes in Economic and Mathematical Systems, Springer-Verlag, New York, 1976.
16. A. W. Tucker, Some topological properties of the disk and sphere, *Proc. First Canad. Math. Congr.*, Montreal, (1945) 285–309.
17. Z. Yang, *Computing Equilibria and Fixed Points*, Kluwer Academic Publishers, Boston, 1999.

KATHRYN L. NYMAN received her B.A. from Carthage College in 1995 and her Ph.D. from Cornell University in 2001. She is an associate professor of mathematics at Willamette University, nestled near waterfalls, wineries, and the occasional alpaca farm in Oregon's Willamette Valley. Her research is in algebraic and enumerative combinatorics. Swing dancing, hiking, painting, and board games keep Kathryn out of trouble. She heartily recommends working with a Los Angeles collaborator to all of her rainy Pacific Northwest and aspiring-extra colleagues; on a research trip for this paper she filmed a webisode of *The Guild*.
Department of Mathematics, Willamette University, Salem, OR 97301
knyman@willamette.edu

FRANCIS EDWARD SU credits Michael Starbird at UT-Austin for stimulating his interest in fixed-point theorems as an undergraduate. He earned his Ph.D. at Harvard and is now professor of mathematics at Harvey Mudd College. His research is in geometric combinatorics and applications to the social sciences. His passion for popularizing mathematics is evident in his hugely popular *Math Fun Facts* website and iPhone app, and in his real analysis course on YouTube. In his spare time, gardening and songwriting give him outlets for pondering deep theological questions. Oregonian collaborators can provide opportunities for stardom as well; his appearance as an extra in the movie *Blue Like Jazz* was filmed in Portland during the writing of this paper, and it may yield him one of the lowest Erdős-Bacon numbers.
Department of Mathematics, Harvey Mudd College, Claremont, CA 91711
su@math.hmc.edu

An Easy Proof of the Divergence of the Harmonic Series Sum

The sum of the harmonic series, 1, 1/2, 1/3, diverges, even as the terms tend to zero. Many proofs of this significant fact are available, such as the well-known proof by N. Oreseme, and the more recent ones (see, for instance, [**1, 2**]). We give another.

Let $t_n = 1 + 1/2 + \cdots + 1/n, n = 1, 2, \ldots$ We may note that

$$t_{k+m} = t_k + 1/(k+1) + 1/(k+2) + \cdots + 1/(k+m) > t_k + m/(k+m),$$

for a finite, fixed k, $m/(k+m) \to 1$, as $m \to \infty$. Therefore, we may consider $\epsilon > 0$, sufficiently small, and get an m such that $p/(k+p) > 1 - \epsilon$, $p = m, m+1, \ldots$ So, $t_{k+m} - t_k > 1 - \epsilon$. Fix ϵ. For such ϵ, we then consider t_{k+m}, and get a finite r, such that $t_{k+m+r} - t_{k+m} > 1 - \epsilon$. This may be continued so that terms, with finite indices, are obtained, each of which exceeds the previous term by at least $1 - \epsilon$. So there cannot exist any upper bound for the series $t_n, n = 1, 2, \ldots$. Hence, the sum of the harmonic series diverges and cannot have a limit.

REFERENCES

1. D. M. Bradley, A note on the divergence of the harmonic series, *Amer. Math. Monthly* **107** (2000) 651, available at http://dx.doi.org/10.2307/2695585.
2. A. Cusumano, Generalization of Oresme's standard method of showing the divergence of the harmonic series, *Amer. Math. Monthly* **119** (2012) 210.

—Submitted by Pritibhushan Sinha
6 A J C Bose Road Kolkata - 700063 India
p_sinha@indiatimes.com