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Alfonso Castro
Harvey Mudd College

J. V. A. Gonçalves
University of Brasília

ON MULTIPLE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS WITH ODD NONLINEARITIES

Alfonso Castro B.

Departamento de Matemáticas
Centro de Investigación del IPN
Apartado Postal 14740
México 14, D.F. México

and

J. V. A. Gonçalves

Universidade de Brasília
Departamento de Matemática
Brasília, DF - Brazil

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Let Δ be the Laplacian and $\lambda_1 < \lambda_2 \leq \ldots, \lambda_j \to +\infty$, the sequence of eigenvalues of the boundary value problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with respective eigenfunctions denoted by $\phi_1, \phi_2, \ldots$. It is well known that λ_1 is simple, positive and ϕ_1 can be chosen positive in Ω.

In this paper we stablish results on multiplicity of solutions for the boundary value problem

$$\begin{cases} \Delta u + \alpha u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ is an odd continuous function. The existence of multiple solutions has been studied by many authors, under various conditions on f, see e.g. Hempel [1], Ambrosetti [2,3], Rabinowitz [4,5], Castro-Lazer [7], Ambrosetti-Mancini [6], Thews [11,12]. We shall be concerned here with the behavior of f both at infinity and at the origin, i.e., we shall explore the condition

$$\limsup_{s\to+\infty} f(s) < 0 \tag{2}$$

and the positions of both α and the limit

$$\ell \equiv \liminf_{s\to 0^+} \frac{f(s)}{s}$$

with respect to the eigenvalues of $-\Delta$. A polinomial growth condition is also required, namely, for all $s \in \mathbb{R}$

$$|f(s)| \le a|s|^\sigma + b \tag{3}$$

with $a,b,\sigma \in [0,+\infty)$ and $1 \le \sigma < \frac{N+2}{N-2}$ if $N > 2$. Our main result is as follows

<u>Theorem 1</u>. *Assume* $f: \mathbb{R} \to \mathbb{R}$ *is an odd continuous function satisfying* (2)-(3) *and suppose* $\alpha \le \lambda_1$. *Then, problem* (1) *has*

(i) *at least* $2j+1$ *solutions if* $\ell > \lambda_j - \alpha$

(ii) *infinitely many solutions if* $\ell = +\infty$.

In order to prove Theorem 1 we associate to (1) the family of problems

$$\begin{cases} \Delta u + \alpha u + f_n(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1$_n$}$$

by truncating the function f conveniently. Then we obtain an L^∞ - a priori bound for the solutions of $(1)_n$, independent of n. Let

$$\lambda_{k-p} < \lambda_{k-p+1} = \dots = \lambda_k < \lambda_{k+1}$$

$k \geq 1$, $p \geq 1$, i.e., we assume λ_k has multiplicity p. The following result will be applied to solve $(1)_n$.

Theorem 2. *Assume* $f: \mathbb{R} \to \mathbb{R}$ *is an odd bounded and continuous function satisfying* (2). *Suppose in addition that* $\alpha = \lambda_k$ *or* $\alpha < \lambda_1$ *in* (1). *Then problem* (1) *has*

(i) *at least* $2(j-k+p)+1$ *solutions if* $\ell > \lambda_j - \alpha$, $j \geq k$

(ii) *infinitely many solutions if* $\ell = +\infty$

Our next result corresponds to the case in that α is between two consecutive eigenvalues of $-\Delta$ and f is sublinear in the sense that

$$\lim_{s \to \infty} \frac{f(s)}{s} = 0. \tag{4}$$

Theorem 3. *Suppose* $f: \mathbb{R} \to \mathbb{R}$ *is an odd continuous function satisfying* (4) *and the following inequality*

$$(f(s)-f(t))(s-t) \leq \gamma(s-t)^2 \tag{5}$$

for all $s,t \in \mathbb{R}$ *and some constant* γ. *Let* $\alpha \in (\lambda_k, \lambda_{k+1})$. *Then problem* (1) *has at least* $2|j-k| + 1$ *solutions provided either* $\ell > \lambda_j > \lambda_k$ *or*

$$\limsup_{s \to 0} \frac{f(s)}{s} < \lambda_j < \lambda_k.$$

Theorems 2 and 3 are in fact an exploration of a result due to Clark [10], (cf. section 2), concerning the existence of critical points for even C^1-functionals. In the proof of Theorem 3 we apply Clark's result in connection with reduction arguments. Our Theorem 1 improves a result by Ambrosetti-Mancini [6], (Cf. Th. 5.7) where it is assumed that $f \in C^1$ and f' is bounded from above. Theorems 2 and 3 are related to results by Rabinowitz [4], Thews [11,12] and Castro-Lazer [7]. Our results remain true for higher order operators.

2. The Abstract Framework and Notations

Let Σ denote the collection of closed, symmetric (with respect to the origin), subsets of $E\backslash\{0\}$, where E is a real Banach space. The genus $\gamma(A)$ of an element $A \in \Sigma$ is defined to be the least integer $j \geq 0$ such that there is an odd $\phi \in C^0(A, R^j\backslash\{0\})$. For the properties of genus see e.g. Rabinowitz [5] or Castro [8]. A C^1-functional $J: E \to \mathbb{R}$ satisfies $(PS)^-$, provided any sequence $u_n \in E$ for which Ju_n is bounded from below, $Ju_n < 0$ and $J'u_n \to 0$ has a convergent subsequence. If J is even we define

$$i_1(J) = \lim_{a\to 0^-} \gamma(J_a)$$

and

$$i_2(J) = \lim_{a\to -\infty} \gamma(J_a)$$

where $J_a = \{u \in E \mid J(u) \leq a\}$. The following theorem is a specialization of a result due to Clark [10] and follows from Clark's version of the Ljusternik-Schnirelman Theory. For a proof see also Castro [8].

Theorem 4 (Clark [10]). *Suppose* $J: E \to \mathbb{R}$ *is an even* C^1*-functional with* $J(0) = 0$, *satisfying* $(PS)^-$. *Then,* J *has at least* $2m$ *critical points* $u \in E$ *with* $J(u) < 0$ *provided* $m \geq 1$ *is an integer such that* $i_1(J) - i_2(J) \geq m$.

In what follows we shall take E to be the Sobolev space H_0^1 whose norm and inner product are given by $|u|_1 = \int|\nabla u|^2$ and $\langle u,v\rangle_1 = \int \nabla u \cdot \nabla v$ respectively. (All integrals here are taken over Ω). We recall the following inequalities

$$\int|\nabla u|^2 \le \lambda_j \int u^2, \quad u \in \langle\phi_1,\dots,\phi_j\rangle \tag{6}$$

$$\int|\nabla v|^2 \ge \lambda_{j+1} \int v^2, \ v \in \langle\phi_1,\dots,\phi_j\rangle^{\perp}. \tag{7}$$

Now, if $f: \mathbb{R} \to \mathbb{R}$ is an odd continuous function satisfying (3) it follows that $J: H_o^1 \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int|\nabla u|^2 - \alpha u^2 - \int F(u) \tag{8}$$

where

$$F(z) = \int_0^z f(s)ds, \quad z \in \mathbb{R}$$

is a well defined even C^1-functional with $J(0) = 0$. In fact,

$$\langle\nabla J(u),v\rangle_1 = J'(u).v = \int \nabla u.\nabla v - \alpha uv - f(u)v, \qquad u,v \in H_o^1.$$

3. Proofs

We associate with the odd, continuous function f a sequence of odd, bounded and continuous functions as follows: for an integer $n \ge 1$,

$$f_n(s) = \begin{cases} -n \ , & f(s) < -n \\ f(s), & |f(s)| \le n \\ n \ , & f(s) > n \end{cases}$$

It is immediate that f_n satisfies (2)-(3) with the same constants a,b,σ. Now, it follows, by applying the Linear Elliptic Theory, that a weak solution u of $(1)_n$ belongs to $W^{2,p}$ $2 \le p < +\infty$. We denote by J_n the energy functional associated with f_n as in (8).

Lemma 5. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be an odd continuous function satisfying* (2)-(3) *and suppose* $\alpha \leq \lambda_1$ *in* (1). *Then, there exists a constant* C, *independent of* n, *such that*

$$|u|_\infty \leq C$$

for all solutions u *of* $(1)_n$ *satisfying* $J_n(u) \leq 0$.

Proof of Lemma 5. Let u be a solution of $(1)_n$. Taking the L^2-inner product with u in $(1)_n$ we find

$$\int u f_n(u) \geq 0. \tag{9}$$

Now, it follows from (2) that

$$f_n(s) \leq C, \quad s \geq 0$$

for some constant C. (We shall use C to denote various constants independent of n). Hence,

$$s f_n(s) \leq C|s|, \qquad s \in \mathbb{R}. \tag{10}$$

Consequently,

$$s f_n(s) \leq -|s f_n(s)| + 2C|s|, \qquad s \in \mathbb{R}$$

and this together with (9) yeld

$$\int |u||f_n(u)| \leq C|u|_{L^2}. \tag{11}$$

The remaining part of the proof is divided into three steps.

1) Assume that $|u_n|_{L^2} \to \infty$ with

$$\Delta u_n + \alpha u_n + f_n(u_n) = 0 \quad \text{in} \quad \Omega \tag{12}$$

and $J_n(u_n) \leq 0$. We write $u_n = t_n\phi_1 + \omega_n$, ω_n orthogonal to ϕ_1. It follows from (2) that

$$F_n(z) \leq C, \quad z \in \mathbb{R}.$$

Therefore,

$$\frac{1}{2}\int|\nabla u_n|^2 - \alpha u_n^2 \leq \int F_n(u_n) \leq C.$$

Hence

$$\left(1 - \frac{\alpha}{\lambda_2}\right)|\omega_n|_1^2 \leq C$$

and $|\omega_n|_1$ is bounded. Then, $|t_n| \to \infty$ and

$$v_n = \frac{u_n}{|u_n|_{L^2}} \overset{L^2}{\to} t^*\phi_1$$

with $t^* = \pm 1$. Suppose $t^* = 1$. By passing to subsequences, if necessary, we may assume that

$$v_n \to \phi_1, \quad u_n \to +\infty \quad \text{and} \quad |v_n| \leq h \qquad \text{a.e. in } \Omega$$

where h is some function in L^2. On the other hand, we find from (2) and (10) that

$$\limsup f_n(u_n) < 0 \quad \text{a.e. in } \Omega$$

and

$$v_n f_n(u_n) \leq C|v_n| \qquad \text{a.e. in } \Omega$$

respectively. Then, by applying Fatou's Theorem in (9) and using the inequality

$$\liminf v_n(C - f_n(u_n)) \geq \phi_1 \liminf(C - f_n(u_n)) \quad \text{a.e. in } \Omega$$

we get

$$0 \leq \limsup \int v_n f_n(u_n) \leq \int \limsup v_n f_n(u_n) \leq \int \phi_1 \limsup f_n(u_n)$$

which is a contradiction. In the case $t^* = -1$ we obtain

$$-v_n \to \phi_1 \quad \text{and} \quad -u_n \to +\infty \qquad \text{a.e. in } \Omega$$

so that the earlier reasoning applies, since f is odd. Assume now, that $|u_m|_{L^2} \to \infty$, where u_m is a solution of $(1)_{n_0}$ for some $n_0 \geq 1$. We arrive at a contradiction by a reasoning similar to the above one.

This proves that the solutions u of $(1)_n$ such that $J_n(u) \leq 0$ are bounded in L^2.

2) We find from (11) and step 1 that

$$\int |\nabla u|^2 - \alpha u^2 \leq \int |u||f_n(u)| \leq C|u|_{L^2}.$$

Therefore,

$$|u|_1 \leq C, \quad \text{(C independent of n)}$$

for all solutions u of $(1)_n$ such that $J_n(u) \leq 0$.

3) We shall use now a bootstrap argument to obtain $|u|_\infty \leq C$. If $N = 2$, we get from Sobolev's Imbedding Theorem and step 2 that $|u|_{L^p} \leq C$, $2 \leq p < +\infty$. Then, we apply the a-priori estimates of Elliptic Theory to $(1)_n$ and use the late inequality together with (3) to get $|u|_{W^{2,p}} \leq C$, $2 \leq p < +\infty$. Thus, $|u|_\infty \leq C$. In the case $N > 2$, Sobolev's Imbedding Theorem and step 2 give $|u|_{L^{2^*}} \leq C$, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$. Let $p_1 = 2^*|\sigma$. Then, using (3) and the a-priori estimates for elliptic operators we get $|u|_{W^{2,p_1}} \leq C$. Next, one studies two cases: $2 \geq \frac{N}{p_1}$ and $2 < \frac{N}{p_1}$. After repeating the argument a finite number of times one arrives at $|u|_\infty \leq C$. This proves Lemma 5.

<u>Proof of Theorem 2</u>. Let $\alpha = \lambda_k$ $k \geq 1$, $Y = \langle\phi_1,\dots,\phi_{k-p}\rangle$, $N = \langle\phi_{k-p+1},\dots,\phi_k\rangle$, $W = \langle\phi_{k+1},\dots\rangle$ and $u_n = y_n + v_n + \omega_n$ with $y_n \in V$, $v_n \in N$ and $w_n \in W$. Assume $|u_n|_1 \to \infty$ and $J(u_n) \to 0$. Then, (6)-(7) and the boundedness of f yeld

$$\langle\nabla J(u_n),\omega_n\rangle_1 = \int |\nabla\omega_n|^2 - \lambda_k\omega_n^2 - f(y_n+v_n+\omega_n)\omega_n \geq \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)|\omega_n|_1^2 - C|\omega_n|_1.$$

Therefore, $|\omega_n|_1$ is bounded. Similarly, we find, by computing

$\langle \nabla J(u_n), y_n \rangle_1$, that $|y_n|_1$ is also bounded. So, $|v_n|_1 \to \infty$. On the other hand,

$$J(u_n) = \frac{1}{2}\int |\nabla \omega_n|^2 - \lambda_k \omega_n^2 + \frac{1}{2}\int |\nabla y_n|^2 - \lambda_k y_n^2 - \int F(y_n + v_n + \omega_n)$$

$$\geq -C - \int F(v_n).$$

Now, it follows from (2) that $F(z) \to -\infty$ as $|z| \to \infty$. Consequently, (Cf. Rabinowitz [4])

$$\int F(v_n) \to -\infty.$$

Hence, $J(u_n) \to +\infty$ so that every sequence u_n such that $-C \leq J(u_n) < 0$ and $\nabla J(u_n) \to 0$ is necessarily bounded. It is easily seen that

$$\nabla J(u) = u - Ku, \quad u \in H_o^1$$

where K is a compact mapping in H_o^1. This proves $(PS)^-$. We shall apply Clark's result. It suffices to show that $i_1(J)-i_2(J) \geq j-k+p$. We find from $\ell > \lambda_j - \lambda_k$ $j \geq k$, that

$$F(z) > \frac{\eta}{2} z^2, \quad |z| \leq \varepsilon$$

for any $\varepsilon > 0$ and some $\eta > \lambda_j - \lambda_k$. Let $u \in \langle \phi_1, \ldots, \phi_k, \ldots, \phi_j \rangle$, $u = u_1 + u_2$ with $u_1 \in \langle \phi_1, \ldots, \phi_k \rangle$ and $u_2 \in \langle \phi_{k+1}, \ldots, \phi_j \rangle$. Then, for $\varepsilon^* > 0$ properly chosen and $|u|_1 = \varepsilon^*$,

$$J(u) = \frac{1}{2}\int |\nabla u_1|^2 - \lambda_k u_1^2 + \frac{1}{2}\int |\nabla u_2|^2 - \lambda_k u_2^2 - \int F(u)$$

$$\leq \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_j}\right) |u_2|_1^2 - \frac{\eta}{2\lambda_j} |u|_1^2.$$

Therefore, $\sup_{u \in S_{\varepsilon^*}} J(u) < 0$, where S_{ε^*} is the sphere of radius ε^* in $\langle \phi_1, \ldots, \phi_j \rangle$. Consequently, (we recall that $\gamma(S_{\varepsilon^*}) = j$), $i_1(J) \geq j$. On the other hand, for $u = v + \omega \in N \oplus W$

$$J(u) = \frac{1}{2}\int|\nabla\omega|^2 - \lambda_k\omega^2 - \int F(v+\omega) \geq \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)|\omega|_1^2 - C|\omega|_1 - \int F(v)$$

$$\geq -C.$$

Let $a < -C$. Then $J_a \cap (N \oplus W) = \emptyset$ so that $\gamma(J_a) \leq k-p$. Thus, $i_2(J) \leq k-p$. In the case $\ell = +\infty$ we find that $i_1(J) \geq j$ for any $j \geq k$. The case $\alpha < \lambda_1$ is handled similarly by observing that J is bounded from below in H_0^1. Theorem 2 is proved.

Proof of Theorem 1. We apply Theorem 2 to solve $(1)_n$ with $\alpha \leq \lambda_1$ and observe that the solutions u obtained via Th. 2 satisfy $J_n(u) \leq 0$. Consequently, $|u|_\infty \leq C$ and from (3),

$$|f(u(x))| \leq C, \quad x \in \bar{\Omega}.$$

Let $n_1 > C$. Then, the solutions of $(1)_{n_1}$ obtained via Th. 2 are also solutions of (1). This proves Theorem 1.

Proof of Theorem 3. It follows from (4) that

$$|f(s)| \leq \epsilon|s| + C_\epsilon \quad \epsilon > 0, \quad s \in \mathbb{R}.$$

Suppose $u_n = v_n + \omega_n$ with $v_n \in \langle\phi_1,\dots,\phi_k\rangle$ and $\omega_n \in W \equiv \langle\phi_{k+1},\dots\rangle$ satisfies:

$$\nabla J(u_n) \to 0 \quad \text{and} \quad |Ju_n| \leq C.$$

Then,

$$|\langle\nabla J(v_n+\omega_n), \quad v_n-\omega_n\rangle_1| \geq$$

$$\geq \left|\int|\nabla v_n|^2 - \alpha v_n^2 - \int|\nabla\omega_n|^2 - \alpha\omega_n^2\right| - \int|f(v_n+\omega_n)||v_n-\omega_n|$$

$$\geq \left(\frac{\alpha}{\lambda_k} - 1\right)|v_n|_1^2 + \left(1 - \frac{\alpha}{\lambda_{k+1}}\right)|\omega_n|_1^2 - \epsilon|v_n+\omega_n|_1^2 - C_\epsilon|v_n+\omega_n|_1.$$

On the other hand,

$$|\langle \nabla J(v_n+\omega_n), v_n-\omega_n|_1 \leq \varepsilon |v_n+\omega_n|_1.$$

Therefore, u_n is bounded, so that J satisfies $(PS)^-$. Now, let $m > k+1$ be such that $\alpha+\gamma < \lambda_{m+1}$, $X = \langle \phi_1,\dots,\phi_m \rangle$ and Y the orthogonal complement of X in H_0^1. By well known results on the reduction method (Cf. Castro [9]) we get, by applying (5), that there exists a continuous mapping $\phi: X \to Y$ such that

$$J(v+\phi(v)) = \min_{\omega \in Y} J(v+\omega), \quad v \in X.$$

Moreover, $u \in H_0^1$ is a critical point of J iff $u = v_0 + \phi(v_0)$ with v_0 a critical point of the functional $\tilde{J}: X \to \mathbb{R}$ given by

$$\tilde{J}(v) \equiv J(v+\phi(v)), \quad v \in X.$$

On the other hand, it is easily seen that $\tilde{J}$ satisfies $(PS)^-$ once J satisfies it. Thus, according to Clark's Theorem it suffices to show that $i_1(\tilde{J}) - i_2(\tilde{J}) \geq |j-k|$. We consider only the case $j > k$. The proof of the case $j < k$ is the same, replacing $\tilde{J}$ by $-\tilde{J}$. Now, it follows by using the condition $\ell > \lambda_j > \lambda_k$, as in the proof of Theorem 2, that

$$\tilde{J}(v) \leq J(v) < 0 \quad v \in \langle \phi_1,\dots,\phi_j \rangle, \quad |v|_1 = \varepsilon^*$$

with $\varepsilon^* > 0$ properly chosen. Consequently,

$$i_1(\tilde{J}) \geq j.$$

Next, we show that

$$i_2(\tilde{J}) \leq k.$$

From (4) it follows that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(z) \leq \frac{\varepsilon}{2} z^2 + C_\varepsilon |z|, \quad z \in \mathbb{R}.$$

Let $\varepsilon > 0$ be such that $\alpha+\varepsilon < \lambda_{k+1}$. Hence, if $\omega \in W$, we find

$$J(\omega) \geq \frac{1}{2} \int |\nabla\omega|^2 - (\alpha+\varepsilon)\omega^2 - C_\varepsilon |\omega|_1$$
$$\geq -C_\varepsilon.$$

Thus,

$$\tilde{J}(v_1) \geq -C_\varepsilon, \quad v_1 \in \langle\phi_{k+1},\ldots,\phi_m\rangle.$$

This implies, as in the proof of Theorem 2, that for $a < -C_\varepsilon$

$$\gamma(\tilde{J}_a) \leq k.$$

Hence, $i_2(\tilde{J}) \leq k$ and Theorem 3 is proved.

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