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REDUCTION METHODS VIA MINIMAX

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1. Introduction

Let H be a real Hilbert space and $J:H \to \mathbb{R}$ a functional of class C^1. That is, there exists a continuous function $\nabla J:H \to H$ such that for $x,y \in H$

$$\lim_{t\to 0} \frac{J(x+ty)-J(x)}{t} = \langle \nabla J(x), y\rangle$$

where $\langle\ ,\ \rangle$ is the inner product in H.

In this note we consider the existence of critical points of J, which are points $u \in H$ such that $\nabla J(u) = 0$. The particular kind of functional J that we study have the property that there exist closed subspaces X and Y with $H = X \oplus Y$ and such that the existence of critical points of J is equivalent to the existence of critical points of a new functional $\tilde{J}:X \to \mathbb{R}$. The functional $\tilde{J}$ is given in the form $\tilde{J}(x) = J(x+r(x))$, where $r:X \to Y$ is a continuous function defined via a "minimax characterization"; that is, some functional takes a minimum, maximum or minimax value at $r(x)$ (see Lemmas 1 and 3).

In section 2 the reader will find the basic abstract tools which will be used throughout the applications. As applications we present the existence of solutions for Hammerstein integral equations, periodic solutions of the forced pendulum equation and solutions to a nonlinear Dirichlet problem.

2. Reduction Lemmas

Lemma 1. *Let* X *and* Y *be closed subspaces of a real Hilbert space* H *such that* $H = X \oplus Y$. *Let* $J:H \to \mathbb{R}$ *be a functional of class* C^1. *If there exists an increasing function* $\phi:(0,\infty) \to (0,\infty)$ *such that* $\phi(t) \to \infty$ *as* $t \to \infty$ *and*

$$\langle \nabla J(x+y) - \nabla J(x+y_1), y-y_1 \rangle \geq \|y-y_1\| \phi(\|y-y_1\|) \tag{2.1}$$

for all $x \in X$, $y, y_1 \in Y$, $y \neq y_1$, *then:*

i) *there exists a continuous function* $r:X \to Y$ *such that* $J(x+r(x)) = \min\{J(x+y);\ y \in Y\}$; *moreover,* $r(x)$ *is the only critical point of the functional* $J_x:Y \to \mathbb{R}$, $y \to J(x+y)$.

ii) *the function* $\tilde{J}:X \to R$, $x \to J(x+r(x))$ *is of class* C^1 *and* $\langle \nabla\tilde{J}(x), x_1 \rangle = \langle J(x+r(x)), x_1 \rangle$ *for all* $x, x_1 \in X$.

Proof: From (2.1) and the assumption that ϕ takes only positive values it follows that J_x has at most one critical point. Also from (2.1) we have

$$\begin{aligned} J_x(y) &= J_x(0) + \int_0^1 \langle \nabla J_x(sy), y \rangle ds \\ &\geq J_x(0) - \|\nabla J_x(0)\| \|y\| + \int_0^1 s\|y\| \phi(\|sy\|) ds. \end{aligned} \tag{2.2}$$

Since we are assuming that $\phi(t) \to \infty$ as $t \to \infty$, there exists $R > 0$ such that $\phi(t) \geq 2(\|\nabla J_x(0)\| + 1)$ for $t > R$. Hence, for $\|y\| \geq 2R$ we have $J_x(y) \geq J_x(0) + 3\|y\| \to \infty$ as $\|y\| \to \infty$. Therefore, in order to prove that J_x has a unique point of minimum it is sufficient to

$$= \int_0^1 \langle \nabla J(x+r(x)+sth), h\rangle ds. \tag{2.5}$$

In a similar manner it can be proved that

$$((\bar{J}(x+th)-\bar{J}(x))/t) \geq \int_0^1 \langle \nabla J(x+r(x+th) + sth), h\rangle ds. \tag{2.6}$$

From (2.5), (2.6) and the continuity of ∇J and r we obtain

$$\langle \nabla\bar{J}(x), h\rangle = \langle \nabla J(x+r(x)), h\rangle.$$

Hence $\bar{J}$ has continuous directional derivatives. Therefore $\bar{J}$ is of class C^1 and the Lemma has been proved.

Remarks. 1. In most of the applications the function ϕ has the form $\phi(t) = mt$.

2. It is not necessary that ϕ be increasing, double checking the proof we see that it is sufficient to assume that $\inf\{\phi(t); t \geq \eta\} > 0$ for $\eta > 0$.

3. It is easy to verify that if (2.1) is replaced by "J_x has a unique point of minimum $r(x)$ and the function r is bounded in bounded sets, and ∇J is continuous in the weak topology of H", then the conclusions of Lemma 1 still hold.

4. The function $\bar{J}$ inherits the degree of differentiability of J. For example we have

Lemma 2. *If in addition to the hypotheses of Lemma* 1 *we assume that* J *is of class* C^2 *then* $\bar{J}$ *is of class* C^2.

Proof. See [13, Theorem 4].

The condition (2.1) says that the graph of J_x looks like a "parabola". There are cases in which the graph of J_x looks like a show that J_x is convex. Let $y_1, y_2 \in Y$, $\psi(t) = J_x(y_1 + t(y_2 - y_1))$ and $0 < \alpha < \beta < 1$. Since J is of class C^1, ψ is of class C^1. From (2.1) we have

$$\psi'(\beta)-\psi'(\alpha) = \langle \nabla J(x+y_1+\beta(y_2-y_1))-\nabla J(x+y_1+\alpha(y_2-y_1)),\ y_2-y_1\rangle$$

$$\geq \|(\beta-\alpha)(y_2-y_1)\|\phi(\|(\beta-\alpha)(y_2-y_1\|)/(\beta-\alpha) > 0.$$

Hence ψ is convex, which implies that J_x is convex. Consequently we have proved that J_x has a unique point of minimum which we denote by $r(x)$.

Now we show that $r(x)$ is continuous. If not, let $\delta > 0$ and $\{x_n\}$ be a sequence converging to some $x \in X$ such that

$$\|r(x_n) - r(x)\| \geq 2\delta. \tag{2.3}$$

Since ∇J is a continuous function and $\langle \nabla J(x+r(x)), y\rangle = 0$ for all $y \in Y$, we have for n sufficiently large

$$\|P^*(\nabla J(x_n+r(x)))\| < \phi(\delta), \tag{2.4}$$

where P^* denotes the adjoint of the operator $P(x+y) = y$ for $x \in X$, $y \in Y$. Thus from (2.1) we obtain

$$\|P^*(\nabla J(x_n+r(x)))\|\|r(x_n)-r(x)\| \geq \langle -\nabla J(x_n+r(x)),\ r(x_n)-r(x)\rangle$$

$$\geq \langle \nabla J(x_n+r(x_n)) - \nabla J(x_n+r(x)),\ r(x_n)-r(x)\rangle$$

$$\geq \|r(x_n) - r(x)\|\phi(2\delta).$$

Since this inequality contradicts (2.4), we have proved that r is continuous. This completes the proof of part i).

Let $t > 0$ and $h \in X$. From the fact that J_x attains its minimum at $r(x)$ we have

$$((\bar{J}(x+th)-\bar{J}(x))/t) = ((J(x+th+r(x+th))-J(x+r(x))$$

$$\leq ((J(x+th+r(x))-J(x+r(x)))/t)$$

"global saddle point". In the following Lemma we consider this situation.

Lemma 3. *Let* X, Y *and* Z *be closed subspaces of* H *such that* $H = X \oplus Y \oplus Z$. *Let* $J:H \to \mathbb{R}$ *be a functional of class* C^1. *If there exists an increasing function* $\phi:(0,\infty) \to (0,\infty)$ *such that* $\phi(t) \to \infty$ *as* $t \to \infty$ *and*

$$-\|z_1-z_2\|\phi(\|z_1-z_2\|) \geq \langle \nabla J(x+y+z_1)-\nabla J(x+y+z_2),\ z_1 - z_2 \rangle, \tag{2.6}$$

$$\langle \nabla J(x+y+z)-\nabla J(x+y_1+z),\ y-y_1 \rangle \geq \|y-y_1\|\phi(\|y-y_1\|) \tag{2.7}$$

for all $x \in X$, $y,y_1 \in Y$, $y \neq y_1$, $z,z_1 \in Z$, $z \neq z_1$, *then:*

i) *there exists a continuous function* $r:X \to Y \oplus Z$ *such that* $J(x+r(x)) = \max\{\min\{J(x+y+z);\ y \in Y\};\ z \in Z\} = \min\{\max\{J(x+y+z);\ z \in Z\};\ y \in Y\}$.

ii) *the function* $\bar{J}:X \to R$, $x \to J(x+r(x))$ *is of class* C^1 *and* $\langle J(x), x_1 \rangle = \langle \nabla J(x+r(x)), x_1 \rangle$ *for* $x,x_1 \in X$.

Proof: From (2.7) and Lemma 1 it follows that there exists a continuous function $\psi:X \oplus Z \to Y$ such that $J(x+z+\psi(x+z)) = \min\{J(x+z+y);\ y \in Y\}$, $J_1(x+z) \equiv J(x+z+\psi(x+z))$ is of class C^1 and $\langle \nabla J_1(x+z), u \rangle = \langle \nabla J(x+z+\psi(x+z)), u \rangle$ for $u \in X \oplus Z$.

From the definition of ψ, (2.6) and the fundamental theorem of calculus we infer

$$\begin{aligned} J_1(x+z) &= J(x+z+\psi(x+z)) \leq J(x+z) \\ &= J(x)+\langle \nabla J(x), z\rangle + \int_0^1 \langle \nabla J(x+tz) - \nabla J(x), z\rangle dt \\ &\leq J(x) + \|\nabla J(x)\|\|z\| - \int_0^1 \|z\|\phi(\|tz\|)dt. \end{aligned}$$

Arguing as in the paragraph following (2.2) we see that

$$J_1(x+z) \to -\infty \quad \text{as} \quad \|z\| \to \infty \tag{2.8}$$

Now we fix $x \in X$. Let $\{z_n\}$ be a sequence in Z such that $\lim_n J_1(z_n) = \sup\{J_1(x+z);\ z \in Z\}$. From (2.8) we see that $\{z_n\}$ is bounded. Therefore, without loss of generality, we can assume that $\{z_n\}$ converges weakly to some $z_0 \in Z$. By the definition of ψ we have

$$\begin{aligned} J_1(x+z_n) &\leq J(x+z_n + \psi(x+z_n)) \\ &\leq J(x+z_n+\psi(x+z_0)). \end{aligned} \tag{2.9}$$

From (2.6) it follows that the function $z \to J(x+z+\psi(x+z_0))$ is concave, therefore (see [20])

$$\begin{aligned} \limsup_n J(x+z_n+\psi(x+z_0)) &\leq J(x+z_0+\psi(x+z_0)) \\ &= J_1(x+z_0). \end{aligned} \tag{2.10}$$

Hence, from (2.9) and (2.10) we see that J_1 attains its maximum value at z_0. Thus $r(x) = z_0 + \psi(x+z_0)$ satisfies

$$J(x+r(x)) = \max\{\min\{J(x+y+z);\ y \in Y\};\ z \in Z\}. \tag{2.11}$$

Now we show that $r(x)$ is the only critical point of $J_x : Y + Z \to R$, $y + z \to J(x+y+z)$. Suppose y_1+z_1 and y_2+z_2 are two different critical points of J_x and say $y_1 \neq y_2$. Since, from (2.7), the function $y \to J_x(z_1+y)$ is convex and y_1 is a critical point of this function we have $J_x(z_1+y_1) < J(z_1+y_2)$. From (2.6) we see that the function $z \to J_x(y_2+z)$ is concave. Hence, since z_2 is a critical point of this function, we have $J(z_1+y_2) \leq J(y_2+z_2)$. From the last two inequalities we have $J_x(y_1+z_1) < J_x(y_2+z_2)$. In a similar manner it can be proved that $J_x(y_2+z_2) < J_x(y_1+z_1)$. Since the last two inequalities contradict each other, we have proved that J_x has a unique critical point. Consequently $r(x)$ is the unique critical point of J_x.

Another way to obtain a critical point for J_x is like this.

For each $y \in Y$ the function $z \to J_x(y+z)$ has a unique point of maximum $\rho(x+y)$. Repeating the analysis we did on the function J_1 it is shown that the function $y \to J_x(y+\rho(x+y))$ attains a minimum value at some point y_0. Therefore we have

$$\begin{aligned} J_x(y_0+\rho(x+y_0)) &= J(x+y_0+\rho(x+y_0)) = \min\left\{J(x+y+\rho(x+y));\ y \in Y\right\} \\ &= \min\left\{\max\left\{J(x+y+z);\ z \in Z\right\};\ y \in Y\right\}. \end{aligned} \tag{2.12}$$

Combining (2.11) with (2.12) and using the fact that $r(x)$ is the unique critical point of J_x we have $r(x) = y_0 + \rho(x+y_0)$; thus we have proved the variational characterization of the conclusion i). The proof of the continuity of r is essentially identical to the proof of the continuity of the function r in Lemma 1. For this reason we leave this as an exercise.

We prove now that $\tilde{J}$ is differentiable. From (2.12) we see that for $t > 0$, $x,h \in X$ we have

$$\begin{aligned} \frac{\tilde{J}(x+th)-\tilde{J}(x)}{t} &= \frac{J(x+th+r_1(x+th)+r_2(x+th)) - J(x+r_1(x)+r_2(x))}{t} \\ &\le \frac{J(x+th+r_1(x)+r_2(x+th))-J(x+r_1(x)+r_2(x))}{t} \\ &\le \frac{J(x+th+r_1(x)+r_2(x+th))-J(x+r_1(x)+r_2(x+th))}{t} \\ &= \int_0^1 \langle \nabla J(x+sth+r_1(x)+r_2(x+th)),\ h\rangle ds. \end{aligned} \tag{2.13}$$

In the above we have written $r(u) = r_1(u) + r_2(u)$ with $r_1(x) \in Y$ and $r_2(u) \in Z$. In the first inequality we have used that $J(x+th+r(x+th)) = \min\{J(x+th+y+r_2(x+th));\ y \in Y\}$. Similar argument justifies the next inequality. We leave it to the reader to show that this type of analysis also leads to

$$\frac{\tilde{J}(x+th)-\tilde{J}(x)}{t} \ge \int_0^1 \langle \nabla J(x+sth+r_1(x+th)+r_2(x)),h\rangle ds. \tag{2.14}$$

Clearly (2.13) and (2.14) imply that J is of class C^1 and

$$\langle \nabla\tilde{J}(x), h\rangle = \langle \nabla J(x+r_1(x)+r_2(x)), h\rangle \qquad (2.15)$$

and the Lemma is proven.

We invite the reader to state for Lemma 3 analogs of Remarks 1-4 of Lemma 1. Several applications of Lemma 3 can be found in [3].

3. Applications to Hammerstein integral equations

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded region and $K:\Omega\times\Omega \to \mathbb{R}$ be such that $K(x,y) = K(y,x)$ for all $x,y \in \Omega$ and $K \in L^2(\Omega\times\Omega)$. We define $k:L^2(\Omega) \to L^2(\Omega)$ by

$$k(u)(x) = \int_\Omega K(x,y)u(y)dy. \qquad (3.1)$$

Since $\|k(u)\|_{L^2(\Omega)} \leq \|K\|_{L^2(\Omega\times\Omega)}\|u\|_{L^2(\Omega)}$, the operator k is continuous. From Fubini's theorem it follows that k is compact (see [16, ch. IV]) and selfadjoint.

Let $g:\Omega\times\mathbb{R} \to \mathbb{R}$, $(x,y) \to g(x,y)$ satisfy the Caratheodory condition, that is, let g be continuous in u for each x and measurable in x for each u. We consider the *Hammerstein integral equation*

$$u(x) = \int_\Omega K(x,y)g(y,u(y))dy. \qquad (3.2)$$

Now we show that finding solutions to (3.2) is equivalent to finding critical points of a certain functional to be defined (see Lemma 5). Since k is compact and $L^2(\Omega)$ is separable there exists a complete orthonormal set $\{\phi_1,\phi_2,...\}$ in $L^2(\Omega)$ and a sequence of real numbers $\{\lambda_1,\lambda_2,...\}$ tending to 0 with $k\phi_i = \lambda_i\phi_i$, $i=1,2,....$ We denote by Y the closed subspace of $L^2(\Omega)$ generated by $\{\phi_i;\ \lambda_i > 0\}$. It is easy to show that if X denotes the orthogonal complement of Y then X is the closed subspace generated by $\{\phi_i;\ \lambda_i \leq 0\}$. Hence, given $y \in Y$ there exists a sequence of real numbers $\{c_i\}$ such that $c_i = 0$ if $\lambda_i \leq 0$ and $y = \sum_{i=1}^{\infty} c_i\phi_i$. For $u \in L^2(\Omega)$ with

$u = \sum_{i=1}^{\infty} d_i\phi_i$ we define $Q_1(u) = \sum_{i=1}^{\infty} a_i\phi_i$ with $a_i = d_i(\lambda_i)^{\frac{1}{2}}$ for $\lambda_i \geq 0$ and $a_i = 0$ for $\lambda_i < 0$. It is easy to verify that

$$Q_1(x) = 0 \text{ for } x \in X, \text{ and } Q_1(Q_1(y)) = k(y) \tag{3.4}$$

for all $y \in Y$. In the same manner it can be shown that there exists a selfadjoint operator $Q: L^2(\Omega) \to L^2(\Omega)$ such that

$$Q(y) = 0 \text{ for } y \in Y, \text{ and } Q(Q(x)) = -k(x) \tag{3.5}$$

for $x \in X$. Now we can prove:

Lemma 4. *For* $w \in L^2(\Omega)$ *let* $g_1(w)(\xi) \equiv g(\xi, w(\xi))$. *If* $(x,y) \in X \times Y$ *satisfies*

$$-x(\xi)+y(\xi) = Q(g_1(Q_1(y)+Q(x))(\xi))+Q_1(g_1(Q_1(y)+Q(x))(\xi)) \tag{3.6}$$

then $u = Q(x) + Q_1(y)$ *is a solution of* (3.2). *Conversely, if* u *is a solution of* (3.2) *then there exists* $(x,y) \in X \times Y$ *which satisfies* (3.6) *and such that* $u = Q(x) + Q_1(y)$.

Proof: Let $P: L^2(\Omega) \to L^2(\Omega)$ be the orthogonal projection onto X and $P_1 = I-P$ the orthogonal projection onto Y. Let (x,y) be a solution to (3.6) and let $u = Q(x) + Q_1(y)$. Applying Q and Q_1 to (3.6) we obtain

$$Q(x) = -Q^2(g_1(u)) = k(P(g_1(u))) \tag{3.7}$$

$$Q_1(y) = k(P_1(g_1(u))) \tag{3.8}$$

respectively. Adding (3.7) and (3.8) we see that u is a solution of (3.2).

Suppose now that u is a solution to (3.2). Let $x_1 = P(u)$ and $y_1 = P(u)$. Hence $x_1 = P(k(g_1(u))) = k(P(g_1(u))) = -Q^2(P(g_1(u)))$. Thus, if we set $x = -Q(P(g_1(u)))$ then $Q(x) = x_1$. Similarly $y_1 = Q_1(y)$ with $y = Q_1(g_1(u))$. Hence $-x = +Q(P(g_1(x_1+y_1))) =$

$+Q(G_1(Q(x) + Q_1(y)))$ and $y = Q_1(g_1(Q(x) + Q_1(y)))$. Adding the last two equations we see that $(x,y) \in X\times Y$ is a solution of (3.6) and the Lemma is proved.

Lemma 5. *Suppose there exist a real number* A *and a function* $B \in L^2(\Omega)$ *such that*

$$|g(x, u)| \le A|u| + B(x) \tag{3.9}$$

for $(x,u) \in \Omega\times\mathbb{R}$. *For* $w \in L^2(\Omega)$ *let* $G_1(w)(\xi) = \int_0^{w(\xi)} g(\xi, s)ds$. *If* $J:L^2(\Omega) \to \mathbb{R}$ *is the functional defined by*

$$J(u) = \int_\Omega (((P_1u)^2 - (Pu)^2)/2)-G_1(Q(u)+Q_1(u))(\xi)d\xi, \tag{3.10}$$

then J *is of class* C^1 *and* u_0 *is a critical point of* J *iff* $(P(u_0), P_1(u_0)) \in X \times Y$ *is a solution to* (3.6).

Proof: For $(x,u) \in \Omega\times\mathbb{R}$ let $G(x,y) = \int_0^u g(x,s)ds$. Since g satisfies the Caratheodory condition, G also satisfies it. From (3.9) we have $|G(\xi,u)| \le (A+1)|u| + B^2(\xi)$. Hence J is well defined for all $u \in L^2(\Omega)$. Let us show that J is of class C^1. Let $u,v \in L^2(\Omega)$. A simple computation shows that

$$\frac{J(u+tv)-J(u)}{t} = \int_\Omega (P_1(u)P_1(v)-P(u)P(v))(1-(t/2))$$
$$-\int_\Omega (G_1((Q+Q_1)(u+tv))-G_1((Q+Q_1)(u)))/t).$$

Hence, using the definition of G_1 and the fact that $(\partial G/\partial u)(x,u) = g(x,u)$ we infer that

$$\lim_{t\to 0} \frac{J(u+tv)-J(u)}{t} = \int_\Omega (P_1(u)P_1(v)-P(u)P(v))$$
$$-\int_\Omega g_1((Q+Q_1)(u))(Q+Q_1)(v).$$
$$= \int_\Omega (P_1(u)-P(u)-(Q+Q_1)(g_1((Q+Q_1)(u))))v. \tag{3.11}$$

We observe that in obtaining the last equality we have used the selfadjointness of Q and Q_1. But P, P_1, Q and Q_1 are bounded operators. Hence because of (3.9) and Nemytskii's theorem (see [20, chapter 2]) the expression $P_1(u) - P(u) - (Q+Q_1)(g_1((Q+Q_1)(u)))$ defines a continuous operator with domain $L^2(\Omega)$ and values in $L^2(\Omega)$. Hence J is of class C^1 and

$$\nabla J(u) = P_1(u)-P(u)-(Q+Q_1)(g_1((Q+Q_1)(u))). \tag{3.12}$$

From (3.12) we see that if $\nabla J(u_o) = 0$ then

$$P(u_o) = -Q(g_1((Q+Q_1)(u_o))) \tag{3.13}$$

$$P_1(u_o) = Q_1(g_1((Q+Q_1)(u_o))). \tag{3.14}$$

Adding the last two equations we see that $(P(u_o), P_1(u_o)) \in X \times Y$ is a solution to (3.6).

Conversely, if $(P(u_o), P_1(u_o))$ is a solution to (3.6) then u_o satisfies (3.13) and (3.14). Hence from (3.12) we see that $\nabla J(u_o) = 0$ and the Lemma has been proved.

Since the eigenvalues $\{\lambda_i;\ i=1,2,\dots\}$ form a sequence tending to zero we can assume that they are ordered in the following way: $\lambda_1 \geq \lambda_2 \geq \dots \geq 0 \geq \dots \geq \lambda_{-2} \geq \lambda_{-1}$. Now we can proceed to apply the results of the previous section to the equation (3.2).

Theorem 6. *If (3.9) holds and there exist an integer N and real numbers γ, γ' and C such that:*

a) $(1/\lambda_N) < \gamma \leq \gamma' < (1/\lambda_{N+1})$,

b) *for* $u,v \in \mathbb{R}$, $x \in \Omega$ $(g(x,u) - g(x,v))(u-v) \leq \gamma'(u-v)^2$,

c) $G(x,u) \geq (\gamma/2)u^2-C$ *for* $(x,u) \in \Omega\times\mathbb{R}$

then:

A) *the equation (3.2) has a solution,*

B) *if the hypothesis c) is replaced by* $(g(x,u) - g(x,v))$

$(u-v) \geq \gamma(u-v)^2$ *for all* $u,v \in \mathbb{R}$, $x \in \Omega$, *then* (3.2) *has a unique solution.*

Proof: In order to prove claim A) it is sufficient to prove that J has a critical point (see Lemma 5). Without loss of generality we can assume that $N > 0$. Let Y_1 be the closed subspace of $L^2(\Omega)$ containing $\{\phi_{N+1}, \phi_{N+2}, \ldots\}$ and X_1 be its orthogonal complement. Thus X_1 is the closed subspace generated by the Kernel of k, X and the set $\{\phi_N, \phi_{N-1}, \ldots, \phi_1\}$. Applying b) and (3.12) we have for $x \in X_1$, $y, y_1 \in Y_1$

$$\langle \nabla J(x+y_1) - \nabla J(x+y), y_1 - y \rangle = \int ((y_1 - y)^2 - (g_1((Q+Q_1)(x+y_1))$$

$$- g_1((Q+Q_1)(x+y)))(Q+Q_1)(x+y))$$

$$\geq \int ((y_1 - y)^2 - \gamma'(Q+Q_1)^2(y_1 - y)). \quad (3.15)$$

From the definition of Q_1 we have that $\int_\Omega (Q_1(y))^2 \leq \lambda_{N+1} \int y^2$. Combining this inequality with (3.15) we have

$$\langle \nabla J(x+y_1) - \nabla J(x+y), y_1 - y \rangle \geq (1 - \gamma'\lambda_{N+1}) \|y_1 - y\|^2_{L^2(\Omega)}. \quad (3.16)$$

Since hypothesis a) implies that $(1-\gamma'\lambda_{N+1}) > 0$, by Lemma 1 there exists a continuous function $r: X_1 \to Y_1$ sucht that $\tilde{J}(x) \equiv J(x+r(x)) = \min\{J(x+y);\ y \in Y_1\}$. Also from Lemma 1 (claim ii)) we know that in order to prove that J has a critial point it is sufficient to prove that $\tilde{J}$ has a critical point.

For $x \in X_1$, using c) we have

$$-2\tilde{J}(x) \geq -2J(x)$$

$$= -\int_\Omega ((P_1(x))^2 - (P(x))^2 - 2G_1((Q+Q_1)(x)))$$

$$\geq -\|P_1(x)\|^2_{L^2(\Omega)} + \|P(x)\|^2_{L^2(\Omega)} + \gamma\|(Q+Q_1)(x)\|^2_{L^2(\Omega)}$$

$$-2C\ \mathrm{meas}(\Omega)$$

$$\geq (\gamma\lambda_N - 1)\|x_1\|^2_{L^2(\Omega)} + \|P(x)\|^2_{L^2(\Omega)} - 2C\ \text{meas}(\Omega), \qquad (3.17)$$

where x_1 denotes the orthogonal projection of x on the subspace generated by $\{\phi_1,\dots,\phi_N\}$. From (3.17) it is clear that

$$\tilde{J}(x) \to -\infty \quad \text{as} \quad \|x\| \to \infty, \quad x \in X_1. \qquad (3.18)$$

Also (3.17) implies that $\tilde{J}$ is bounded above.

Let $\{x_n\}$ be a sequence in X_1 such that $\tilde{J}(x_n) \to \sup\{\tilde{J}(x);\ x \in X_1\}$. From (3.18) we see that $\{x_n\}$ is bounded. Thus without loss of generality we can assume that $\{x_n\}$ converges weakly to $x_0 \in X_1$. Let x_n' denote the orthogonal projection of x_n on the subspace generated by $\{\phi_1,\dots,\phi_N\}$. Clearly $\{x_n'\}$ converges to x_0', where x_0' is the orthogonal projection of x_0 on that subspace. Let us see that $\{r(x_n)\}$ converges to $r(x_0)$. For $y \in Y_1$ we have

$$\begin{aligned} 0 &= \lim_n \langle \nabla J(x_n + \phi(x_n)), y\rangle \\ &= \lim_n \int ((x_n' + r(x_n))y - g_1((Q+Q_1)(x_n + r(x_n)))Q_1(y)). \end{aligned} \qquad (3.19)$$

Setting, in (3.16), $x = x_n$, $y_1 = r(x_n)$ and $y = 0$ we see that $\{r(x_n)\}$ is bounded. Hence we can also assume that $\{\phi(x_n)\}$ converges weakly to some element $y_0 \in Y_1$. Hence using the compactness of Q and Q_1 we obtain

$$0 = \int y_0 y - g_1((Q+Q_1)(x_0 + y_0)))Q_1(y) \qquad (3.20)$$

for all $y \in Y_1$. Since by Lemma 1 the only critical point of $J_{x_0}(y) \equiv J(x_0 + y)$ is $r(x_0)$, from (3.2) we see that $y_0 = r(x_0)$. Therefore, setting in (3.16), $x = x_n$, $y_1 = r(x_n)$ and $y = r(x_0)$, we have

$$\begin{aligned} (1-\gamma'\lambda_{N+1})\|r(x_n) - r(x_0)\|^2_{L^2(\Omega)} &\leq \langle \nabla J(x_n + r(x_0)) - \nabla J(x_n + r(x_n)), \\ r(x_0) - r(x_n)\rangle &= \int ((r(x_0))(r(x_0) - r(x_n)) - g_1((Q+Q_1)(x_n + r(x_0))) \\ &\qquad Q_1(r(x_0) - r(x_n))). \end{aligned} \qquad (3.21)$$

Since Q_1 is compact the last expression in (3.21) tends to zero. Consequently $\{r(x_n)\}$ converges to $r(x_0)$. Hence we have

$$\begin{aligned}\sup\{\tilde{J}(x);\ x \in X_1\} &= \lim \tilde{J}(x_n)\\ &= (\int (r(x_n))^2 - (P(x_n))^2 - (x_n')^2 - 2G_1((Q+Q_1)(x_n + r(x_n))))/2\\ &\le J(x_0 + r(x_0)) = \tilde{J}(x_0).\end{aligned} \tag{3.22}$$

In obtaining the last inequality we have used that $\{r(x_n)\}$ converges to $r(x_0)$, that x_n' converges to x_0' and that $\int (P(x_0))^2 \le \liminf \int (P(x_n))^2$. From (3.22) it follows that x_0 is a point of maximum of $\tilde{J}$, which proves that J has a critical point. Thus claim A) has been proved.

Now we prove claim B). Imitating the proof of (3.15) we have

$$<\nabla J(x_1+y) - \nabla J(x+y),\ x_1 - x \le (1-\gamma\lambda_N)\|x_1 - x\|^2_{L^2(\Omega)},$$

for $x, x_1 \in X_1$ and $y \in Y_1$. Hence using Lemma 3 with $X = \{0\}$, $Z = X_1$, $Y = Y_1$ we infer that J has a unique critical point and, therefore, that (3.2) has a unique solution.

Remark. If in addition to a), b), c) we assume that $g(x,0) = 0$, $(\partial g/\partial u)(x,0) < (1/\lambda_N)$ for all $x \in \Omega$, $\phi_N \in L^\infty(\Omega)$ and $r(s\phi_N) \in L^\infty(\Omega)$ for $|s|$ small, then (3.2) has a nontrivial solution. The proof of this follows the following pattern. From the proof of the claim A) of Theorem 5 we know that (3.2) has a solution which comes from a point of maximum of $\tilde{J}$. The additional hypothesis permits showing that for some small s, $\tilde{J}(s\phi_1) > 0 = \tilde{J}(0)$. Hence such a point of maximum is not 0.

4. On a result by A. Ambrosetti and G. Prodi

In this section we present a version of a classical result proved in [4]. The proof of the next theorem is essentially taken from [6].

Let Ω be a bounded region in $\mathbb{R}^n$. Let (λ_i, ϕ_i) be the ith eigenvalue-eigenfunction pair of the problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{in} \quad \partial\Omega,$$

where Δ denotes the Laplacian operator $\partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$. We assume $\int_\Omega \phi_i^2 = 1$ for all $i=1,2,\ldots$.

We consider the problem

$$\Delta u + g(u) = \rho\phi_1 + h \quad \text{in} \quad \Omega, \tag{4.1}$$

$$u = 0 \quad \text{in} \quad \partial\Omega, \tag{4.2}$$

where ρ is a real parameter, $h \in L^2(\Omega)$, $\int_\Omega h\phi_1 = 0$ and g is a continuous function satisfying

(I) $\lim_{x\to-\infty} g(x)/x = \mu < \lambda_1$

(II) $\lim_{x\to\infty} g(x)/x = \nu \in (\lambda_1, \lambda_2)$

(III) $(g(u) - g(v))/(u-v) \le \gamma < \lambda_2$ if $u \neq v$.

Let H denote the Sobolev space of square integrable functions in Ω vanishing on $\partial\Omega$ and having generalized first order partial derivatives in $L^2(\Omega)$ (see [1, ch. III]). In order to state our next theorem we recall that $u \in H$ is a *weak solution* to (4.1)-(4.2) iff u is a critical point of the functional $J:H \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega (\|\nabla u\|^2/2 - G(u) + \rho\phi_1 u + hu)$$

where $G(u) = \int_0^u g(s)ds$.

Theorem 7. *If* g *and* h *are as above then there exists* $\rho(h)$ *such that the problem* (4.1)-(4.2) *has*

(A) *at least two solutions for* $\rho > \rho(h)$

(B) *at least one solution for* $\rho = \rho(h)$

(C) *no solution for* $\rho < \rho(h)$.

Further, if $\{h_n\}$ *converges weakly to* h *in* $L^2(\Omega)$ *then* $\rho(h_n) \to \rho(h)$. *If, in addition,* (IV) g *is strictly convex, then* (A) *and* (B) *are valid with "at least" replaced by "precisely".*

Proof: By choosing $\varepsilon < \lambda_1 - \mu$, C large and considering the equation

$$\Delta u + (\lambda_1-\varepsilon)u + g(u)-(\lambda_1-\varepsilon)u+C) = \rho\phi_1+h_1$$

it is clear that there is no loss of generality in assuming $\mu < 0$ and $g \geq 0$. We will also take $\phi_1 \geq 0$.

Let X denote the subspace generated by ϕ_1 and let Y be its orthogonal complement in H. Since Δ is selfadjoint it is easy to see that Y is the closure of the subspace generated by $\{\phi_2, \phi_3, \ldots\}$. We leave it to the reader to prove that the functional J is of class C^1. From the hypothesis (III) follows it that for $x \in X$, $y, y_1 \in Y$

$$\langle \nabla J(x+y_1)-\nabla J(x+y), y_1-y\rangle \geq (1-\gamma/\lambda_2)\|\nabla(y_1 - y)\|^2_{L^2(\Omega)}. \qquad (4.3)$$

Since X is one dimensional, from (4.3) and Lemma 1 we see that for each real number t there exists $r(t) \in Y$ such that $J(t\phi_1+r(t)) = \min\{J(t\phi_1+y);\ y \in Y\}$. We wish to show that $d(\bar{J}(t))/dt \to -\infty$ as $|t| \to \infty$. Note that, by claim ii) of Lemma 1,

$$\frac{d\bar{J}(t)}{dt} = \langle \nabla J(t\phi_1+r(t)), \phi_1\rangle = t\int_\Omega |\nabla\phi_1|^2+\rho-\int_\Omega g(t\phi_1+r(t))\phi_1. \qquad (4.4)$$

Since g is nonnegative, $\bar{J}'(t) \to -\infty$ as $t \to -\infty$. Also

$$J(t\phi_1+r(t)) \leq J(t\phi_1) = t^2\int_\Omega |\nabla\phi_1|^2/2 - \int_\Omega G(t\phi_1) + t\rho. \qquad (4.5)$$

From hypothesis (II) we see that for any $\varepsilon > 0$ there exists C such that for all $t > 0$, $2G(t) \geq (\nu-\varepsilon)t^2 - 2C$. Hence, from (4.5) we obtain

$$J(t\phi_1+r(t)) \leq (t^2\lambda_1-(\nu-\varepsilon)t^2)/2+C \to -\infty \quad \text{as} \quad t \to \infty.$$

Hence, if $\tilde{J}'(t)$ does not tend to $-\infty$ as $t \to \infty$, there must exist a sequence $t_n \to \infty$ such that $J'(t_n)$ is bounded. From the inequality in (3.4), the definition of J, and (III) we have

$$\int_\Omega \|\nabla r(t_n)\|^2 \leq \gamma\int_\Omega (t_n\phi_1+r(t_n))^2 + g(0)\int_\Omega (t_n\phi_1+r(t_n))$$

$$\leq \gamma t_n^2 + \gamma\int_\Omega (r(t_n))^2 + |g(0)|(t_n\int_\Omega \phi_1 + (\text{meas } \Omega)\ \frac{1}{\sqrt{\lambda_2}}\ \|\nabla r(t_n)\|^2_{L^2(\Omega)}).$$

This implies that $\{\|\nabla r(t_n)\|_{L^2(\Omega)}/t_n\}$ is bounded and by taking subsequence, we can suppose that $\{\phi(t_n)/t_n\}$ converges weakly in Y to ψ say. Replacing t by t_n in (4.4), dividing by t_n and taking limit as $n \to \infty$ give

$$0 = \int |\nabla\phi_1|^2 - g_1(\phi_1+\psi)\phi_1 \tag{4.6}$$

where $g_1(s) = \mu s$ if $s \leq 0$, $g_1(s) = \nu s$ if $s > 0$. Notice that $\lim_{n\to\infty} \langle\nabla J(t_n\phi_1 + r(t_n))/t_n, y\rangle = \lim_n 0 = 0$ for all $y \in Y$, i.e.,

$$0 = \lim_n \int_\Omega (\nabla r(t_n)\cdot\nabla y - g(t_n\phi_1+r(t_n))y+hy)/t_n$$

$$= \int_\Omega \nabla\psi\cdot\nabla y - g_1(\phi_1+\psi)y.$$

But putting $y = \psi$ gives

$$\lambda_2\|\psi\|^2_{L^2(\Omega)} \leq \|\nabla\psi\|^2_{L^2(\Omega)} = \int g_1(\phi_1+\psi)\psi \leq \nu\int\psi^2$$

which implies that $\psi = 0$. Now (4.6) becomes $0 = \lambda_1-\nu$, a contradiction. Thus $\tilde{J}'(t) \to -\infty$ as $|t| \to \infty$, which implies the existence of $\rho(h)$ satisfying (A), (B) and (C). Observe that this also implies that the set of zeros of $\tilde{J}'$ is bounded.

The fact $\rho(h)$ depends continuously on h will follow by showing that $r(t)$ depends continuously on h. Let $h_1, h_2 \in L^2(\Omega)$ with $\int_\Omega \phi_1 h_1 = \int_\Omega \phi_1 h_2 = 0$ and for fixed $t \in \mathbb{R}$ let ψ_1 and ψ_2 be the $r(t)$ corresponding to replacing h in J by h_1 and h_2 respectively. Hence

$$0 = \int_\Omega (\nabla\psi_1 \cdot \nabla(\psi_1 - \psi_2) - (g(t\phi_1+\psi_1)-h_1)(\psi_1-\psi_2))$$

and

$$0 = \int_\Omega (\nabla\psi_2 \cdot \nabla(\psi_1 - \psi_2) - (g(t\phi_1+\psi_2)-h_2)(\psi_1-\psi_2)).$$

Subtraction gives

$$\begin{aligned} 0 &= \|\nabla(\psi_1-\psi_2)\|^2 - \int_\Omega (g(t\phi_1+\psi_1)-g(t\phi_1+\psi_2))(\psi_1-\psi_2) \\ &\quad + \int_\Omega (h_1-h_2)(\psi_1-\psi_2) \\ &\geq \|\nabla(\psi_1-\psi_2)\|^2 - \gamma\|\psi_1-\psi_2\|^2 - \|h_1-h_2\|\|\psi_1-\psi_2\| \end{aligned} \tag{4.7}$$

where all the above norms are in $L^2(\Omega)$. From (4.7) we have

$$\|\nabla(\psi_1-\psi_2)\| \leq \sqrt{\lambda_2}/(\lambda_2-\gamma)\|h_1 - h_2\|. \tag{4.8}$$

Thus, for fixed t, the mapping $h \to r(t)$ is globally Lipschitzian from $L^2(\Omega)$ to H. Now (4.8) shows that if $h_n \to h$ weakly, and if ψ_n, ψ denote the corresponding $r(t)$'s, then $\{\psi_n\}$ is bounded in H. Hence by the Sobolev embedding Theorem ([1, pp. 97]) we can assume that $\psi_n \to \psi$ in $L^2(\Omega)$. Finally (4.7) shows that $\psi_n \to \psi$ in H. Thus we have proved that $\rho(h)$ is continuous from the weak topology in $L^2(\Omega)$ to the reals.

The proofs of the implications of the hypothesis (IV) do not depend on the abstract developements of section 2, so we rather refer the reader to [6] for these details.

5. Periodic solutions of the forced pendulum equation

Let $g:\mathbb{R} \to \mathbb{R}$ be a continuous T-periodic function such that $(g(u) - g(v))(u-v) < (u-v)^2$ for $u \neq v$. Let $p_0:\mathbb{R} \to \mathbb{R}$ be a continuous 2π-periodic function such that $\int_0^{2\pi} p_o = 0$. In [9], using the methods of section 2 as we applied them in the previous two sections it was proved that the 2π-periodic solvability of the *pendulum equation*

$$x''(t) + g(x(t)) = p(t) = p_o(t) + p_1 \tag{5.1}$$

where p_1 is a constant, is given by:

Theorem 8: *If* p_o, p_1 *and* g *are as above, then there exist real numbers* $d(p_o)$, $D(p_o) \in [\min\{g(s): s \in \mathbb{R}\}, \max\{g(s): s \in \mathbb{R}\}]$ *such that the equation* (5.1) *has a solution iff* $d(p_o) \leq p_1 \leq D(p_o)$. *The functions* d *and* D *are continuous in the weak topology of* $L^2[0, 2\pi]$, *i.e., if* $\{p_o^n\}$ *is a sequence converging weakly to* p_o *in* $L^2(\Omega)$ *then* $d(p_o^n) \to d(p_o)$, $D(p_o^n) \to D(p_o)$. *Moreover,* $d(p_o) \leq \left(\int_0^T g(s)ds\right)/T \leq D(p_o)$. *If, in addition,* $\{x: g(x) = \min\{g(s): s \in \mathbb{R}\}\}$ *and* $\{x: g(x) = \max\{g(s): s \in \mathbb{R}\}\}$ *are discrete then* $d(p_o) = 0$ *or* $D(p_o) = 0$ *iff* $p_o = 0$.

For details of the proof we refer the reader to [9].

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