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On Periodic Solutions of Weakly Coupled Systems of Differential Equations.

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Sunto. – *Si dimostra che il sistema (3) possiede infinite soluzioni 2T-periodiche sotto la condizione di superlinearità (1) quando g è dispari e le p_k sono limitate e dispari.*

The scalar differential equation

$$x''(t) + g(x(t)) = p(t)$$

has been widely investigated under the suppositions that g is sufficiently regular and $\lim_{|x|\to\infty} (g(x)/x) = +\infty$ and that $p(t)$ is continuous and periodic. It is known that these hypotheses imply the existence of at least one periodic solution having the same period as p (see [3]). If g and p satisfy certain symmetry conditions—for example, if g and p are odd or if p is even—then it is known that there are infinitely many periodic solutions having the same period as p (see, for example, [1], [2], [3], [4] and [6]). The purpose of this note is to establish results of the latter type for weakly coupled systems of differential equations. Our principal result is

THEOREM 1. – *For $k = 1, ..., n$ let $g_k: \mathbf{R} \to \mathbf{R}$ be locally Lipshitzian and satisfy the growth condition*

$$\frac{g_k(x)}{x} \to \infty \quad as\ |x| \to \infty, \tag{1}$$

and suppose g_k is odd. If for $k = 1, ..., n$ $p_k: \mathbf{R}^{n+1} \to \mathbf{R}$ satisfies

$$p_k(t + T, x_1, ..., x_n) = p_k(t, x_1, ..., x_n),$$

where $T > 0$, and

$$p_k(-t, -x_1, ..., -x_n) = -p_k(t, x_1, ..., x_n)$$

for all $(t, x_1, \ldots, x_n) \in R^{n+1}$, *if* p_k *is continuous in* t *and locally Lipshitzian in* $x_1, \ldots, x_n$, *and if there exists* $M > 0$ *such that for each* k

$$|p_k(t, x_1, \ldots, x_n)| \leqslant M \tag{2}$$

on R^{n+1}, *then there exist infinitely many vector solutions of the system*

$$x_k'' + g_k(x_k) = p_k(t, x_1, \ldots, x_n) \tag{3}$$

such that each component is odd and $2T$*-periodic.*

The motivation for the theorem is as follows: In the case, where for $k = 1, \ldots, n$, p_k depends only on t, the system is uncoupled so infinitely many odd $2T$-periodic solution vectors exist according to what is known about the scalar equation. Since the coupling terms in (3) are bounded and, for $k = 1, \ldots, n$, g_k is superlinear, it would seem that the effect of coupling terms becomes insignificant on a time interval in which $x_k(t)^2 + x_k'(t)^2$ is large for all $k = 1, \ldots, n$. This will be the idea of our proof.

In the proof of Theorem 1 it may be assumed without loss of generality that for $k = 1, \ldots, n$

$$g_k(x)x > 0 \quad \text{if } x \neq 0 \tag{4}$$

Indeed, according to (1), (4) holds for $|x|$ sufficiently large. Therefore if g_k is replaced by the sum of g_k and a suitable odd function with compact support (4) will hold for all x. Since the sum of p_k and an odd bounded function of x_k alone will satisfy the same hypotheses as p_k the claim is established. The proof of Theorem 1 will be established via a sequence of Lemmas.

LEMMA 1. – *If* $a_1, \ldots, a_n$ *are arbitrary numbers then the solution of the system* (3) *satisfying the initial conditions* $x_k(0) = 0$, $x_k'(0) = a_k$ $k = 1, \ldots, n$ *exists for all* t.

PROOF. – For each $k = 1, \ldots, n$ let $G_k(x) = \int g_k(s)\,ds$. Clearly (1) implies that $G_k(x) \to \infty$ as $|x| \to \infty$. If the assertion of the lemma were not true there would exist a number $t_1 > 0$ such that the solution $x_1(t), \ldots, x_n(t)$ exists for $0 \leqslant t < t_1$ and such that

$$\lim_{t \to t_1 -} \sum_{k=1}^{n} (x_k(t)^2 + x_k'(t)^2) = +\infty .$$

If

$$E(t) = \sum_{k=1}^{n} \left(\frac{x_k'(t)^2}{2}\right) + G_k(x_k(t))$$

then $E(t) \to +\infty$ as $t \to t_1-$.

But $E'(t) = \sum_{k=1}^{n} x_k'(t)\, p_k(t, x_1(t), \dots, x_n(t))$ so by (2) and the Schwarz inequality $E'(t) \le \left(nM^2(x_1'(t)^2 + \dots + x_n'(t)^2)\right)^{\frac{1}{2}}$. Since (4) implies that $G_k(x) > 0$ for all x, $E'(t) \le M(2nE(t))^{\frac{1}{2}}$ for all $t \in [0, t_1)$, and hence $(E(t))^{\frac{1}{2}} \le (E(0))^{\frac{1}{2}} + M(n/2)^{\frac{1}{2}} t_1$ in the same interval. This contradiction proves the lemma.

In the following if $a = (a_1, \dots, a_n)$ denotes a point in R^n we let $x_k(t, a)$ denote the k-th component of the solution vector of (3) defined by the initial conditions

$$x_k(0, a) = 0\,, \quad x_k'(0, a) = a_k \quad k = 1, \dots, n \tag{5}$$

LEMMA 2. – *Fix k with $1 \le k \le n$. Given $r > 0$ there exists a number $R_k(r) > 0$ such that if $|a_k| \ge R_k(r)$ and $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$ aer arbitrary, then $x_k(t, a)^2 + x_k'(t, a)^2 \ge r^2$ for $0 \le t \le 2T$.*

PROOF. – Let $E_k(t, a) = x_k'(t, a)^2/2 + G_k(x_k(t, a))$. We have $E_k(0, a) = a_k^2/2$ and $(d/dt)\, E_k(t, a) = x_k'(t, a)\, p_k(t, x_1(t, a), \dots, x_n(t, a)) \ge -M(2E_k(t, a))^{\frac{1}{2}}$ and hence, for $0 \le t \le 2T$,

$$(E_k(t, a))^{\frac{1}{2}} \ge (E_k(0, a))^{\frac{1}{2}} - \frac{M}{\sqrt{2}}\, t \ge \frac{|a_k|}{\sqrt{2}} - \sqrt{2}\, MT\,.$$

Given $r > 0$, let $L(r)$ denote a number so large that $y^2/2 + G_k(x) \ge L(r)$ implies that $x^2 + y^2 \ge r^2$—the existence follows from the fact that $G_k(x) \to \infty$ as $|x| \to \infty$. It follows from the above that if $|a_k| \ge 2MT + \sqrt{2L(r)} \equiv R(r)$ then $E_k(t, a) \ge L(r)$ for $0 \le t \le 2T$. As this implies that $x_k'(t, a)^2 + x_k(t, a)^2 \ge r^2$ on $[0, T]$, the Lemma in proved.

By virtue of the preceeding lemma, if $r_0 > 0$ and if $|a| \ge R(r_0)$ and $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$ are arbitrary then there exist functions $r_k(t, a) > 0$ and $\theta_k(t, a)$ continuous for $t \in [0, 2T]$ and $|a_k| \ge R_k(r_0)$ and continuously differentiable in t such that for $t \in [0, 2T]$

$$x_k(t, a) = r_k(t, a) \sin \theta_k(t, a) \tag{6}$$

$$x_k'(t, a) = r_k(t, a) \cos \theta_k(t, a)\,. \tag{7}$$

The function $\theta_k(t, a)$ is only determined within an integral multiple of 2π. However, if $a_k > 0$ then the initial conditions $x_k(0, a) = 0$, $x_k'(0, a) = a_k$ show that we may take $\theta_k(0, a) = 0$ so $\theta_k(t, a)$ will be uniquely determined. We do this in what follows.

LEMMA 3. – *Fix k with $1 \leqslant k \leqslant n$. Given any number $c > 0$ there exists a number $A_k(c) > 0$ such that if $a_k \geqslant A_k(c)$ and $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n$ are arbitrary then $\theta_k(T, a) \geqslant c$ and such that $\theta_k'(t, a) > 0$ on $[0, T]$.*

PROOF. – From (6) and (7) we see that

$$\theta_k'(t, a) = \frac{x_k'(t, a)^2 - x_k(t, a)x_k''(t, a)}{x_k(t, a)^2 + x_k'(t, a)^2} \tag{8}$$

$$= \left(\cos^2 \theta_k(t, a)\right) + \frac{g_k\left(r_k(t, a)\sin\theta_k(t, a)\right)\sin\theta_k(t, a)}{r_k(t, a)}$$

$$- p_k\left(t, x_1(t, a), \ldots, x_n(t, a)\right)\frac{\sin\theta_k(t, a)}{r_k(t, a)} .$$

Given $c > 0$ let N be an integer such that $\pi N > c$ and let δ satisfy

$$0 < \delta < \min\{\pi/4, T/9N\} . \tag{9}$$

If $x_0 > 0$ and

$$m_k(x_0) \equiv \min_{|x| \geqslant x_0} \frac{g_k(x)}{x} , \tag{10}$$

then according to (1), $m_k(x_0) \to \infty$ as $x_0 \to +\infty$. Let r_0 be so large that the following three inequalities hold for a fixed δ satisfying (9)

$$\frac{M}{r_0} < \frac{1}{4} , \tag{11}$$

$$m_k(r_0 \sin\delta)\sin^2\delta > \frac{2M}{r_0} , \tag{12}$$

$$\frac{2\pi - 4\delta}{m_k(r_0\sin\delta)\sin^2\delta} < \delta . \tag{13}$$

We claim that *if $a_k \geqslant R_k(r_0)$ then $\theta_k(T, a) \geqslant c$*. To see this first let us suppose that $t \in [0, 2T]$ and that $\theta_k(t, a)$ belongs to an interval of the form $[j\pi - \delta, j\pi + \delta]$ where j is an integer. By (2), (4) and (8) we have for $r_k > r_0$, $\theta'_k(t, a) \geq \cos^2 \delta - M/r_0$. Hence by (9) and (11) we have for $t \in [0, 2T]$

(14) $$a_k \geq R_k(r_0)\,, \quad \theta_k(t, a) \in \bigcup_{-\infty}^{\infty} [j\pi - \delta, j\pi + \delta]\,,$$

implies $\theta'_k(t, a) \geq 1/4$, where we have used Lemma 2.

On the other hand if $\theta_k(t, a)$ belongs to an interval of the form $[j\pi + \delta, (j+1)\pi - \delta]$ where j is an integer and $r_k(t, a) \geq r_0$ it follows by (8) and (10) that

$$\theta'_k(t, a) \geq (\sin^2 \theta_k(t, a)) \frac{g_k(r_k(t, a) \sin \theta_k(t, a))}{r_k(t, a) \sin \theta_k(t, a)} - \frac{M}{r_0}\,.$$

Therefore, by (10) and (12) we see that

(15) $$a_k \geq R_k(r_0)\,, \quad \theta_k(t, a) \in \bigcup_{-\infty}^{\infty} [j\pi + \delta, (j+1)\pi - \delta]$$

implies $\theta'_k(t, a) > m_k(r_0 \sin \delta)(\sin^2(\delta))/2$.

In the following we assume that $a_k \geq R_k(r_0)$. Since $\theta_k(t, a)$ must belong to one of the two types of intervals appearing in (14) and (15) we have

(17) $$\theta'_k(t, a) > 0 \quad \text{for } t \in [0, 2T].$$

Suppose that $\bar{t} \in [0, T)$ is a number such that

(18) $$\bar{t} + 9\delta < T\,, \quad \theta_k(\bar{t}, a) = j\pi$$

where j is a nonnegative integer. From (14) we see that for $j\pi < \theta_k(t, a) < j\pi + \delta$, $\theta'_k(t, a) > \frac{1}{4}$ therefore $\theta_k(t, a)$ cannot remain in the interval $[j\pi, j\pi + \delta]$ for a duration of time longer than 4δ. Thus there exist a number t_1 such that

(19) $$\bar{t} < t_1 < \bar{t} + 4\delta\,, \quad \theta_k(t_1, a) = j\pi + \delta\,.$$

For $j\pi + \delta < \theta_k(t, a) < (j+1)\pi - \delta$ it follows from (15) that $\theta'_k(t, a) > m_k(r_0 \sin \delta)(\sin^2(\delta))/2$ and therefore $\theta_k(t, a)$ cannot remain in the interval $[j\pi + \delta, (j+1)\pi - \delta]$ for a duration longer than

$$\frac{2(\pi - 2\delta)}{m_k(r_0 \sin \delta)\, sm^2 \delta} < \delta$$

where the last inequality follows from (13).

Therefore there exists t_2 such that

$$(20) \qquad t_1 < t_2 < t_1 + \delta\,, \quad \theta_k(t_2, a) = (j+1)\pi - \delta\,.$$

For $(j+1)\pi - \delta \leqslant \theta_k(t, a) \leqslant (j+1)\pi$ it follows from (14) that $\theta_k(t, a) \geqslant \frac{1}{4}$ so $\theta_k(t, a)$ cannot remain in the interval $[(j+1)\pi - \delta, (j+1)\pi]$ onger than 4δ. Hence from (18), (19) and (20) we see that there exists number t^* such that $t_2 < t^* < t_2 + 4\delta < \bar{t} + 9\delta = T$ and $\theta_k(t^*, a) = (j+1)\pi$. In summary we have shown that whenever $0 \leqslant \bar{t} < \bar{t} + 9\delta < T$ and $\theta_k(\bar{t}, a) = j\pi$, j an integer, then there exists $t^* \in (\bar{t}, \bar{t} + 9\delta)$ with $\theta_k(t^*, a) = (j+1)\pi$.

Since $\theta(0, a) = 0$ and since, by (9), $N(9\delta) < T$, there exist numbers s_j, $j = 1, \ldots, N$ such that $0 < s_1 < \ldots < s_N < T$, $s_{j+1} - s_j < 9\delta$, $j = 1, \ldots, N-1$ and $\theta_k(s_j, a) = j\pi$. Since $\theta_k(t, a)$ in increasing on $[0, T]$ it follows that $\theta_k(T, a) > \theta_k(s_N, a) = N\pi > c$. Hence, if r_0 satisfies (11), (12) and (13) for some δ satisfying (9) the lemma follows by setting $A_k(c) = R_k(r_0)$.

LEMMA 4. – *Fix k with $1 \leqslant k \leqslant n$. Let a_{k_0} be a number such that $a_{k_0} \geqslant R_k(r_0)$, for some number $r_0 > 0$.* (See Lemma 2). *There exists a number $\Omega_k(a_{k_0})$ such that if $a_k = a_{k_0}$, and a_j, $j = 1, \ldots, n$ $j \neq k$ are arbitrary then $\theta_k(T, a) \leqslant \Omega_k(a_{k_0})$.*

PROOF. – Letting $E_k(t, a)$ have the same meaning as in the proof of Lemma 2 we see that

$$\frac{d}{dt} E_k(t, a) = x(t, a) p_k)t, x_1(t, a), \ldots, \ldots, x_n(t, a) < M(2E_k(t, a))^{\frac{1}{2}}\,,$$

so

$$(E_k(t, a))^{\frac{1}{2}} < (E_k(0, a))^{\frac{1}{2}} + Mt/(2)^{\frac{1}{2}}\,, \qquad t > 0.$$

Hence

$$(21) \qquad (E_k(t, a))^{\frac{1}{2}} < (a_{k_0} + MT)/(2)^{\frac{1}{2}}$$

Since $y^2/2 + G_k(x) \to \infty$ as $x^2 + y^2 \to \infty$ there exist a number $\sigma(a_{k_0})$ such that $y^2/2 + G_k(x) < a_{k_0}/\sqrt{2} + MT^2/\sqrt{2}$ implies $x^2 + y^2 < \sigma(a_{k_0})^2$. Hence (21) for $0 < t < T$. From (2), (8) and the hypothesis that $a_{k_0} > R_k(r_0)$ that if $L_k(a_{k_0}) = \max_{|x| < \sigma(a_{k_0})} |g_k(x)|$ then for $t \in [0, T]$

$$\theta_k'(t, a) < 1 + \frac{L_k(a_{k_0})}{r_0} + \frac{M}{r_0}\,,$$

and therefore, since $\theta_k(0, a) = 0$,

$$\theta_k(T, a) \leqslant \left(1 + L_k(a_{k_0}) \frac{L_k(a_k)}{r_0} + \frac{M}{r_0}\right) T = \Omega_k(a_k)\,.$$

This proves the Lemma.

To complete the proof of Theorem 1 we use the following theorem which was given by Miranda in [5].

Generalized Intermediate Value Theorem. – *Let $f: R^n \to R^n$ be a continuous function defined on the set $K = \{x = (x_1, \dots, x_n)$: $a_k \leqslant x_k \leqslant b_k$, $1 \leqslant k \leqslant n\}$. If $f(x) = (f_1(x), \dots f_n(x))$ and for each $k = 1, \dots, n$ we have*

$$(22)\qquad f_k(x_1, \dots, x_{k-1}, a_k, x_{k+1}, \dots, x_n) < 0 \qquad a_j \leqslant x_j \leqslant b_j \qquad j \neq k,$$

and

$$(23)\qquad f_k(x_1, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_n) > 0 \qquad a_j \leqslant x_j \leqslant b_j \qquad j \neq k,$$

then there exists $(\bar{x}_1, \dots, \bar{x}_n) \in K$ such that $f_k(\bar{x}_1, \dots, \bar{x}_n) = 0$ for $k = 1, \dots, n$.

Although Miranda showed that this result is equivalent to Brouwer's fixed point theorem, a simple proof can be based on the degree of a mapping. By a linear change of variables we may assume $a_k = -1$, $b_k = 1$, $k = 1, \dots, n$. Since for each $s \in [0, 1]$ the vector field $H(x, s) = (1 - s)x + sf(x)$ does not vanish on ∂K, $d(f, \operatorname{int} K, 0) = d(Id, \operatorname{int} K, 0) = 1$ so f has a zero in the interior of K. See also [7, pag. 178].

To prove Theorem 1 we observe that the n-tuple $x(t) = (x_1(t), \dots, x_n(t))$ is an odd T-periodic solution of (3) if and only if x satisfies (3) and $x(0) = x(T) = 0$. This condition is certainly necessary.

Conversely if $x(t)$ is a solution of (3) with $x(0) = 0$, then, since $y(t) = -x(-t)$ is also a solution because of (3), and since $x(0) = y(0) = 0$, $x'(0) = y'(0)$, $x(t) = -x(-t)$ by uniqueness. If, in addition, $x(T) = 0$ then since $x(t + 2T) = w(t)$ is also a solution with $w(-T) = x(T) = 0 = x(-T)$ and $x'(-T) = x'(T) = x'(-T)$, $x(t + 2T) \equiv x(t)$.

Theorem 1 will follow from the following

Lemma 5. – ***There exist integers $m_{1_0} > 0, \dots, m_{n_0} > 0$ such that if $m_1, \dots, m_n$ are integers with $m_k \geqslant m_{k_0}$, $k = 1, \dots, n$, then there exists a solution vector $(x_1(t), \dots, x_n(t))$ of (3) such that $x_k(0) = x_k(T) = 0$ for $k = 1, \dots, n$ and such that $x_k(t)$ has exactly m_k simple zeros on the open interval $(0, T)$.***

PROOF. – By lemmas 2 and 3 there exist numbers $a_{1_0}, \ldots, a_{n_0}$ such that, for fixed k, if $a_k = a_{k_0}$ and a_j, $j = 1, \ldots, n$, $j \neq k$, are arbitrary, then $a_k = a_{k_0}$ implies that $x_k(t, a)^2 + x_k'(t, a)^2 > 0$ for $t \in [0, T]$ and such that $\theta_k(t, a)$ defined by (6) and (7) and the condition $\theta_k(0, a) = 0$ satisfies

$$\theta_k'(t, a) > 0 \qquad t \in [0, T]. \tag{24}$$

By Lemma 4, there exists an integer $m_{k_0} > 0$ such that if $a_k = a_{k_0}$ and a_j, $j = 1, \ldots, n$, $j \neq k$ then $\theta_k(T, a) < (m_{k_0} - 1)\pi$. If m_k is an integer with $m_k \geq m_{k_0}$, then, according to lemma 3, there exists $a_k^* > a_{k_0}$ such that if $p_k = a_k^*$ and a_j, $j = 1, \ldots, n$, $j \neq k$, are arbitrary then $\theta_k(T, a) > (m_k + 1)\pi$. Let K be the set of points $(a_1, a_2, \ldots, a_n)$ in R^n such that $a_{k_0} \leq a_k \leq a_k^*$ for $k = 1, \ldots, n$. For $1 \leqslant k \leqslant n$ let $\Gamma_k(a) = \Gamma_k(a_1, \ldots, a_n) = \theta_k(T, a) - (m_k + 1)\pi$. These functions are continuous on K and, according to the above, for any k between 1 and n we have the inequalities

$$\Gamma_k(a_1, \ldots, a_{k-1}, a_{k_0}, a_{k+1}, \ldots, a_m) < 0 \qquad a_{j_0} \leqslant a_j \leqslant a_j^*, \qquad j \neq k \tag{25}$$

and

$$\Gamma_k(a_1, \ldots, a_{k-1}, a_j^*, a_{k+1}, \ldots, a_n) > 0 \qquad a_{j_0} \leqslant a_j \leqslant a_j^*, \qquad j \neq k.$$

Hence by the intermediate value theorem there exists $(\overline{a}_1, \ldots, \overline{a}_n) = \overline{a}$ such that $\theta_k(T, \overline{a}) = (m_k + 1)\pi$ for $1 \leqslant k \leqslant n$. Since for $1 \leqslant k \leqslant n$, $\theta_k'(t, \overline{a}) > 0$ and $\theta_k(0, \overline{a}) = 0$, we see that $\theta_k(t, a)$ assumes each of the values $l\pi$ $l = 1, \ldots, m_k$ exactly once on $(0, T)$. Hence, by (6) and (7), for $k = 1, \ldots, n$, $x_k(t, \overline{a})$ has exactly m_k simple zeros on $(0, T)$ and satisfies $x_k(0, \overline{a}) = x_k(T, \overline{a}) = 0$. This proves the lemma and by earlier remarks this completes the proof of Theorem 1.

By examining the proof of lemmas 1 through 4 we see that these lemmas do not depend on the fact that g_k is odd for $k = 1, \ldots, n$ nor on the conditions that, for $1 \leqslant k \leqslant n$, p_k be odd and $2T$ periodic in t.

By obvious modifications in the proof of theorem 1 we have

THEOREM 2. – *Suppose for each* $k = 1, \ldots, n$ $g_k\colon \mathbf{R} \to \mathbf{R}$ *is locally Lipshitzian and satisfies* (1). *Suppose that for each* $k = 1, \ldots, n$ $p_k\colon \mathbf{R}^{n+1} \to \mathbf{R}$ *is continuous, locally Lipshizian in* $x_1, \ldots, x_n$, *and satisfies* (2). *If* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ *are arbitrary numbers the boundary* *value problem*

$$x_k'' + g_k(x_k) = p_k(t, x_1, \ldots, x_n)$$

$$x_k(0) \cos \alpha_k - x_k'(0) \sin \alpha_k = 0$$

$$x_k(T) \cos \beta_k - x_k'(T) \sin \beta_k = 0 \qquad T > 0$$

$k = 1, \ldots, n$ has infinitely many solutions. In fact if $N_1, \ldots, N_n$ are sufficiently large positive integers then there exists a solution such that x_k has exactly N_k simple zeros on $(0, T)$.

Finally let us suppose that for $1 \leq k \leq n$ g_k and p_k satisfy the conditions of Theorem 2 and in addition that

$$p_k(t, x_1, \ldots, x_n) = p_k(-t, x_1, \ldots, x_n) \tag{25}$$

$$p_k(t + 2T, x_1, \ldots, x_n) = p_k(t, x_1, \ldots, x_n) \tag{26}$$

for all $k = 1, \ldots, n$ and all $(t, x_1, \ldots, x_n) \in R^{n+1}$. In this case a symmetry argument similar to the one used to prove Theorem 1 shows that $(x_1(t), \ldots, x_n(t))$ is an even $2T$-periodic solution of (3) iff it satisfies (3) and the boundary conditions

$$x_k'(0) = x_k'(T) \qquad k = 1, \ldots, n.$$

Therefore we have

THEOREM 3. – *If for $k = 1, \ldots, n$ g_k and p_k satisfy the hypotheses of Theorem 2 and in addition p_k satisfies* (25) *and* (26) *then there exist infinitely many $2T$-periodic even solutions of* (3).

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