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Results on Periodic Solutions of Parabolic Equations Suggested by Elliptic Theory.

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Sunto. – *Si introduce la nozione di* autovalore principale *per operatori parabolici che agiscono su funzioni soddisfacenti la condizione al contorno periodica di Dirichlet. Sfruttando tale nozione si estendono alcuni risultati della teoria delle equazioni ellittiche a equazioni di tipo parabolico.*

1. – Introduction.

Let $\Omega \subset \boldsymbol{R}^N$ $(N \geqslant 1)$ be a bounded domain with boundary of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$. Let $f\colon \bar{\Omega} \times \boldsymbol{R} \to \boldsymbol{R}$ be sufficiently regular. Boundary value problems of the form $-\Delta u = f(x, u)$ in Ω, $u|\partial\Omega = 0$ have been studied intensively, especially during the past decade. Such a problem arises when one looks for equilibrium solutions of the semilinear diffusion equation $u_t - \Delta u = f(x, u)$ subject to the condition $u(x, t) = 0$ for $(x, t) \in \partial\Omega \times \boldsymbol{R}$. The semilinear diffusion equation has several natural interpretations. For example, $u(x, t)$ could represent the population density of a single species and $f(x, u)$ the natural growth rate with Δ accounting for migration (see, for example, [3, p. 325]). Other possible interpretations of u are neutron density in a reactor [18] and the temperature distribution in a body heated by a steady electric current where $f(x, u)$ represents a nonlinear resistance function (see [8]). A natural generalization of this type of model would be to let f depend also on the time t in a periodic way and to replace the operator $\partial/\partial t - \Delta$ by a general second-order parabolic operator L whose coefficients depend on x and t and are periodic in t with the same period as f. Such a model would arise if one considered population density in a non homogeneous medium for which both the growth rate and the diffusion rate are subject to seasonal variations. One could also consider neutron density in reactor with periodic enrichment or the temperature in an inhomogeneous medium heated by an alternating current. For such models it would be

natural to look for solutions of the periodic Dirichlet problem $Lu = f(x, t, u)$, $u|\partial\Omega \times \mathbf{R} = 0$, $u(x, t + T) = u(x, t)$ where T is the period of f and the coefficients of L. In this paper we apply an extension of the so-called subsolution-supersolution method for elliptic equations to study the periodic-Dirichlet problem. If one analyses most applications of this method to the study of the semi-linear Dirichlet problem (D) $-\Delta u = f(x, u)$, $u|\partial\Omega = 0$ one sees that a key role is played by the fact that there exists a number $\lambda_1 > 0$ and φ satisfying $-\Delta\varphi = \lambda_1\varphi$, $\varphi|\partial\Omega = 0$ with $\varphi > 0$ in Ω. (See for example [17, chapt. 2]). For this reason a large portion of this paper is used to introduce the concept of the *principal eigenvalue* of a parabolic operator acting on functions satisfying periodic-Dirichlet boundary conditions. This concept was used by the second author in [14].

We consider three types of conditions of f and L. First, we show that if f grows more slowly in u than a certain linear function of u then there exists a solution. We call this type of condition a Hammerstein-type condition since it was first used to study nonlinear integral equations by Hammerstein in the classical paper [5]. We also show that if the partial derivative of f with respect to u is bounded above by a number less than λ_1 then the problem has a unique solution which is, in a certain sense, globally asymptotically stable. Second, we give conditions on the behavior of f for small values of u and for large positive values of u which guarantee the non uniqueness of periodic solutions. Third, we shall consider the case in which the linear part has nontrivial solutions and for which the nonlinear part is bounded. Conditions of this type were considered for the elliptic case in [13].

One type of condition used to study the elliptic problem which we do not consider here is the Ambrosetti-Prodi or Kazdan-Warner type conditions (see [2] or [7]). This type of condition was considered for the periodic-Dirichlet problem by the second author in [14]. A good bibliography of the literature on periodic solutions of parabolic equations can be found in [1].

Let $\Omega \subset R^N$ $(N \geq 1)$ be as above. If $[a, b]$ is a compact interval and u is a real-valued function defined on $D \equiv \bar{\Omega}\times[a, b]$ we write $u \in C^{\alpha,\alpha/2}(D)$ if the number

$$\bar{H}^D_\alpha(u) = \sup_{\substack{(x_k, t_k)\in D \\ (x_1,t_1)\neq(x_2,t_2)}} \frac{|u(x_1, t_1) - u(x_2, t_2)|}{[|x_1 - x_2|^2 + |t_1 - t_2|]^{(\alpha/2)}}$$

is finite. The set of all such functions is a Banach space with norm

$$\|u\|_{C^{\alpha,\alpha/2}(\bar{\Omega}\times[a,b])} = \max_{(x,t)\in D} |u(x, t)| + \bar{H}^D_\alpha(u) .$$

We write $u \in C^{2+\alpha,1+\alpha/2}(D)$ if u, u_t, u_{x_i}, $u_{x_ix_j}$ belong to $C^{\alpha,\alpha/2}(D)$ for $1 \leqslant i, j \leqslant N$ and the $C^{2+\alpha,1+\alpha/2}(D)$ norm of u is defined to be the sum $C^{\alpha,\alpha/2}(D)$ norms of all these functions. Similarly $C_{1+\alpha,\alpha/2}(D)$ is defined to be the set of functions u defined on D such that u and u_{x_i}, $i = 1, \ldots, N$ belong to $C^{\alpha,\alpha/2}(D)$. We define $C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times R)$ and $C^{\alpha,\alpha/2}(\bar{\Omega}\times R)$ to be the functions which belong $C^{2+\alpha,1+\alpha}(\bar{\Omega}\times[a,b])$ and $C^{\alpha,\alpha/2}(\bar{\Omega}\times[a,b])$ respectively for all compact intervals $[a,b]$. (see [4] for further discussion of these concepts).

In the following L will denote the differential operator defined by

$$Lu = u_t - \sum_{i,j=1}^{N} a_{ij}(x,t)u_{x_ix_j} - \sum_{i=1}^{N} b_i(x,t)u_{x_i} - c(x,t)u$$

where $x = (x_1, \ldots, x_N)$ and $u \in C^{2,1}(\bar{\Omega}\times I)$ for some interval I. Throughout this paper T will denote a fixed positive number. In Theorems 1, 4, 5 and 6 below we assume:

(A_1) The coefficients of L are periodic in t with period T.

(A_2) The coefficients of L belong to $C^{\alpha,\alpha/2}(\bar{\Omega}\times R)$.

(A_3) L is *uniformly parabolic*, that is there exists a constant $m > 0$ such that for all $(x,t) \in \bar{\Omega}\times R$ and all N-tuples of real numbers $(\xi_1, \xi_2, \ldots, \xi_N)$ the inequality $\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geqslant m\sum_{i=1}^{N}\xi_i^2$ holds.

In Theorems 2, 3 and 7 below we replace (A_2) with the stronger regularity assumption

(A_2') $a_{ij} \in C^{2+\alpha,\alpha/2}(\bar{\Omega}\times R)$ $1 \leqslant i, j \leqslant N$

$b_i \in C^{1+\alpha,\alpha/2}(\bar{\Omega}\times R)$ $1 \leqslant i \leqslant N$

$c \in C^{\alpha,\alpha/2}(\bar{\Omega}\times R)$.

We now describe the subsolution-supersolution method for parabolic periodic-Dirichlet problems which was first given by Kolesov [9]. This method has been extended by Amann [1] to give conditions for the existence of periodic solutions of parabolic differential equations subject to other types of homogeneous boundary conditions. All of the theorems we give below are also true for more general boundary conditions but, for simplicity of presentation we shall only consider the Dirichlet condition. Let $F\colon \bar{\Omega}\times R\times R \to R$ satisfy $F(x, t+T, u) = F(x,t,u)$ for all $(x,t,u) \in \bar{\Omega}\times R\times R$. We assume that for u in bounded subintervals of R the function $F(x,t,u)$

is uniformly of class $C^{\alpha,\alpha/2}(\bar{\Omega}\times[0, T])$. Moreover, we assume that $\partial F/\partial u$ is continuous on $\bar{\Omega}\times R$. We consider the periodic-Dirichlet problem

(P-D) $Lu = F(x, t, u)$, $u(x, t+T) \equiv u(x, t)$, $u(x, t) = 0$ for $(x, t) \in \partial\Omega\times R$.

A *supersolution* of (P-D) is a function $v \in C^{2,1}(\bar{\Omega}\times[0, T_1])$ where $T_1 > T$ such that $Lv \geqslant F(x, t, v)$ on $\bar{\Omega}\times T_1$, $v(x, 0) \geqslant v(x, T)$, $v(x, t) \geqslant 0$, $(x, t) \in \partial\Omega\times[0, T_1]$.

A *subsolution* of (P-D) is defined analogously by reversing all of the inequalities in the definition of a supersolution. Kolesov actually gave sufficient conditions for the existence of a solution of a periodic-Dirichlet problem for second-order parabolic differential equation in which the nonlinearity may even depend on the first-order partial derivatives of u. We state these conditions only as they apply to the less general problem (P-D).

KOLESOV'S THEOREM. – *If there exists a supersolution v and a subsolution w of the problem* (P-D) *such that $w(x, 0) \leqslant v(x, 0)$ for all $x \in \bar{\Omega}$ then the problem* (P-D) *has a solution $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times R)$. Moreover, $w \leqslant u \leqslant v$ on $\bar{\Omega}\times[0, T_1]$.*

Amann, showed that if $u \in C^{2,1}(\bar{\Omega}\times R)$ and if u is a solution of (P-D) then $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times R)$.

2. – Principal eigenvalues and eigenfunctions of periodic-parabolic differential operators.

The main purpose of this section is to prove the following result.

THEOREM 1. – *There exists a number λ_1 and a function $\phi \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times R)$ such that*

$$L\phi = \lambda_1\phi\,, \quad \phi(x, t+T) \equiv \phi(x, t)\,. \tag{1}$$

$$\phi(x, t) = 0 \quad \text{if } (x, t) \in \partial\Omega\times R\,. \tag{2}$$

Moreover $\phi(x, t) > 0$ for all $(x, t) \in \Omega\times R$ and if $x \in \partial\Omega$ and $\bar{n}$ denotes the unit outer normal to $\partial\Omega$ at x then $(\partial\phi/\partial\bar{n})(x, t) < 0$ for all t. If (1) *and* (2) *hold with ϕ replaced by ψ then $\psi = k\phi$ for some constant k. Moreover λ_1 is the smallest number λ for which the problem $Lu = \lambda u$, $u|\partial\Omega\times R = 0$, $u(x, t+T) \equiv u(x, t)$ has a nontrivial solution.*

Throughout this section d will denote a fixed constant which is chosen so that

$$d - c(x, t) \geqslant 1 \quad \text{for all } (x, t) \in \bar{\Omega} \times R . \tag{3}$$

Let F denote the Banach space consisting of functions which satisfy $f \in C^{\alpha,\alpha/2}(\bar{\Omega} \times R)$ and $f(x, t + T) = f(x, t)$ with norm $\| \|_F$ defined by $\|f\|_F = \|f\|_{C^{\alpha,\alpha/2}(\bar{\Omega} \times [0,T])}$. Let E denote the Banach space consisting of functions $u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega} \times R)$ which satisfy $u(x, t + T) \equiv u(x, t)$ and $u(x, t) = 0$ for all $(x, t) \in \partial\Omega \times R$ with norm $\| \|_E$ defined by $\|u\|_E = \|u\|_{C^{2+\alpha,1+\alpha/2}(\bar{\Omega} \times [0,T])}$.

LEMMA 1.1. – *If $M: E \to F$ is the linear map defined by*

$$Mu = Lu + du .$$

Then M is continuous, one-to-one and onto. Hence $M^{-1}: F \to E$ is continuous.

PROOF. – The continuity of M follows immediately from the definitions of E and F and the regularity assumptions on the coefficients of the differential operator L. To show that M is one-to-one suppose that for some $u \in E$ $(Mu)(x, t) = 0$ for all $(x, t) \in \bar{\Omega} \times R$. From the condition (3) it follows that the coefficient of u in $-Mu$ is negative. Thus since $-Mu \equiv 0$ it follows by the maximum principle for parabolic differential equations due to Nirenberg [15] (or see Protter and Weinberger [16, p. 173]) and by the periodicity of u in t that u cannot have a non negative maximum anywhere on $\Omega \times R$ unless $u \equiv 0$. Applying the same argument to $-u$, we see that u cannot have a nonpositive minimum anywhere on $\Omega \times R$ unless $u \equiv 0$. Since $u|\Omega \times R \equiv 0$ and u is periodic in t, u must attain either a maximum or a minimum somewhere on $\Omega \times R$. Hence $u \equiv 0$. This shows that M is one-to-one.

Let $f \in F$ and $r = \max_{\bar{\Omega} \times R} |f(x, t)|$. If $v(x, t) = r$ and $w(r, t) = -r$ for all $(x, t) \in \bar{\Omega} \times R$ then according to equation (3) $(Mv)(x, t) = r(d - c(x, t)) \geqslant r \geqslant f(x, t)$ and $(Mw)(x, t) = -r(d - c(x, r)) \leqslant -r \leqslant f(x, t)$ for all $(x, t) \in \Omega \times R$. By Kolesov's theorem there exists $u \in E$ such that $-r \leqslant u(x, t) \leqslant r$ for all $(x, t) \in \bar{\Omega} \times R$ and such that $Mu = f$. Hence M is onto. Since $M: E \to F$ is one-to-one, onto, and continuous, it follows from the open mapping theorem that M^{-1} is continuous. This proves the lemma.

Let $i: E \to F$ be the imbedding of E into F. By the way that the norms in E and F were defined and the Arzela-Ascoli theorem

it is easy to see that i is compact. Therefore by the above lemma it follows that if $T\colon F \to F$ is defined by $Tf = i(M^{-1}f)$ then T is compact.

LEMMA 1.2. – *There exists a number* $\mu_0 > 0$ *and* $\phi \in F$ *such that* $T\phi = \mu_0\phi$, $\phi(x, t) > 0$ *for all* $(x, t) \in \Omega \times R$, *and* $(\partial\phi/\partial\overline{n})(x, t) < 0$ *where* $\overline{n}$ *denotes the unit outer normal to* $\partial\Omega$ *at* x. *If* $\psi \in F$ *and* $T\psi = \mu_0\psi$ *then* $\psi = k\phi$ *for some constant* k. *Moreover if* $\theta \in F$, $\theta \neq 0$, *and* $T\theta = \gamma\theta$ *then* $\gamma \leqslant \mu_0$.

PROOF. – If K denotes the set of functions in F which are nonnegative everywhere on $\overline{\Omega} \times R$ then K *is a cone in the Banach space* F (see [11] or [10, chapt. 5]). Suppose that $f \in K$, $f \neq 0$, and $u = M^{-1}f$. Since according to (3) the coefficient of u in $-Mu$ is negative and since $-Mu = -f \leqslant 0$ it follows by the periodicity of $u(x, t)$ in t, the parabolic maximum principle, referred to above, and the geometry of $\Omega \times R$ that u cannot have a nonpositive absolute minimum anywhere on $\Omega \times R$ unless u is constant. Since $f \not\equiv 0$ and $u|\partial\Omega \times R = 0$ it is impossible that u is a constant. Therefore, since $u(x, t + T) = u(x, t)$ and u vanishes on $\partial\Omega \times R$, we see at once that $u(x, t) > 0$ for all $(x, t) \in \Omega \times R$. Let $(x, t) \in \partial\Omega \times R$. Since $u(x, t) = 0$ and $u > 0$ on $\Omega \times R$, it follows from the periodicity of u in t, the geometry of $\Omega \times R$, and the strong form of the parabolic maximum principle (see Protter and Weinberger [16, 0. 174]) that $(\partial u/\partial\overline{n})(x, t) < 0$ where $\overline{n}$ is the unit outer normal to $\partial\Omega$ at x. Let f_0 be a fixed element of K with $f_0 \not\equiv 0$. If $f \in K$ and $f \not\equiv 0$ it follows from the above that there exist numbers $\alpha(f) > 0$ and $\beta(f) > 0$ such that $\alpha(f)(M^{-1}f_0)(x, t) < (M^{-1}f)(x, t) < \beta(f)(M^{-1}f_0)(x, t)$ for all $(x, t) \in \Omega \times R$. Since $T = i(M^{-1})$ it follows that $\alpha(f)v_0 \leqslant Tf \leqslant \beta(f)v_0$ where $v_0 = Tf_0 \in K$. Since T is compact, $T(K) \subset K$, and, in the terminology of linear positive operators, T *is* v_0*-positive*, it follows from the theory of linear positive operators (see [10, p. 261-267] or [11, p. 67-80]) that there exists $\mu_0 > 0$ and $\phi \in K$ satisfying the assertions in the lemma and the proof is complete.

The proof of Theorem 1 now follows from Lemma 1.2. Since $\mu_0\phi = M^{-1}\phi$, $M\phi = (L + d)\phi = (1/\mu_0)\phi$ so (1) and (2) hold with $\lambda_1 = (1/\mu_0) - d$. Of course λ_1 need not be positive. If $V \in C^{2,1}(\overline{\Omega} \times R)$, $LV = \lambda V$ for some $\lambda \in R$, $V(x, t + T) \equiv V(x, t)$, and $V/\partial\Omega \times R = 0$ then according to a result of Amann (see Introduction) $V \in E$. Hence $MV = (\lambda + d)V$ so if $V \neq 0$, $\lambda + d \neq 0$. It follows that if $V \neq 0$ $TV = (1/(\lambda + d))V$ so, by the previous lemma, $1/(\lambda + d) \leqslant \mu_0$. Hence $\lambda \geqslant 1/\mu_0 - d = \lambda_1$. This proves the Theorem.

We now replace assumption (A_2) by (A_2'). We define the *adjoint* L^* of L *by the equation* $L^*u = -(\partial u/\partial t + A^*u)$ where,

$u \in C^{2,1}(\bar{\Omega}\times R)$ and

$$A^* u = \sum_{i,j=1}^{N} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^*(x, t)\frac{\partial u}{\partial x_i} + c^*(x, t)u ,$$

where

$$b_i^* = -b_i + 2\sum_{j=1}^{N}\frac{\partial a_{ij}}{\partial x_j}$$

and

$$c^* = c - \sum_{i=1}^{N}\frac{\partial b_i}{\partial x_i} + \sum_{i,j=1}^{N}\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} .$$

Because of the assumption (A_2') we see that $a_{ij}, b_i^*, c^* \in C^{\alpha,\alpha/2}(\bar{\Omega}\times R)$. If $u, v \in C^{2,1}(\bar{\Omega}\times R)$ are T-periodic and $u(x, t) = v(x, t) = 0$ for all $(x, t) \in \partial\Omega\times R$, then integration by parts shows that

$$\int_0^T\Big(\int_\Omega uLv\,dx\Big)dt = \int_0^T\int_\Omega (vL^*u\,dx)\,dt . \tag{4}$$

The following result can probably be obtained using Lemma 1.2 by considering the dual cone K^* of K which is invariant under the adjoint T^* of T. However it seems easier to give a direct proof based on Theorem 1.

For $f, g \in L^2(\Omega\times[0, T])$ we set

$$(f, g)_0 = \int_0^T\Big(\int_\Omega fg\,dx\Big)dt .$$

Theorem 2. – *There exists $\phi^* \in E$ such that $L^*\phi^* = \lambda_1\phi^*$, $\phi^*(x, t) > 0$ for all $(x, t) \in \Omega\times R$ and $(\phi, \phi^*)_0 = 1$.*

Proof. – We define a differential operator on $C^{2,1}(\bar{\Omega}\times R)$ by means of the equation $\tilde{L}u = \partial u/\partial t - \tilde{A}u$ where

$$\tilde{A}u = \sum_{i,j=1}^{N} a_{ij}(x, -t)\frac{\partial^2 u}{\partial x_j \partial x_i} + \sum_{i=1}^{N} b_i^*(x, -t)\frac{\partial u}{\partial x_i} + c^*(x, -t)u$$

It is easy to see that $\tilde{L}$ is uniformly parabolic and satisfies the hypothesis of Kolesov's lemma. Hence, by Theorem 1 and Remark 1 there exists a number $\tilde{\lambda}_1$ and a function $\tilde{\phi} \in E$ such that $\tilde{L}\tilde{\phi} = \tilde{\lambda}_1\tilde{\phi}$ and $\tilde{\phi}(x, t) > 0$ for all $(x, t) \in \Omega\times R$. If $\phi^*(x, t) \equiv \tilde{\phi}(x, -t)$ then

$\phi^* \in E$ and

$$\frac{\partial \phi^*}{\partial t}(x, t) = -\frac{\partial \tilde{\phi}}{\partial t}(x, -t) = \left[\sum_{i,j=1}^{N} a_{ij}(x, t)\frac{\partial^2 \tilde{\phi}}{\partial x_i \partial x_j}(x, -t)\right.$$
$$\left. + \sum_{i=1}^{N} b_i^*(x, t)\frac{\partial \tilde{\phi}}{\partial x_i}(x, -t) + c^*(x, t)\tilde{\phi}(x, -t)\right] - \tilde{\lambda}_1 \tilde{\phi}(x, -t)$$
$$= -[(A^*\varphi^*)(x, t) + \tilde{\lambda}_1 \phi^*(x, t)]$$

Hence

$$L^*\phi^* = -\left(\frac{\partial \phi^*}{\partial t} + A^*\phi^*\right) = \tilde{\lambda}_1 \phi^* \quad \text{and } \phi^*(x, t) = \tilde{\phi}(x, -t) > 0$$

for all $(x, t) \in \Omega \times R$.

From the relation (4) we have

$$\lambda_1(\phi, \phi^*)_0 = (\phi^*, L\phi)_0 = (\phi, L^*\phi^*)_0 = \tilde{\lambda}_1(\phi, \phi^*)_0$$

and since $(\phi, \phi^*)_0 > 0$ it follows that $\tilde{\lambda}_1 = \lambda_1$. By replacing ϕ^* by $k\phi^*$ where $k > 0$ is suitably chosen we may assume that $(\phi, \phi^*)_0 = 1$. This proves the theorem.

Finally we prove a Fredholm-type result which will be needed in the final section.

THEOREM 3. – *Given $f \in F$ there exists $w \in E$ such that $Lw - \lambda_1 w = f$ if and only if $(f, \phi^*)_0 = 0$. Also there exists $w \in E$ such that $L^*w - \lambda_1 w = f$ if and only if $(f, \phi)_0 = 0$. If V is a continuous T-periodic function defined on $\bar{\Omega} \times \boldsymbol{R}$ such that $(f, V)_0 = 0$ for all f in the range of $L^* - \lambda_1 I: E \to F$ then $V = c\phi$ for some constant c.*

PROOF. – Let T and $\mu_0 > 0$ be as in Lemma 1.2. If $w \in E$ and $Lw - \lambda_1 w = f$ then $Mw = (\lambda_1 + d)w + f$ so $\mu_0 w - Tw = \mu_0 Tf$. Hence $f \in \text{Range}(L - \lambda_1 I)$ iff $\mu_0 Tf \in \text{Range}(\mu_0 I - T)$. By the Riesz-Fredholm-Schauder theory of compact operators dim kernel $(\mu_0 I - T^*) = \dim \text{kernel}(\mu_0 I - T) = 1$ where $T^*: F^* \to F^*$ is the adjoint of T and if $\Psi \in \text{kernel}(\mu_0 I - T^*)$ and $\Psi \neq 0$ then $g \in \text{Range}(\mu_0 I - T)$ if and only if $\Psi(g) = 0$. Thus $f \in \text{Range}(L - \lambda_1 I)$ if and only if $0 = \Psi(Tf) = T^*\Psi(f) = \mu_0 \Psi(f)$, or $\Psi(f) = 0$. The embedding of F in $L^2(\bar{\Omega} \times [0, T])$ is obviously continuous. Therefore if we define $\Psi_1: F \to \boldsymbol{R}$ by $\Psi_1(f) = (f, \phi^*)_0$ $\Psi_1 \in F^*$ and $\Psi_1 \neq 0$ since $\Psi_1(\phi) = 1$. If $Lw - \lambda_1 w = g$ for some $w \in E$ then $\Psi_1(g) = (Lw - \lambda_1 w, \phi^*)_0 = (w, L\phi^* - \lambda_1 \phi^*)_0 = 0$. Let $f_0 \in F$ satisfy $\Psi(f_0) = 1$. If $f \in F$ and $f_1 = f - \Psi(f) f_0$ then $\Psi(f_1) = 0$ so by what we have shown $f_1 \in \text{Range}(L - \lambda_1 I)$. Therefore $\Psi_1(f_1) = 0$ so $\Psi_1(f) = \Psi(f)\Psi_1(f_0)$.

Hence $\Psi_1 = \Psi_1(f_0)\Psi$ and $\Psi_1(f_0) \neq 0$ since $\Psi_1 \neq 0$. It follows that $f \in \text{Range}(L - \lambda_1 I)$ if and only if $0 = \Psi_1(f) = (f, \phi^*)_0$. This proves the first assertion and the second assertion is proved in exactly the same way.

To prove the final assertion suppose that $V \in C(\bar{\Omega} \times [0, T])$ and that $(V, g)_0 = 0$ for all $g \in \text{Range}(L^* - \lambda_1 I)$. If $f \in F$ and $f_1 = f_1 - (f, \phi)_0 \phi^*$ then $(f_1, \phi)_0 = 0$ so by the second assertion $f_1 \in \text{Range}(L^* - \lambda_1 I)$. Therefore

$$(V, f)_0 = (f, \phi)_0 (V, \phi^*)_0 \quad \text{so } (V - (\phi^*, V)_0 \phi, f)_0 = 0 .$$

Since F is dense in $L^2(\bar{\Omega} \times [0, T])$, $V = (\phi^*, V)_0 \phi$ and the Theorem is proved.

3. – Hammerstein-type conditions.

Let $f \colon \bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ satisfy $f(x, t + T, u) \equiv f(x, t, u)$ and the regularity assumptions of Kolesov's theorem. Let λ_1 be as in the previous section.

THEOREM 4. – *Let $\gamma < \lambda_1$: If there exists a number $r_0 > 0$ such that for all $(x, t) \in \bar{\Omega} \times \boldsymbol{R}$*

$$f(x, t, u)/u \leqslant \gamma \quad \textit{if } |u| \geqslant r_0 \tag{5}$$

then there exists at least one solution of the periodic-Dirichlet problem $Lu = f(x, t, u)$, $u(x, t + T) \equiv u(x, t)$, $u|\partial\Omega \times \boldsymbol{R} = 0$.

PROOF. – If a constant $k \geqslant 0$ is chosen so that $f(x, t, u) - \gamma u \leqslant k$ for $(x, t) \in \bar{\Omega} \times \boldsymbol{R}$ and $0 \leqslant u \leqslant r_0$ and so that $f(x, t, u) - \gamma u \geqslant -k$ for $(x, t) \in \bar{\Omega} \times \boldsymbol{R}$ and $-r_0 \leqslant u \leqslant 0$, then the condition (5) implies that for $(x, t) \in \bar{\Omega} \times \boldsymbol{R}$

$$f(x, t, u) - \gamma u \leqslant k \quad \text{if } u \geqslant 0 \tag{6}$$

and

$$f(x, t, u) - \gamma u > -k \quad \text{if } u \leqslant 0 . \tag{7}$$

Let d be as in Lemma 1.1 and let $z \in E$ satisfy $Mz = Lz + dz = k$. If ϕ is as in Theorem 1 then since $\phi(x, t) > 0$ for $(x, t) \in \bar{\Omega} \times \boldsymbol{R}$, $(\partial\phi/\partial\bar{n})(x, t) < 0$ for $(x, t) \in \partial\Omega \times \boldsymbol{R}$, and since ϕ is T-periodic in t,

there exists a constant $c > 0$ such that

$$c\phi(x,t) + z(x,t) \geqslant 0 \tag{8}$$

and

$$c(\lambda_1 - \gamma)\phi(x,t) - (\gamma + d)z(x,t) \geqslant 0 \quad \text{on } \bar{\Omega}\times\boldsymbol{R}. \tag{9}$$

If $v(x,t) = c\phi(x,t) + z(x,t)$ then according to (6), (8), and (9) $Lv - f(x,t,v) \geqslant Lv - (\gamma v + k) = c(\lambda_1 - \gamma)\phi - (\gamma + d)z \geqslant 0$ on $\bar{\Omega}\times\boldsymbol{R}$. If $w(x,t) = -v(x,t)$ then according to (7), (8), and (9) $Lw - f(x,t,w) \leqslant Lw - (kw - k) = -c(\lambda_1 - \gamma)\phi + (\gamma + d)z \leqslant 0$ on $\bar{\Omega}\times\boldsymbol{R}$. Since $w(x,t) \leqslant 0 \leqslant v(x,t)$ on $\bar{\Omega}\times\boldsymbol{R}$, it follows from Kolesov's theorem that there exists $u_0 \in E$ such that for $(x,t) \in \bar{\Omega}\times\boldsymbol{R} - v(x,t) \leqslant u_0(x,t) \leqslant v(x,t)$ and such that $Lu_0 = f(x,t,u_0)$. This proves the theorem.

We now give a condition which implies uniqueness and stability of the solution u_0.

THEOREM 5. – *Let γ and f be as in the statement of Theorem 4. If f satisfies the more stringent condition*

$$\frac{\partial f}{\partial u}(x,t,s) \leqslant \gamma\,, \qquad (x,t,s) \in \bar{\Omega}\times\boldsymbol{R}\times\boldsymbol{R} \tag{10}$$

then there exists a unique $u_0 \in E$ such that $Lu_0 = f(x,t,u_0)$. Moreover u_0 is globally, exponentially, asymptotically stable in the following sense: If $z(x,t)$ is solution of the initial-value boundary-value problem $Lz = f(x,t,z)$, $z(\cdot,0) \in c^{2+\alpha}(\bar{\Omega})$, $z(x,t) = 0$, $(x,t) \in \partial\Omega\times[0,\infty)$, defined on $\bar{\Omega}\times[0,\infty)$, then for $0 < \alpha < \lambda_1 - \gamma$ there exists a constant $D = D(z)$ such that

$$|z(x,t) - u_0(x,t)| \leqslant D\exp[-\alpha t]$$

for $(x,t) \in \bar{\Omega}\times[0,\infty)$.

PROOF. – Let $c > 0$ be so large that

$$u_0(x,0) - c\varphi(x,0) \leqslant z(x,0) \leqslant u_0(x,0) + c\phi(x,0) \tag{11}$$

for all $x \in \bar{\Omega}$. Such a choice is possible since $u_0(x,0) = \phi(x,0) = z(x,0) = 0$ for $x \in \partial\Omega$, $\phi(x,0) > 0$ for $x \in \Omega$, and $(\partial\phi/\partial\bar{n})(x,0) < 0$ for $x \in \partial\Omega$. Let $0 < \alpha < \lambda_1 - \gamma$ and define $V(x,t) = u_0(x,t) + c\phi(x,t)\exp[-\alpha t]$ $W(x,t) = u_0(x,t) - c\phi(x,t)\exp[-\alpha t]$. Since $L\phi = \lambda_1\phi$ and $\phi \geqslant 0$ it follows from (10) and the mean value theorem

that

$$LV - f(x, t, V) = c(\lambda_1 - \alpha)\phi \exp[-\alpha t] - [f(x, t, u_0 + c\phi \exp[-\alpha t]) - f(x, t, u_0)] \geqslant c(\lambda_1 - \alpha - \gamma)\phi \exp[-\alpha t] \geqslant 0$$

for all $(x, t) \in \bar{\Omega} \times [0, \infty]$. Similarly

$$LW - f(x, t, W) = -c(\lambda_1 - \alpha)\phi \exp[-\alpha t] - [f(x, t, u_0 - c\phi \exp[-\alpha t]) - f(x, t, u_0)] \leqslant -c(\lambda_1 - \alpha - \gamma)\phi \exp[-\alpha t] \leqslant 0$$

for all $(x, t) \in \bar{\Omega} \times [0, \infty)$. Since $V(x, t) = W(x, t) = z(x, t) = 0$ for all $(x, t) \in \partial\Omega \times [0, \infty)$, since $Lz = f(x, t, z)$ on $\bar{\Omega} \times [0, \infty)$, and since by (11) $W(x, 0) \leqslant z(x, 0) \leqslant V(x, 0)$ on $\bar{\Omega}$, it follows by standard results on parabolic differential inequalities (see for example [17, p. 186-188] or [19, 24. VI p. 187]) that $W(x, t) \leqslant z(x, t) \leqslant V(x, t)$ on $\bar{\Omega} \times [0, \infty)$ and hence that $|z(x, t) - u_0(x, t)| \leqslant D \exp[-\alpha t]$ where $D = c \max_{\bar{\Omega} \times R} |\phi(x, t)|$. This proves the second assertion of the theorem and, since the first assertion is an obvious consequence of the second, the theorem is proved.

REMARK. – It can be shown that the existence of V and W defined on $\bar{\Omega} \times [0, \infty)$, such that $LV - f(x, t, V) \geqslant 0$, $LW - f(x, t, W) \leqslant 0$, $V|\partial\Omega \times [0, \infty) = W|\partial\Omega \times [0, \infty) = 0$ implies that if $\theta \in C^{2+\alpha}(\bar{\Omega})$ satisfies $V(x, 0) \leqslant \theta(x) \leqslant W(x, 0)$ on $\bar{\Omega}$ then there exists a solution of the initial-value-boundary-value problem $Lz = f(x, t, z)$, $z|\partial\Omega \times [0, \infty) = 0$, $z(x, 0) = \theta(x)$ defined on $\bar{\Omega} \times [0, \infty)$. The proof is indicated in [17, p. 26-27].

Finally we give a result concerning nonuniqueness and the existence of positive periodic solutions. This result is motivated in part by the discussion of the elliptic boundary value $-\Delta u = f(u)$, $u|\partial\Omega = 0$ given in [18].

THEOREM 6. – *Let $f \in C^1(\bar{\Omega} \times \boldsymbol{R} \times (a, \infty))$ for some $a < 0$ and satisfy $f(x, t + T, u) \equiv f(x, t, u)$. If $f(x, t, 0) = 0$ and $(\partial f/\partial u)(x, t, 0) > \lambda_1$ for all $(x, t) \in \bar{\Omega} \times \boldsymbol{R}$ and if there exists $r_0 > 0$ such that $(f(x, t, u)/u) \leqslant \gamma$ for $(x, t, u) \in \bar{\Omega} \times \boldsymbol{R} \times [r_0, \infty)$, where γ is as in Theorem 4, then there exists $u_0 \in E$ such that $u_0(x, t) > 0$ for $(x, t) \in \Omega \times \boldsymbol{R}$ and such that $Lu_0 = f(x, t, u_0)$.*

PROOF. – By the mean value theorem, the positivity of ϕ on $\Omega \times \boldsymbol{R}$, and the T-periodicity of f and ϕ in t there exists $\varepsilon > 0$ such that $\varepsilon\lambda_1\phi(x, t) - f(x, t, \varepsilon\phi(x, t)) < 0$ on $\Omega \times \boldsymbol{R}$. Therefore if $w(x, t) = \varepsilon\phi(x, t)$, $w \in E$ and $Lw \leqslant f(x, t, w)$. Let z be as in the proof of Theorem 4. Since $\partial\phi/\phi\bar{n} < 0$ on $\partial\Omega \times \boldsymbol{R}$ we may choose $c > 0$ so large that both (6) and (7) hold and in addition $c\phi + z \geqslant \varepsilon\phi = w$ on $\bar{\Omega} \times \boldsymbol{R}$. The argument in the proof of Theorem 4 shows that if $v = c\phi + z$ then $Lv \geqslant f(x, t, v)$. Thus by Kolesov's theorem there exists $u_0 \in E$ such that $0 < \varepsilon\phi \leqslant u_0 \leqslant v$ on $\Omega \times \boldsymbol{R}$ and $Lu_0 = f(x, t, u_0)$. This proves the theorem.

4. – Resonance.

This term was first used in [13] to describe the elliptic analogue of the situation described in the following

THEOREM 7. – *Let $h \in F$ and let $g\colon (-\infty, \infty) \to \boldsymbol{R}$ be of class C^1. Assume that the limits $g(\pm\infty) = \lim\limits_{s\to\pm\infty} g(s)$ exist and are finite and that*

$$g(-\infty) < g(s) < g(\infty) \quad \text{for } s \in (-\infty, \infty)\,. \tag{12}$$

Let ϕ^* be as in Theorems 2 and 3, and let

$$c_0 = \int\limits_0^T \Big(\int\limits_\Omega \phi^*(x, t)\, dx\Big)\, dt\,.$$

A necessary and sufficient condition that there exists $u \in E$ such that

$$Lu = \lambda_1 u + h(x, t) - g(u) \tag{13}$$

is that

$$c_0 g(-\infty) < (h, \phi^*)_0 < c_0 g(\infty)\,. \tag{14}$$

PROOF. – As for the elliptic case in [13] the proof of the necessity of (14) is almost immediate. If $u \in E$ then $(Lu - \lambda_1 u, \phi^*)_0 = (u, L^*\phi^* - \lambda_1\phi^*)_0 = 0$ so if u satisfies (13) then $(h, \phi^*)_0 = (g(u), \phi^*)_0$. Since $\phi^* > 0$ on $\Omega \times [0, T]$, (12) implies that

$$g(-\infty)\int\limits_0^T \Big(\int\limits_\Omega \phi^*(x, t)\, dx\Big)\, dt < (g(u), \phi^*)_0 < g(\infty)\int\limits_0^T \Big(\int\limits_\Omega \phi^*(x, t)\, dx\Big)\, dt$$

which proves (14) is necessary.

The proof of the sufficiency of condition (14) is based on Theorem 3, a bootstrap argument, and a perturbation argument used by Hess [6] in the elliptic case. Let $\{\gamma_m\}_1^\infty$ be a sequence of numbers such that $\lim_{m\to\infty} \gamma_m = \lambda_1$ and such that $\gamma_m < \lambda_1$ for all m. If $f_m(x, t, u) = \gamma_m u + h(x, t) - g(u)$ then there exists a number r_m such that

$$f_m(x, t, u)/u \leqslant (\gamma_m + \lambda_1)/2 < \lambda_1$$

for $|u| \geqslant r_m$. Consequently, by Theorem 4, for each $m \geqslant 1$ there exists $u_m \in E$ satisfying

$$Lu_m = \gamma_m u_m + h(x, t) - g(u_m). \tag{15}$$

We claim that there exists a number R_1 such that

$$\|u_m\|_\infty = \max_{\bar{\Omega}\times \mathbf{R}} |u_m(x, t)| \leqslant R_1 \tag{16}$$

for all m. If not, by replacing $\{u_m\}_1^\infty$ by a suitable subsequence, we may assume that $\|u_m\|_\infty \to \infty$ as $m \to \infty$. If for each m we set $v_m = u_m/\|u_m\|_\infty$ then

$$Mv_m = Lv_m + dv_m = (d + \gamma_m)v_m + (h - g(u_m))/\|u_m\|_\infty \tag{17}$$

where d is as in section 2. If for each m we set

$$H_m = (d + \gamma_m)v_m + (h - g(u_m))/\|u_m\|_\infty$$

then there exists $R_2 > 0$ such that $\|H_m\|_\infty \leqslant R_2$ for all m. Let $\theta \in C^2[0, \infty)$ satisfy $\theta(t) = 0$ for $0 \leqslant t \leqslant T/2$ and $\theta(t) = 1$ for $T \leqslant t < \infty$. Setting $z_m = \theta v_m$ for $m = 1, 2, \ldots$, we have

$$\begin{cases} Mz_m = \theta' v_m + \theta H_m \equiv G_m \\ z_m(x, 0) = 0\,, \quad z_m|\partial\Omega \times \mathbf{R}^+ = 0\,. \end{cases} \tag{18}$$

Let p be defined by $2 - (N + 2)/p = \alpha$. Since z_m satisfies (18), there exists a constant k independent of m such that

$$\|z_m\|_{W_p^{2,1}(\Omega\times[0,2T])} \leqslant k\|G_m\|_{L^p(\Omega\times[0,2T])}\,. \tag{19}$$

(see [12, p. 4-5] for the definition of $W_p^{2,1}$ and [12, p. 342] for the

estimate.) Since the right hand side of (18) is bounded independently of m, and since $W_p^{2,1}(\bar{\Omega}\times[0,2T])$ is continuously embedded in $C^{\alpha,\alpha/2}(\bar{\Omega}\times[0,2T])$ (see [12, p. 80]) there exists $R_3>0$ such that $\|z_m\|_{C^{\alpha,\alpha/2}(\Omega\times[0,2T])}<R_3$ for all m. Thus from the T-periodicity of v_m and the fact that $\theta(t)=1$ for $t\geqslant T$, we have $\|v_m\|_F=\|v_m\|_{C^{\alpha,\alpha/2}(\Omega\times[0,2T])}=\|z_m\|_{C^{\alpha,\alpha/2}(\Omega\times[T,2T])}\leqslant\|z_m\|_{C^{\alpha,\alpha/2}(\Omega\times[0,2T])}\leqslant R_3$ for all m. By Ascoli's lemma the embedding $F\to C(\bar{\Omega}\times\boldsymbol{R})$ is compact so without loss of generality we may assume that $\{v_m\}_1^\infty$ converges uniformly on $\bar{\Omega}\times\boldsymbol{R}$ to a continuous T-periodic function v satisfying $\|v\|_\infty=\lim_{m\to\infty}\|v_m\|_\infty=1$. Returning to (17) we see that

$$Lv_m-\gamma_m v_m=[h(x,t)-g(u_m)]/\|u_m\|_\infty \tag{20}$$

Let $z\in E$ be arbitrary. From (20) it follows that

$$(L^*z-\lambda_1 z,v)_0=\lim_{m\to\infty}(L^*z-\gamma_m z,v_m)_0=\lim_{m\to\infty}(z,Lv_m-\gamma_m v_m)_0=\lim_{m\to\infty}\left(z,\frac{h-g(u_m)}{\|u_m\|_\infty}\right)_0=0$$

since g is bounded. From Theorem 3 it follows that $v=c\phi$. We consider only the case $c>0$, the case $c<0$ is treated similarly. Since $\phi>0$ on $\Omega\times\boldsymbol{R}$ and since $u_m=\|u_m\|_\infty v_m$, it follows that $\lim_{m\to\infty}u_m(x,t)=+\infty$ at each $(x,t)\in\Omega\times\boldsymbol{R}$. From (20) we have

$$0=(L^*\varphi\phi^*-\lambda_1\phi^*,v_m)_0=(\gamma_m-\lambda_1)(\phi^*,v_m)_0+\frac{1}{\|u_m\|_\infty}(h-g(u_m),\phi^*)_0\,. \tag{21}$$

By construction $\gamma_m-\lambda_1<0$ for all m. Moreover $(\phi^*,v_m)_0\to c(\phi^*,\phi)=c>0$ as $m\to\infty$ and by Lebesgue's bounded convergence theorem

$$\lim_{m\to\infty}(h-g(u_m),\phi^*)_0=(h,\phi^*)_0-g(\infty)\int_0^T\left(\int_\Omega\phi^*(x,t)\,dx\right)dt<0,$$

assuming (15). Therefore the righthand side of (21) is negative for large m and we have a contradiction. This contradiction shows that (15) implies the existence of a constant R_1 such that (16) holds for all m. Rewriting (15) in the form

$$Mu_m=(d+\gamma_m)u_m+h(x,t)-g(u_m)\equiv k_m(x,t) \tag{22}$$

we see that there exists a constant R_4 such that

$$\|k_m\|_\infty = \max_{\bar{\Omega}\times R} |k_m(x, t)| \leqslant R_4$$

for all m. Repeating the bootstrap argument that was used before we infer the existence of a constant R_5 such that $\|u_m\|_F \leqslant R_5$ for all m. The fact that $g \in C^1$ implies the existence of R_6 such that $\|k_m\|_F \leqslant R_6$ for all m. The continuity of $M^{-1}: F \to E$ and (22) implies the existence of a constant R_7 such that $\|u_m\|_F \leqslant R_7$ for all m. Therefore since the embedding of E into F is compact we may assume without loss of generality that $\|u_m - u\|_F \to 0$ as $m \to \infty$ for some $u \in F$. Since

$$u_m = M^{-1}[(d + \gamma_m)u_m + h - g(u_m)]$$

it follows that $u \in E$ and $Lu = \lambda_1 u + h - g(u)$. This proves the Theorem.

REMARK. – If instead of (12) we assume $g(\infty) < g(s) < g(-\infty)$ then $c_0 g(\infty) < (h, \phi^*)_0 < c_0 g(-\infty)$ is a necessary and sufficient condition for the solvability of (13). In this case one chooses a sequence $\{\gamma_m\}_1^\infty$ such that $\gamma_m > \lambda_1$ and $\lim_{m\to\infty} \gamma_m = \lambda_1$. Using the fact that the spectrum of the mapping T os section 2 is discrete one shows that $(L - \gamma_m I): E \to F$ is a homeomorphism for large m. For such m, the mapping $(L - \gamma_m I)^{-1}[h - g(u)]$ regarded as a map from F into itself is compact and maps a closed ball into itself. Therefore, there exists $u_m \in E$ satisfying (13) and one proceeds as before.

REFERENCES

[1] H. AMANN, *Periodic solutions of semilinear parabolic equations*, in Nonlinear Anal., a volume in honor of E. H. Rothe, Academic Press, (1978), 1-29.

[2] A. AMBROSETTI - G. PRODI, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Math. Pura Appl. (4), 93 (1972), 231-247.

[3] C. CLARK, *Mathematical bioeconomics*, John Wiley, New York, 1976.

[4] A. FRIEDMAN, *Partial differential equations of parabolic type*, Prentice Hall, Englewood Cliffs, N.J., 1964.

[5] A. HAMMERSTEIN, *Nichtlineare Integralgleichungen nebst Anwendungen*, Acta Math., 54 (1930), 117-176.

[6] P. Hess, *On a theorem of Landesman and Lazer*, Indiana Univ. Math. J., **23** (1974), 827-927.

[7] J. Kazdan - F. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math., **28** (1975), 567-597.

[8] H. Keller, *Some positone problems suggested by nonlinear heat eration*, in *Bifurcation and nonlinear eigenvalue problems*, J. Keller and S. Antman eds., Benjamin, New York, (1969), 217-255.

[9] Ju S. Kolesov, *A test for the existence of periodic solutions to parabolic equations*, Soviet Math. Dokl., **7** (1966), 1318-1320.

[10] M. Krasnoselski, *Topological methods in the theory of nonlinear integral equations*, McMillan Co., New York, 1934.

[11] M. Krasnoselski, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.

[12] O. Ladysenskaja - V. Solonnikov - N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence Rhode Island, 1968.

[13] E. Landesman - A. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech., **19** (1970), 609-623.

[14] A. Lazer, *Some remarks on periodic solutions of parabolic differential equations*, Proceedings of Second International Symposium on Dynamical Systems, Gainsville, Florida (to appear).

[15] L. Nirenberg, *A strong maximum principle for parabolic equations*, Comm. Pure Appl. Math., **6** (1953), 167-177.

[16] M. Protter - H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

[17] D. Sattinger, *Topics in stability and bifurcation theory*, Springer Lecture Notes in Mathematics, Vol. 309, New York, 1973.

[18] I. Stakgold - L. Payne, *Nonlinear problems in nuclear reactors analysis*, in *Nonlinear problems in the physical sciences and biology*, Springer Lecture Notes in Mathematics, Vol. 322, New York, 1973.

[19] W. Walter, *Differential and integral inequalities*, Springer Verlag Berlin, Heidelberg, New York, 1970.

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