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INFINITELY MANY RADIAL SOLUTIONS FOR A p-LAPLACIAN PROBLEM WITH INDEFINITE WEIGHT

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Abstract. We prove the existence of infinitely many sign changing radial solutions for a p-Laplacian Dirichlet problem in a ball. Our problem involves a weight function that is positive at the center of the unit ball and negative in its boundary. Standard initial value problems-phase plane analysis arguments do not apply here because solutions to the corresponding initial value problem may blow up near the boundary due to the fact that our weight function is negative at the boundary. We overcome this difficulty by connecting the solutions to a singular initial value problem with those of a regular initial value problem that vanishes at the boundary.

1. Introduction.

We study the quasilinear Dirichlet problem

$$\begin{cases} \Delta_p u + W(x)g(u) = 0 & \text{in } B_1(0) \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{1}$$

where $N \geq 2$, $p > 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, and $B_1(0)$ denotes de unit ball in $\mathbb{R}^N$ centered at the origin.

We assume that g is a non-decreasing locally Lipschitzian continuous function and there exists $C > 0$ such that

$$|g(s)| \leq C|s|^{p-1} \quad \text{for all } \ s \in [-1, 1]. \tag{2}$$

For the sake of simplicity in the calculations we assume that $sg(s) > 0$ for $s \neq 0$. We also assume that there exist $q_1, q_2 \in (p-1, \infty)$ and $A_1, A_2 \in (0, \infty)$ such that

$$\lim_{s\to\infty} \frac{g(s)}{|s|^{q_1-1}s} := A_1, \text{ and } \lim_{s\to-\infty} \frac{g(s)}{|s|^{q_2-1}s} := A_2. \tag{3}$$

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If $p \in (1, N)$ we assume that either

$$(i) \quad q_1 < \frac{N(p-1)}{N-p} \qquad \text{or} \qquad (ii) \quad p-1 < q_1, q_2 < p^* - 1, \tag{4}$$

where $p^* = Np/(N-p)$. Note that for $p \geq N$ the assumption $q_1, q_2 \in (p-1, \infty)$ implies

$$N + \frac{q_i(p-N)}{p-1} \geq p, \quad \text{for } i = 1, 2. \tag{5}$$

Also,

$$\text{if } p < N \text{ and } q_1 < N(p-1)/(N-p) \text{ then } N + \frac{q_1(p-N)}{p-1} > 0. \tag{6}$$

Finally, we assume that the weight function $W \in C^1[0,1]$ and there exists $X \in (0,1)$ such that

$$W(X) = 0, \; W'(X) < 0, \; W > 0 \text{ in } [0, X), \text{ and } W < 0 \text{ in } (X, 1]. \tag{7}$$

For the sake of simplicity in the presentation we assume W is decreasing in $[0, X)$ (see Remark 2).

Over the last fifty years the study of radial solutions to elliptic boundary value problems has been very active going back to papers such as [2] and [4]. Our approach here is inspired by the methods in [4], where *Pohozaev energy* and *phase plane* arguments applied to the solutions to a related singular ordinary differential equations are used to prove the existence of solutions to the boundary value problem by a simple application of the intermediate value theorem (see also [8]). The main difficulty of the problem we study here is that, because the weight function W changes sign, some of the solutions to a related initial value problem blow up preventing the use of continuity properties for such problems. We overcome such a difficulty by following the arguments in [4] in a region where the solutions to the initial value problem do not blow up and connecting them to solutions that satisfy the boundary condition. For examples of applications of problem with indefinite weight the reader is referred to [9]. For recent results on quasilinear problems with weight see [1, 5, 11, 14]. For related results on the existence of infinitely many radial solutions to quasilinear problems see [6, 3, 10].

Our main result is the following theorem.

Theorem 1.1. *If* (3), (4) *and* (7) *hold, then there exists* $k_0 \in \mathbb{N}$ *such that for every* $k \geq k_0$*, the problem* (1) *has a solution with* k *nodal sets in the unit ball with* $u(0) > 0$. *In particular, the problem* (1) *has infinitely many radial solutions satisfying* $u(0) > 0$.

Remark 1. Interchanging q_1 and q_2 en (4) we have $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ the problem (1) has a solution with k nodal sets and $u(0) < 0$. In particular, the problem (1) has infinitely many radial solutions satisfying $u(0) < 0$.

This article is organized as follows. In Section 2 we show that all solutions to (10) below are defined in $[0, X]$ and that for each $a \in \mathbb{R}$ there exists a unique ζ such that the solution to (17) below satisfies the boundary condition $u(1) = 0$, see Theorem 2.7. In Section 3, we prove that our hypotheses imply if u is a solution to (10) with large d then $u^2(r) + (u'(r))^2$ remains large in an interval $[0, T_1] \subset [0, X]$ with $T_1 > 0$ independent of d. In Section 4, we present the phase plane analysis of the solutions to (10) in $[0, X]$, and in Section 5 we prove our main result.

2. **The initial value problem.** The radial solutions to (1) are the solutions to

$$\begin{cases} \left(\left|u'\right|^{p-2}u'\right)'(r) + \dfrac{N-1}{r}\left|u'(r)\right|^{p-2}u'(r) + W(r)g\big(u(r)\big) = 0, & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{8}$$

That is, $v : B_1(0) \to \mathbb{R}$ is a radial solution to (1) if and only if the function $u : [0,1] \to \mathbb{R}$ defined by $u(\sqrt{x_1^2 + \cdots + x_N^2}) := v(x_1, \ldots, x_N)$ satisfies (9). Due to the singularity given by the zeros of u' the solutions to (8) need not be of class C^2. In fact, regularity theory for quasilinear problems indicates that the solutions to (8) may only be expected to be in the Holder space $C^{1,\mu}$ for some $\mu \in (0,1)$, see [7, 12].

It fits our purposes to regard (8) as

$$\begin{cases} \left(r^{N-1}\left|u'(r)\right|^{p-2}u'(r)\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0. \end{cases} \tag{9}$$

Our technique is based on the analysis of the solutions to the initial value problem

$$\begin{cases} \left(r^{N-1}\left|u'\right|^{p-2}u'\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, & 0 < r < 1, \\ u(0) = d, \quad u'(0) = 0. \end{cases} \tag{10}$$

Throughout this paper we write $u(r,d) := u(r)$ if the dependence of u on d is clear from the context. Letting $\Gamma(x) = x|x|^{p-2}$ one sees that, for each $d \in \mathbb{R}$, a continuous function u satisfies the integral equation

$$u(r) = d - \int_0^r \Gamma^{-1}\left(s^{1-N}\int_0^s t^{N-1}W(t)g(u(t))dt\right)ds \tag{11}$$

if and only if it is a solution to (10). More generally, for any $r_0 \in [0,1), a \in \mathbb{R}, b \in \mathbb{R}$, a continuous function u satisfies

$$u(r) = a + \int_{r_0}^r \Gamma^{-1}\left(s^{1-N}\left[r_0^{N-1}\Gamma(b) - \int_{r_0}^s t^{N-1}W(t)g(u(t))dt\right]\right)ds \tag{12}$$

if and only if it satisfies

$$\begin{cases} \left(r^{N-1}\left|u'\right|^{p-2}u'\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, & r_0 \le r < 1, \\ u(r_0) = a, \quad u'(r_0) = b. \end{cases} \tag{13}$$

Given $d_0 \in \mathbb{R} - \{0\}$, since g is a locally Lipschitzian function, there exists $\tau > 0$ such that for each $d \in [d_0 - \tau, d_0 + \tau]$, equation (11) has a unique solution u_d in the space of continuous functions defined on $[0,\tau]$. This and the continuity of the right hand side in (11) on (d,u), imply that u_d continuously depends on d. If $\tau = 1$ such a solution is a solution to (10). If $\tau \in (0,1)$, we obtain a solution on $[0,\tau_1]$ for some $\tau_1 > \tau$ by applying the same argument to (12) with $a = u_d(\tau)$ and $b = u_d'(\tau)$. The function u_d may be extended to a maximal interval which is either $[0,1]$ or $[0,\hat{\tau}(d))$ with $\lim_{t\to\hat{\tau}(d)^-}[u^2(t) + (u'(t))^2] = +\infty$. We note that, due to hypothesis (2), no solution to (13) satisfies $\lim_{t\to\hat{\tau}(d)^-}[u^2(t) + (u'(t))^2] = 0$ if $(a,b) \neq (0,0)$. For a comprehensive study of existence, uniqueness and continuous dependence, we refer the reader to [13]. See also [6] for some details in the case $W = 1$.

In our next lemma we prove that $\hat{\tau}(d) > X$. Since $d_0 \in \mathbb{R} - \{0\}$ is arbitrary, this show the existence of a unique solution to (10) on $[0,X]$ that depends continuously on d.

From now on we define

$$G(t) = \int_0^t g(s)ds \quad \text{and} \quad p' = p/(p-1). \tag{14}$$

Lemma 2.1. *For each $d \in \mathbb{R}$ the solution to (10) is defined in $[0, X]$.*

Proof. Let u be a solution to (10) defined in $[0, t)$ with $t \leq X$, and

$$\mathcal{E}(r, d) \equiv \mathcal{E}(r) := \frac{p-1}{p}|u'(r)|^p + W(r)G\big(u(r)\big). \tag{15}$$

Observe $|u'|^p = \big|\,|u'|^{p-2}u'\,\big|^{p/(p-1)}$ and the function $|u'|^{p-2}u'$ is differentiable in $(0, t)$ (see (10)). Moreover, function $h(s) = |s|^{p/(p-1)}$ is differentiable on $\mathbb{R}$ and $h'(s) = \frac{p}{p-1}|s|^{p/(p-1)-2}s = \frac{p}{p-1}|s|^{(2-p)/(p-1)}s$ for all $s \neq 0$ and $h'(0) = 0$. Hence, $\mathcal{E}$ is differentiable on $(0, t)$ and

$$\begin{aligned}
\mathcal{E}'(r) &= \left(\frac{p-1}{p}|u'(r)|^{p-2}u'(r)|^{p/(p-1)}\right)' + W'(r)G(u(r)) + W(r)g(u(r))u'(r) \\
&= \big||u'(r)|^{p-2}u'(r)\big|^{(2-p)/(p-1)} |u'(r)|^{p-2}u'(r) \left(|u'(r)|^{p-2}u'(r)\right)' \\
&\quad + W'(r)G(u(r)) + W(r)g(u(r))u'(r) \\
&= |u'(r)|^{2-p}|u'(r)|^{p-2}u'(r) \left(|u'(r)|^{p-2}u'(r)\right)' \\
&\quad + W'(r)G(u(r)) + W(r)g(u(r))u'(r) \\
&= u'(r)\left(-\frac{N-1}{r}|u'(r)|^{p-2}u'(r) - W(r)g(u(r))\right) \\
&\quad + W'(r)G(u(r)) + W(r)g(u(r))u'(r) \quad \text{(from (8))} \\
&= -\frac{N-1}{r}|u'(r)|^p + W'(r)G(u(r)) \\
&= -\frac{p(N-1)}{(p-1)r}\mathcal{E}(r) + G\big(u(r)\big)\left[\frac{p(N-1)}{(p-1)r}W(r) + W'(r)\right] \qquad \text{(from (15))} \\
&\leq W'(r)G(u(r)).
\end{aligned} \tag{16}$$

Hence $\mathcal{E}$ decreases on $[0, t)$ which implies $|u'(r)|^p \leq p'W(0)G(d)$ for all $r \in [0, t)$. Thus $\lim_{r\to t-} u(r) := u(t) \in \mathbb{R}$ and hence, $\lim_{r\to t-} u'(r) := u'(t) \in \mathbb{R}$. Therefore u may be extended to an interval $[0, t + \varepsilon_0)$ for some $\varepsilon_0 > 0$. Since this is valid for any $t \in [0, X]$ we conclude that the solution to (10) may be extended to $[0, X + \varepsilon_0)$ with ε_0 depending on d. This proves the lemma. □

Remark 2. The assumption $W'(r) \leq 0$ in $[0, X]$ may be eliminated by observing that $\dfrac{p(N-1)}{(p-1)r}W(r) + W'(r) < 0$ in an interval of the form $[X - \delta, X + \delta]$ and that $W'(r)G(u(r)) \leq C\mathcal{E}(r)$ for $r \in [0, X - \delta]$ for some constant C depending only on W.

For $a, \zeta \in \mathbb{R}$ let us consider

$$\begin{cases} \left(r^{N-1}\left|u'\right|^{p-2}u'\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, & X < r < 1, \\ u(X) = a, \quad u'(X) = \zeta. \end{cases} \tag{17}$$

As mentioned above, due to our assumptions on g, the initial value problem (17) has a unique solution $u = u(a, \zeta)$ on a maximal interval I, which is denoted by $[X, R_{a,\zeta}) := I$.

Lemma 2.2. *For $a > 0$ let*

$$\eta_a = \max\left\{ \frac{2^{p/(p-1)} a}{(1-X)X^{(N-1)/(p-1)}}, \left(\frac{2\|W\|_\infty (1-X) g(a)}{X^{N-1}} \right)^{1/(p-1)} \right\}. \tag{18}$$

If u is the solution to (17) with $\zeta = -\eta_a$ then there exists $\hat{r} \in I \cap [0,1)$ such that $u(\hat{r}) = 0$ and u decreases in $[X, \hat{r}]$.

Proof. Let $r > 0$ be such that $0 \le u(s) \le a$ for every $s \in [X, r]$. The existence of such an r is guaranteed by the initial conditions in (17) and the fact that $\zeta = -\eta_a < 0$. Thus, integrating the differential equation in (17) on $[X, r]$,

$$r^{N-1}|u'(r)|^{p-2}u'(r) - X^{N-1}|u'(X)|^{p-2}u'(X) = -\int_X^r s^{N-1} W(s) g\big(u(s)\big) ds.$$

Hence, the definition of η_a implies

$$\begin{aligned} |u'(r)|^{p-2}u'(r) &= -\left(\frac{X}{r}\right)^{N-1} \eta_a^{p-1} - \int_X^r \left(\frac{s}{r}\right)^{N-1} W(s) g(u(s)) ds \\ &\le -X^{N-1}\eta_a^{p-1} + (1-X)\|W\|_\infty g(a) \\ &\le -\frac{X^{N-1}\eta_a^{p-1}}{2}. \end{aligned} \tag{19}$$

Thus u decreases in $[X, r]$. Therefore u is bounded in $[X, r]$, which implies that $[X, r] \subset I$. Let

$$\hat{r} = \sup\{r \in I \colon 0 \le u(s) \le a \quad \text{for all} \quad s \in [X, r]\} := \sup B.$$

Due to the continuity of u, if $r \in B$ then $u(r) \ge 0$. Applying again the continuity of u we have $u(\hat{r}) \ge 0$. Since $[X, r] \subset I$ for all $r \in B$, we have $[X, \hat{r}] \subseteq I$. Assuming that $u(\hat{r}) > 0$, the continuity of u implies that there exists $\delta > 0$ such that $u(s) > 0$ for all $s \in [X, \hat{r} + \delta)$ contradicting the definition of $\hat{r}$. Hence $u(\hat{r}) = 0$.

From (19),

$$-u'(r) \ge \left(\frac{X^{N-1}\eta_a^{p-1}}{2} \right)^{1/(p-1)} \quad \text{for all} \quad r \in [X, \hat{r}].$$

Integrating on $[X, \hat{r}]$,

$$0 = u(\hat{r}) = u(X) + \int_X^{\hat{r}} u'(r) dr \le a - \frac{(\hat{r} - X) X^{(N-1)/(p-1)} \eta_a}{2^{1/(p-1)}}. \tag{20}$$

This and the definition of η_a yield

$$\hat{r} \le X + \frac{2^{1/(p-1)} a}{X^{(N-1)/(p-1)} \eta_a} \le X + \frac{1-X}{2} < 1. \tag{21}$$

Thus, from (21) and (19), $\hat{r} \in (X, 1)$, $u(\hat{r}) = 0$ and u decreases in $(X, \hat{r})$ proving the lemma. ∎

Lemma 2.3 (Comparison principle)**.** *Let $a, y_1, y_2 \in \mathbb{R}$. Let u_1 satisfy*

$$\begin{cases} \left(r^{N-1}|u_1'|^{p-2}\, u_1'\right)' + r^{N-1} W(r) g\big(u_1(r)\big) = 0, \quad X < r < R_1 := R_{a,y_1}, \\ u_1(X) = a, \quad u_1'(X) = y_1, \end{cases} \tag{22}$$

and u_2 satisfy

$$\begin{cases} \left(r^{N-1}\left|u_2'\right|^{p-2}u_2'\right)' + r^{N-1}W(r)g\big(u_2(r)\big) = 0, \quad X < r < R_2 := R_{a,y_2}, \\ u_2(X) = a, \quad u_2'(X) = y_2. \end{cases} \tag{23}$$

If $y_1 < y_2$, then $u_1(t) < u_2(t)$ for every $t \in [X, R_1) \cap [X, R_2)$.

Proof. Assuming to the contrary there exists $t \in [X, R_1) \cap [X, R_2)$ such that

$$u_1(t) = u_2(t) \quad \text{and } u_1(r) < u_2(r) \text{ for all } r \in (X, t). \tag{24}$$

Then, $u_2'(t) \le u_1'(t)$. Since Γ is an increasing function,

$$|u_2'(t)|^{p-2}u_2'(t) \le |u_1'(t)|^{p-2}u_1'(t).$$

This, (22) and (23) yield

$$\begin{aligned} X^{N-1}|y_2|^{p-2}y_2 - &\int_X^t s^{N-1}W(s)g\big(u_2(s)\big)ds \\ &\le X^{N-1}|y_1|^{p-2}y_1 - \int_X^t s^{N-1}W(s)g\big(u_1(s)\big)ds. \end{aligned} \tag{25}$$

On the other hand, since $y_1 < y_2$ and Γ is strictly increasing,

$$X^{N-1}|y_1|^{p-2}y_1 < X^{N-1}|y_2|^{p-2}y_2. \tag{26}$$

Moreover, since $-W \ge 0$ on $[X, 1]$ and g is non-decreasing,

$$-\int_X^t s^{N-1}W(s)g\big(u_1(s)\big)ds \le -\int_X^t s^{N-1}W(s)g\big(u_2(s)\big)ds. \tag{27}$$

Since (25) together with (26) contradict (27) the lemma is proven. □

Lemma 2.4. *Let $r_* \in [X, 1)$, $b > 0$ and $y \ge 0$. If u satisfies*

$$\begin{cases} \left(r^{N-1}\left|u'\right|^{p-2}u'\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, \qquad r_* \le r < R := R_{b,y}, \\ u(r_*) = b, \quad u'(r_*) = y, \end{cases} \tag{28}$$

then $u'(r) > 0$ for all $r \in [r_*, R)$.

Proof. Let $t \in [r_*, R)$. From (28),

$$\begin{aligned} t^{N-1}|u'(t)|^{p-2}u'(t) &= r_*^{N-1}|u'(r_*)|^{p-2}u'(r_*) - \int_{r_*}^t s^{N-1}W(s)g\big(u(s)\big)ds \\ &= r_*^{N-1}y^{p-1} + \int_{r_*}^t s^{N-1}(-W(s))g\big(u(s)\big)ds. \end{aligned} \tag{29}$$

Since $b > 0$, if t is close to r_*, $g(u(r)) \approx g(b) > 0$ for all $r \in [r_*, t]$. Hence, (29) implies $u'(t) > 0$. Now, assume $t \in [r_*, R)$ satisfies $u'(t) = 0$ and $u'(r) > 0$ for every $r \in [r_*, t)$. Then $u(r) \ge u(r_*) = b$. From (29),

$$0 = t^{N-1}|u'(t)|^{p-2}u'(t) = r_*^{N-1}y^{p-1} + \int_{r_*}^t s^{N-1}(-W(s))g\big(u(s)\big)ds > 0.$$

This contradiction shows $u'(t) > 0$ for every $t \in [r_*, R)$, proving the lemma. □

From now on let $a > 0$ and for $y > -\eta_a$, let us denote by u_y the unique solution of

$$\begin{cases} \left(r^{N-1}\left|u'\right|^{p-2}u'\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, \\ u(X) = a, \quad u'(X) = y, \end{cases} \tag{30}$$

which is defined on a maximal interval $[X, R_y), (R_y := R_{a,y})$.

Let

$$A := \{y \geq -\eta_a \colon u_y \text{ has a zero } \ r_y \text{ in } \ (X,1)\} \ \text{ and } \ \hat{\zeta}(a) = \sup A. \tag{31}$$

Applying Lemma 2.4 with $r_* = X$ and $b = a$, we observe $A \subseteq (-\infty, 0)$. Thus, $\hat{\zeta}(a) \leq 0$ for all $a > 0$.

Remark 3. From the comparison principle (Lemma 2.3), if $y_1, y_2 \in A$ with $y_1 < y_2$ and $r_1 \equiv r_{y_1}$, $r_2 \equiv r_{y_2}$ in $(X,1)$ are the corresponding first zeros of u_{y_1} and u_{y_2}, then $r_1 < r_2$.

Theorem 2.5. *Let $a > 0$. If u is the solution to (17) with $u'(X) = \hat{\zeta}(a)$ then $u(1) = 0$ and u is positive in $[X, 1)$.*

Proof. Let $\{y_j\}_j \subset A$ be an increasing sequence converging to $\hat{\zeta}(a)$. Let u be the solution to the initial value problem (30) with $y = \hat{\zeta}(a)$ and let u_j be the solution to the initial value problem (30) with $y = y_j$. Let $r_j \in (X, 1)$ be such that $u_j(s) > 0$ for all $s \in (X, r_j)$ and $u_j(r_j) = 0$. By Lemma 2.3, $\{r_j\}_j$ is an increasing sequence bounded above by 1. This and the continuous dependence of solutions on initial conditions imply $\hat{\zeta}(a) \leq 1$. Let $\tau = \lim_{j\to\infty} r_j \in [X, 1]$. By the continuity of u and Lemma 2.3,

$$u(\tau) = \lim_{j\to\infty} u(r_j) \geq \liminf_{j\to\infty} u_j(r_j) = 0. \tag{32}$$

Let $\varepsilon > 0$. By the continuous dependence of solutions to initial value problems on initial conditions, there exists j_0 such that if $j \geq j_0$ then $r_j \in (\tau - \varepsilon, \tau)$ and $|u'(t) - u_j'(t)| < \varepsilon$ for all $j \geq j_0$ and $t \in [X, \tau - \varepsilon]$. From

$$t^{N-1}|u_n'(t)|^{p-2}u_n'(t) = X^{N-1}|y_n|^{p-2}y_n - \int_X^t s^{N-1}W(s)g\big(u_n(s)\big)ds,$$

and the fact that $\{y_j\}_j$ is a bounded sequence, there exists $M > 0$ such that

$$|u_j'(t)| \leq M, \ |u'(t)| \leq M \text{ for all } \quad j \geq j_0, \ t \in [X, \tau - \varepsilon]. \tag{33}$$

Hence, for $j \geq j_0$, $u_j(\tau - \varepsilon) \leq M(r_j - \tau + \varepsilon) \leq M\varepsilon$. Thus

$$\begin{aligned} u(\tau) &= a + \int_X^\tau u'(s)ds = a + \int_X^{\tau-\varepsilon} u_j'(s)ds + \int_X^{\tau-\varepsilon} (u'(s) - u_j'(s))ds \\ &\quad + \int_{\tau-\varepsilon}^\tau u'(s)ds \\ &\leq u_j(\tau - \varepsilon) + \varepsilon(\tau - X) + M\varepsilon \\ &\leq \varepsilon(2M + \tau - X). \end{aligned} \tag{34}$$

Since $\varepsilon > 0$ is arbitrary we have $u(\tau) = 0$. Since $u(t) > u_j(t)$ for all $t \in [X, r_j]$, u is positive in $[X, \tau)$.

By the uniqueness of solutions to initial value problems and the assumption $g(0) = 0$, we have $u'(\tau) < 0$. Assuming that $\tau < 1$, there exists $\varepsilon \in (0, 1 - \tau)$ such that $u(x) < 0$ for $x - \tau \in (0, \varepsilon)$. Let $\{z_j\}_j$ be a decreasing sequence converging to

$\hat{\zeta}(a)$ and let v_j be the solution to the initial value problem (30) with $y = z_j$. By continuous dependence of solutions to initial value problems on initial conditions, there exists j such that $v_j(\tau + \varepsilon/2) < 0$. Since v_j is positive in $[X, \tau]$, there exists $r_1 \in (\tau, \tau + \varepsilon/2) \subset [X, 1)$ such that v_j is positive in $[X, r_1)$ and $v_j(r_1) = 0$. Hence $z_j \in A$ and $z_j > \hat{\zeta}(a)$ contradicting the definition of $\hat{\zeta}(a)$. This contradiction proves that $\tau = 1$ and, therefore, the theorem. □

Theorem 2.6. *The function $\hat{\zeta} : [0, \infty) \to (-\infty, 0]$ defined on $(0, +\infty)$ by Theorem 2.5 and by $\hat{\zeta}(0) = 0$ is a decreasing continuous function.*

Proof. Let $a_1 < a_2$. Let u be the solution to the second order differential equation in (30) that satisfies the initial condition $u(X) = a_1$, $u'(X) = \hat{\zeta}(a_1)$ and similarly v for $(a_2, \hat{\zeta}(a_2))$. Because $a_1 < a_2$, $v(r) > u(r)$ for r near X. This and $u(1) = v(1)$ imply that there exists $\sigma \in (X, 1]$ such that $v(s) > u(s)$ for all $s \in (X, \sigma)$ and $v(\sigma) = u(\sigma)$. By uniqueness of solutions to initial value problems, $v'(\sigma) < u'(\sigma)$. Assuming that $\hat{\zeta}(a_1) \le \hat{\zeta}(a_2)$, we have

$$\begin{aligned} 0 &< \sigma^{N-1}(|u'(\sigma)|^{p-2}u'(\sigma) - |v'(\sigma)|^{p-2}v'(\sigma)) \\ &= X^{N-1}(|\hat{\zeta}(a_1)|^{p-2}\hat{\zeta}(a_1) \\ &\quad - |\hat{\zeta}(a_2)|^{p-2}\hat{\zeta}(a_2)) - \int_X^\sigma s^{N-1}W(s)(g(u(s)) - g(v(s)))ds \\ &< 0. \end{aligned} \tag{35}$$

This contradiction proves that $\hat{\zeta}$ is a decreasing function.

Let $\{a_n\}$ be a decreasing sequence of non-negative numbers converging to $a \ge 0$. Let $\hat{\zeta}(a_n) := \hat{\zeta}_n$, and u_n be the solution to the second order differential equation in (30) that satisfies the initial condition $u_n(X) = a_n$, $u_n'(X) = \hat{\zeta}_n$. Let u be the solution to the second order differential equation in (30) that satisfies the initial condition $u(X) = a$, $u'(X) = \hat{\zeta}(a)$. Since $\{\hat{\zeta}_n\}$ is an increasing sequence bounded by $\hat{\zeta}(a)$, $c = \lim_{n\to+\infty} \hat{\zeta}(a_n) \le \hat{\zeta}(a)$. Let w denote the solutions to the second order differential equation in (30) with $w(X) = a$ and $w'(X) = c$. By continuous dependence on initial conditions we have

$$0 = \lim_{n\to\infty} u_n(1) = w(1). \tag{36}$$

Therefore $c = \hat{\zeta}(a)$. Thus $\lim_{n\to+\infty} \hat{\zeta}(a_n) = \zeta(a)$. Similarly, if $\{a_n\}$ is an increasing sequence converging to a then $\lim_{n\to+\infty} \hat{\zeta}(a_n) = \zeta(a)$. This proves that $\hat{\zeta}$ is continuous on $[0, \infty)$. □

Imitating the proofs in Theorem 2.5 and Theorem 2.6 one proves that $\hat{\zeta}$ may be extended to $(-\infty, \infty)$. That is we have the following result.

Theorem 2.7. *There exists a continuous function $\hat{\zeta} : \mathbb{R} \to \mathbb{R}$ with $\hat{\zeta}(0) = 0$ such that if u is the solution to (17) with $u'(X) = \hat{\zeta}(a)$ then $u(1) = 0$ and, u is positive in $[X, 1)$ if $a > 0$ and u is negative in $[X, 1)$ if $a < 0$.*

3. Energy analysis.

If (4) (ii) is satisfied then there exists $\delta > 0$ such that

$$(\delta + 1)(q_i + 1) < p^*, \quad i = 1, 2. \tag{37}$$

We choose $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{ \frac{A_1[p^* - (\delta+1)(q_1+1)]}{p^* + (\delta+1)(q_1+1)}, \frac{A_2[p^* - (\delta+1)(q_2+1)]}{p^* + (\delta+1)(q_2+1)} \right\}. \tag{38}$$

Letting $C_1 := A_1 - \varepsilon, C_2 := A_1 + \varepsilon, C_3 := A_2 - \varepsilon, C_4 := A_2 + \varepsilon$, by (3) there exists $M > 0$ such that

$$\forall\, s \geq 0, \quad C_1 s^{q_1+1} - M \leq \quad sg(s) \quad \leq C_2 s^{q_1+1} + M, \tag{39}$$

$$\forall\, s \leq 0, \quad C_3 |s|^{q_2+1} - M \leq \quad sg(s) \quad \leq C_4 |s|^{q_2+1} + M, \tag{40}$$

$$\forall\, s \geq 0, \quad \frac{C_1}{q_1+1} s^{q_1+1} - M \leq \quad G(s) \quad \leq \frac{C_2}{q_1+1} s^{q_1+1} + M, \tag{41}$$

$$\forall\, s \leq 0, \quad \frac{C_3}{q_2+1} |s|^{q_2+1} - M \leq \quad G(s) \quad \leq \frac{C_4}{q_2+1} |s|^{q_2+1} + M, \tag{42}$$

where G is a primitive of g such that $G(0) = 0$. Hence there exist $D > 0, \tilde{C}_1, \tilde{C}_2$ such that

$$\tilde{C}_1 |s|^{q_i+1} \leq sg(s) \leq \tilde{C}_2 |s|^{q_i+1} \quad i = 1, 2, \text{ for } |s| \geq D. \tag{43}$$

Note that the monotonicity of g implies $G(s) \geq 0$ for all $s \in \mathbb{R}$ and $sg(s) \geq 0$ for all $s \neq 0$.

Due to the continuity of W at zero, there exists $T \in (0, X)$ (see (7)) so that

$$\forall\, r \in [0, T], \quad W(r) \geq \frac{W(0)}{2} := \frac{m}{2}. \tag{44}$$

Given $d > 0$, let u be the solution to (10) defined on $[0, X]$ (see Lemma 2.1 above). It follows that

$$-\, r^{N-1} \big|u'(r)\big|^{p-2} u'(r) = \int_0^r s^{N-1} W(s) g\big(u(s)\big)\, ds. \tag{45}$$

Due to $d > 0$ and the continuity of u, we have $u > 0$ near $r = 0$. Since g is an increasing function, $g(0) = 0$, (44) and (45) then $u'(r) < 0$ for $r > 0$ small. Let

$$r_0 = r_0(d) := \sup \{r > 0 : \forall\, s \in [0, r],\ u(s) \geq d/2\}.$$

Note that, from (45), $u'(r) < 0$ for all $r \in (0, r_0)$.

Lemma 3.1. *There exist positive constants K_0 and K_1 independent of d such that*

$$K_0\, d^{\frac{p-1-q_1}{p}} \leq r_0 \leq K_1\, d^{\frac{p-1-q_1}{p}} \quad \textit{for} \quad d \gg 1. \tag{46}$$

Proof. For $d > 2D$, let us define $\tau = \tau(d) := \min\{r_0(d), T\}$. By (43), (44) and the fact that $u'(r) < 0$ for $r \in (0, \tau]$,

$$\begin{aligned} r^{N-1}\big|u'(r)\big|^{p-1} &= \int_0^r s^{N-1} W(s) g\big(u(s)\big)\, ds \geq \frac{m\tilde{C}_1}{2} \int_0^r s^{N-1} \big(u(s)\big)^{q_1}\, ds \\ &\geq \frac{m\tilde{C}_1}{2} \left(\frac{d}{2}\right)^{q_1} \frac{r^N}{N} \equiv \tilde{K}_1 d^{q_1} r^N. \end{aligned}$$

Hence, $-u'(r) \geq \tilde{K}_1^{1/(p-1)} d^{\frac{q_1}{p-1}} r^{\frac{1}{p-1}}$. Integrating on $[0, \tau]$, we have

$$\begin{aligned} d - d/2 \geq d - u(\tau) &= -\int_0^\tau u'(r) dr \geq \frac{p-1}{p} \tilde{K}_1^{1/(p-1)} d^{\frac{q_1}{p-1}} r^{\frac{p}{p-1}} \Big]_0^\tau \\ &= \frac{p-1}{p} \tilde{K}_1^{1/(p-1)} d^{\frac{q_1}{p-1}} \tau^{\frac{p}{p-1}}. \end{aligned}$$

Thus,

$$\tau \leq K_1\, d^{\frac{p-1-q_1}{p}} \quad \text{for} \quad d > 2D,$$

where $K_1 = \frac{p-1}{p}\tilde{K_1}^{1/(p-1)}$. Therefore $\tau(d) \to 0$ as $d \to +\infty$. Then, for $d \gg 1$,

$$\tau \le K_1\, d^{\frac{p-1-q_1}{p}} \le T/2.$$

Hence, $\tau < T$ and thus $\tau = r_0$. Consequently, for $d \gg 1$,

$$r_0 \le K_1\, d^{\frac{p-1-q_1}{p}}.$$

On the other hand, for $d \gg 1$ and $r \in [0, r_0]$, using again (43),

$$\begin{aligned} r^{N-1}\big|u'(r)\big|^{p-1} &= \int_0^r s^{N-1}W(s)g\big(u(s)\big)\,ds \le \tilde{C_2}\|W\|_\infty \int_0^r s^{N-1}\big(u(s)\big)^{q_1}\,ds \\ &\le \tilde{C_2}\|W\|_\infty d^{q_1}\frac{r^N}{N} \equiv \tilde{K_0} d^{q_1} r^N. \end{aligned}$$

As above, integrating on $[0, r_0]$ we have $d/2 \le \tilde{K_0}^{1/(p-1)} d^{\frac{q_1}{p-1}} r_0^{\frac{p}{p-1}}$ which proves the first inequality in (46). Hence the lemma has been proved. □

Lemma 3.2. *There exists $C > 0$, independent of d, such that for $r \in [0, r_0]$,*

$$\mathcal{E}(r, d) \ge C\, d^{q_1+1}, \quad \textit{for} \;\; d \gg 1.$$

Proof. Without loss of generality we can assume that $r_0 < T$. For every $r \in [0, r_0]$ we have

$$\begin{aligned} G(u(r)) &\ge C|u(r)|^{q_1+1} - M \quad (\text{ see } (41)) \\ &\ge C\, d^{q_1+1} - M \ge \frac{C}{2} d^{q_1+1} \quad (d \gg 1). \end{aligned}$$

Since, $\mathcal{E}(r, d) \ge W(r)G(u(r)) \ge mCd^{q_1+1}/4$, the lemma follows. □

Since W is of class C^1, there exists $T_1 \le T$ such that for $r \in (0, T_1]$,

$$\frac{p'(N-1)}{r}W(r) + W'(r) > 0, \tag{47}$$

where $p' = p/(p-1)$. Note that by Lemma 3.1 we may assume $r_0(d) < T_1$ for $d \gg 1$.

Lemma 3.3. *If either $p \ge N$ or $p < N$ and (i) in (4) hold, then $\lim_{d\to+\infty} \mathcal{E}(r, d) = \infty$ uniformly for $r \in [0, T_1]$.*

Proof. From Lemma 3.2, it follows that $\lim_{d\to+\infty} \mathcal{E}(r, d) = +\infty$, uniformly for $r \in [0, r_0]$. Due to (16), (47) and $G(t) \ge 0$ for all $t \in \mathbb{R}$, we have

$$\mathcal{E}'(r) + \frac{p'(N-1)}{r}\mathcal{E}(r) = G(u)\Big[\frac{p'(N-1)}{r}W(r) + W'(r)\Big] \ge 0, \tag{48}$$

for every $r \in (0, T_1]$. Therefore, $\Big(r^{p'(N-1)}\mathcal{E}(r)\Big)' \ge 0$. From Lemmas 3.1 and 3.2, since $N + q_1(p-N)/(p-1) > 0$ (see (5) and (6)), we get

$$\mathcal{E}(r) \ge r^{p'(N-1)}\mathcal{E}(r) \ge r_0^{p'(N-1)}\mathcal{E}(r_0) \ge C\, d^{N+q_1(p-N)/(p-1)} \;\to +\infty, \tag{49}$$

as $d \to +\infty$ uniformly for $r \in [r_0, T_1]$, which proves the lemma. □

For $u(r,d) := u(r)$ the solution to (10) we define:

$$\begin{aligned} H(r,d) &:= r\mathcal{E}(r,d) + \frac{N-p}{p}|u'(r,d)|^{p-2}u'(r,d)u(r,d), \\ P(r,d) &:= \int_0^r s^{N-1}\left[\Big(NW(s)+sW'(s)\Big)G\big(u(s)\big) - \frac{N-p}{p}W(s)g\big(u(s)\big)u(s)\right] ds. \end{aligned} \tag{50}$$

The quantities in (50) are related by the Pohozaev-type identity (see [3, 4, 11]):

$$\begin{aligned} &r^{N-1}H(r,d) - t^{N-1}H(t,d) \\ &= \int_t^r s^{N-1}\Big[\Big(NW(s)+sW'(s)\Big)G(u) - \frac{N-p}{p}W(s)g(u)u\Big]ds\,. \end{aligned} \tag{51}$$

Taking $t = 0$ in equation (51), we have the following Pohozaev identity

$$r^{N-1}H(r,d) = P(r,d),$$

equivalently

$$\begin{aligned} &r^N\left[\frac{p-1}{p}\big|u'(r)\big|^p + W(r)G\big(u(r)\big)\right] + \frac{N-p}{p}\,r^{N-1}\big|u'(r)\big|^{p-2}\,u'(r)\,u(r) \\ &\quad = \int_0^r s^{N-1}\Big[\Big(NW(s)+sW'(s)\Big)G(u) - \frac{N-p}{p}W(s)g(u)u\Big]ds \;= P(r,d). \end{aligned} \tag{52}$$

We recall that $r_0(d) \to 0$ as $d \to \infty$. Let $\delta > 0$ be as in (61). By further restricting T_1 we may assume that

$$N + s\frac{W'(s)}{W(s)} > \frac{N}{1+\delta}, \qquad \text{for } s \in [0, T_1]. \tag{53}$$

In this case, $W(s) > m/2 > 0$ and hence (52) is equivalent to

$$P(r,d) = \int_0^r s^{N-1}W(s)\Big[\Big(N + s\frac{W'(s)}{W(s)}\Big)G(u) - \frac{N-p}{p}g(u)u\Big]ds. \tag{54}$$

Lemma 3.4. *If $p < N$ and (ii) in (4) holds then $P(r_0,d) \to \infty$ as $d \to \infty$.*

Proof. Due to (53), $G(\cdot) \geq 0$, (41), (43) and $u(s) \geq d/2$, we have

$$\begin{aligned} &\Big(N + s\frac{W'(s)}{W(s)}\Big)G(u) - \frac{N-p}{p}g(u)u \\ &\qquad\qquad \geq \left(\frac{d}{2}\right)^{q_1+1}\left[\frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p}C_2\right] - \frac{CN}{1+\delta}. \end{aligned}$$

By (38), the expression inside brackets is positive. Hence, for $d \gg 1$,

$$\Big(N + s\frac{W'(s)}{W(s)}\Big)G(u) - \frac{N-p}{p}g(u)u > 0, \quad \text{for every } \; s \in [0, r_0].$$

Hence, by (46), for $d \gg 1$,

$$\begin{aligned}
P(r_0, d) &\geq \frac{m}{2N}\left(\frac{d}{2}\right)^{q_1+1}\left[\frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p}C_2\right] r_0^N - \frac{m}{2}\frac{C}{1+\delta}r_0^N \\
&\geq \frac{mC}{2^{q_1+1}N}\left[\frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p}C_2\right] d^{q_1+1+\frac{N}{p}(p-1-q_1)} \\
&\quad - \frac{m\tilde{C}_1}{2(1+\delta)} d^{\frac{N}{p}(p-1-q_1)} \\
&= d^{N[1-(q_1+1)/p^*]}\big(C - C_0 d^{-(q_1+1)}\big) \geq C d^{N[1-(q_1+1)/p^*]}.
\end{aligned}$$

This proves the lemma. □

Lemma 3.5. *If $p < N$ and (ii) in (4) holds then $P(r, d) \to \infty$ as $d \to \infty$ uniformly for $r \in [r_0, T_1]$.*

Proof. Note that

$$\begin{aligned}
P(r, d) &= P(r_0, d) + \int_{r_0}^{r} s^{N-1}W(s)\Big[\Big(N + s\frac{W'(s)}{W(s)}\Big)G(u) - \frac{N-p}{p}g(u)u\Big]ds \qquad (55)\\
&= P(r_0, d) + I^+ + I^-,
\end{aligned}$$

where

$$I^+ = \int_{\{u(s)\geq 0\}} s^{N-1}W(s)\Big[\Big(N + s\frac{W'(s)}{W(s)}\Big)G(u) - \frac{N-p}{p}g(u)u\Big]ds,$$

and

$$I^- = \int_{\{u(s)\leq 0\}} s^{N-1}W(s)\Big[\Big(N + s\frac{W'(s)}{W(s)}\Big)G(u) - \frac{N-p}{p}g(u)u\Big]ds.$$

Using (39), (41) and arguing as above, we get

$$\begin{aligned}
I^+ &\geq \int_{\{u(s)\geq 0\}} s^{N-1}W(s)\left[\frac{NC_1}{(1+\delta)(q_1+1)} - \frac{N-p}{p}C_2\right]u^{q_1+1}ds \qquad (56)\\
&\quad - \int_{\{u(s)\geq 0\}} s^{N-1}W(s)\left[\frac{NM}{(1+\delta)(q_1+1)} - \frac{N-p}{p}M\right]ds \\
&\geq - \int_{\{u(s)\geq 0\}} s^{N-1}W(s)\left[\frac{NM_3}{(1+\delta)(q_1+1)} - \frac{N-p}{p}M_1\right]ds \\
&\geq - \left|\frac{NM_3}{(1+\delta)(q_1+1)} - \frac{N-p}{p}M_1\right| \|W\|_\infty \int_0^1 s^{N-1}ds = -C.
\end{aligned}$$

In a similar way, using (40) and (42), we have $I^- \geq -C$. This, (55) and (56) imply $P(r, d) \to \infty$ as $d \to \infty$ for every $r \in [r_0, T_1]$. □

4. **Phase plane analysis.** Recall that, given $d > 0$, the problem (10) has a unique solution $u(r, d)$ defined for all $r \in [0, X]$.

Since $g(0) = 0$, $(u(r, d), u'(r, d)) \neq (0, 0)$ for all $r \in [0, X]$. Hence there exists a continuous function $\phi(r, d)$, for $r \in [0, X]$, such that $\phi(0, d) = 0$,

$$\begin{aligned}
u(r, d) &= \rho(r, d)\,\cos\phi(r, d), \\
u'(r, d) &= -\rho(r, d)\,\sin\phi(r, d),
\end{aligned} \qquad (57)$$

where $\rho(r, d) = \sqrt{\big(u(r, d)\big)^2 + \big(u'(r, d)\big)^2}$. Moreover, $\phi(\cdot, d)$ is differentiable at every $r \in [0, X]$ such that $u'(r) \neq 0$.

Differentiating the first equation in (57) with respect to r, for $u'(r) \neq 0$,

$$u'(r) = \rho'(r,d)\cos\big(\phi(r,d)\big) - \rho(r,d)\,\sin\big(\phi(r,d)\big)\cdot\phi'(r,d). \tag{58}$$

Let $T > 0$ and m be as in (44). We recall that our problem has a singularity at $r = 0$ (see (8)) and, if $u'(r) = 0$, $u''(r)$ may not exist since

$$u''(r) = -\frac{N-1}{(p-1)r}u'(r) - \frac{W(r)g\big(u(r)\big)}{(p-1)|u'(r)|^{p-2}}. \tag{59}$$

However, if $u'(r) \neq 0$ then $u''(r)$ is defined by (59). Combining (57) and the first equation in (10), we have

$$\phi'(r,d) = \frac{(u'(r,d))^2}{\rho^2(r,d)} + \frac{W(r)u(r)g(u(r))}{(p-1)\rho^2(r,d)|u'(r)|^{p-2}} + \frac{(N-1)u(r)u'(r)}{r(p-1)\rho^2(r,d)}, \tag{60}$$

for $r \in (0, X]$ with $u'(r) \neq 0$.

Remark 4. (i) By Lemmas 3.3, 3.4 and 3.5, $\mathcal{E}(r,d) \to +\infty$ as $d \to +\infty$ uniformly for $r \in [0, T_1]$, and therefore $\rho(r,d) \to +\infty$ as $d \to +\infty$ uniformly for $r \in [0, T_1]$.

(ii) From (58), if j is a non-negative integer and $\phi(r_1,d) = j\pi + \pi/2$ for some $r_1 \in (0, X]$ then $\phi'(r_1,d) = 1$. Hence $\phi(r,d) > j\pi + \pi/2$ for every $r \in [r_1, X]$ (see also [3, p. 756] and Corollary 1 below).

(iii) If u has no zero in $(0, T_1/2)$ then $u'(t) < 0$ for all $t \in (0, T_1/2]$, which implies $\sin(\phi(t)) \in (0,1)$, see (57). Hence $\phi(t) > 0$ for all $t \in (0, T_1/2]$. On the other hand, if u vanishes in $r_1 \in (0, T_1/2]$ then taking r_1 as the smallest zero of u we have $\phi(r_1,d) = \pi/2$. This and (ii) imply $\phi(T_1/2, d) > \pi/2$. Thus in any case $\phi(T_1/2, d) > 0$.

Let k be a positive integer. For $x_0 > 0$, let us define

$$\tilde{m}(x_0) = \min\left\{\frac{g(x)}{|x|^{p-2}x} : |x| \geq x_0\right\}.$$

Due to the p-superlinearity of g we have $\tilde{m}(x_0) \to +\infty$ as $x_0 \to +\infty$. For $\rho > 0$ and $\eta > 0$ we define $\omega(\rho,\eta) := \tilde{m}(\rho\sin(\eta))\sin^p(\eta)/(p-1)$. Now we choose $\rho_0 > 0$ and $\delta \in (0, \pi/4)$ such that

$$\begin{aligned} &(i) \quad 0 < \delta < \frac{(p-1)T_1}{32(N-1)}, \qquad (ii) \quad \omega(\rho_0,\delta) > \frac{4(N-1)}{m(p-1)T_1}, \\ &(iii) \quad \tilde{m}(\rho_0/2) \geq \frac{2^{(p/2)+5}k(p-1)}{m}, \qquad (iv) \quad 16\delta + \frac{8\pi}{m\omega(\rho_0,\delta)} \leq \frac{T_1}{2k}. \end{aligned} \tag{61}$$

Since $\lim_{d\to+\infty}\rho(r,d) = \infty$ uniformly for $r \in [0, T_1]$, there exists $d_0 > 0$ such that if $d > d_0$ then $\rho(r,d) \geq \rho_0$ for every $r \in [0, T_1]$.

Lemma 4.1. *If $T_1/2 \leq r \leq T_1$ and $\phi(r,d) \in [\frac{j\pi}{2} - \delta, \frac{j\pi}{2} + \delta]$ with $j > 0$ an odd integer, then $\phi'(r,d) > 1/4$.*

Proof. From (60),

$$\phi'(r,d) \geq \sin^2\phi + \frac{W(r)u(r)g(u)}{(p-1)\rho^2(r,d)|u'(r)|^{p-2}} - \frac{(N-1)|\cos\phi\sin\phi|}{r(p-1)}.$$

Taking into account that $|\sin(\phi(r,d))| \geq \cos\delta$ and $|\cos(\phi(r,d))| \leq \sin\delta \leq \delta$,

$$\phi'(r,d) \geq \cos^2\delta + \frac{W(r)u(r)g(u)}{(p-1)\rho^2(r,d)|u'(r)|^{p-2}} - \frac{2(N-1)\delta}{(p-1)T_1}.$$

Since $\delta < \min\{\pi/4, (p-1)T_1/(32(N-1))\}$, see (61)-(i),

$$\phi'(r,d) \geq \cos^2(\pi/4) + \frac{W(r)u(r)g(u)}{(p-1)\rho^2(r,d)|u'(r)|^{p-2}} - \frac{1}{16} \geq \frac{7}{16} > \frac{1}{4}. \tag{62}$$

Thus, the lemma is proved. □

Lemma 4.2. *If $T_1/2 \leq r \leq T_1$ and $\phi(r,d) \in [\frac{j\pi}{2} + \delta, \frac{(j+1)\pi}{2} - \delta]$ with $j > 0$ an integer, then $\phi'(r,d) > m\omega(\rho_0,\delta)/4$.*

Proof. We carry out the details of the proof for $p \geq 2$. The case $1 < p < 2$ follows similarly. From (60),

$$\begin{aligned}\phi'(r,d) &\geq \frac{W(r)u(r)g(u(r))}{(p-1)\rho^2(r,d)|u'(r)|^{p-2}} - \frac{(N-1)}{2r(p-1)} \\ &\geq \frac{W(r)}{p-1}\frac{g(u(r))}{|u|^{p-2}u(r,d)}\frac{|u(r,d)|^p}{\rho^2(r,d)|u'|^{p-2}} - \frac{N-1}{(p-1)T_1}.\end{aligned}$$

Due to $|\cos\phi(r,d)| \geq \sin\delta$ and $\omega(\rho_0,\delta) > \frac{4(N-1)}{m(p-1)T_1}$, see (61)-(ii), it follows that

$$\begin{aligned}\phi'(r,d) &> \frac{W(r)}{p-1}\frac{g(u(r))}{|u|^{p-2}u(r,d)}\frac{|\cos\phi(r,d)|^p}{|\sin\phi(r,d)|^{p-2}} - \frac{m\omega(\rho_0,\delta)}{4} \\ &\geq \frac{W(r)}{p-1}\frac{g(u(r))}{|u|^{p-2}u(r,d)}\sin^p\delta - \frac{m\omega(\rho_0,\delta)}{4}.\end{aligned}$$

Since $|u| = \rho|\cos\phi| \geq \rho_0\sin\delta$, $g(u)/(|u|^{p-2}u) \geq \tilde{m}(\rho_0\sin\delta)$. This and the definition of $\omega(\rho_0,\delta)$ yield

$$\phi'(r,d) > W(r)\omega(\rho_0,\delta) - \frac{m\omega(\rho_0,\delta)}{4} \geq \frac{m\omega(\rho_0,\delta)}{4}. \tag{63}$$

In the latter inequality we have used $W(r) \geq m/2$ for any $r \in [0,T_1]$. Thus, (63) proves the lemma. □

Lemma 4.3. *If $T_1/2 \leq r \leq T_1$ and $\phi(r,d) \in [j\pi-\delta, j\pi) \cup (j\pi, j\pi+\delta]$ for some positive integer j, then*

$$\phi'(r,d) \geq 8k|\sin(\phi(r,d))|^{2-p}. \tag{64}$$

Proof. From $\delta < \pi/4$, (57), and $|\cos\phi(r,d)| \geq \cos\delta$, it follows

$$u^2(r) = \rho^2(r,d)(1-\sin^2(\delta)) \geq \rho^2(r,d)/2.$$

This, (61)-(iii), and (60) imply

$$\begin{aligned}\phi'(r,d) &\geq \frac{W(r)u(r)g(u(r))}{(p-1)\rho^p(r,d)|\sin(\phi(r,d))|^{p-2}} - \frac{(N-1)|\sin(\phi(r,d))|}{r(p-1)} \\ &\geq \frac{W(r)u(r)g(u(r))|\sin(\phi(r,d))|^{2-p}}{2^{p/2}(p-1)|u(r)|^p} - \frac{2(N-1)|\sin(\phi(r,d))|}{T_1(p-1)} \\ &\geq \left(\frac{W(r)u(r)g(u(r))}{2^{p/2}(p-1)|u(r)|^p} - \frac{1}{16}|\sin(\phi(r,d))|^{p-1}\right)|\sin(\phi(r,d))|^{2-p} \\ &\geq \frac{m\tilde{m}(\rho_0/2)}{2^{(p/2)+2}(p-1)}|\sin(\phi(r,d))|^{2-p} \\ &\geq 8k|\sin(\phi(r,d))|^{2-p},\end{aligned} \tag{65}$$

which completes the proof of the lemma. □

Corollary 1. *Let j be non-negative integer. If $\hat{r} \in [T_1/2, T_1]$ and $\phi(\hat{r}, d) = j\pi/2$ then $\phi(r, d) > j\pi/2$ for all $r \in (\hat{r}, T_1]$.*

Proof. For j odd, see Remark 4. The case j even follows from Lemma 4.3. □

Proposition 1. *For any $p > 1$, $\lim_{d\to+\infty} \phi(T_1, d) = +\infty$.*

Proof. Let $d > d_0$ and k as in Lemmas 4.1, 4.2 and 4.3. Hence $\phi(\cdot, d)$ increases in $[T_1/2, T_1]$. Let $r_0 \in [T_1/2, T_1]$. Since $\phi(r_0, d) > 0$, there exists a non-negative integer j such that either

$$\begin{aligned} &\phi(r_0, d) \in [j\pi/2, j\pi/2 + \delta],\ \phi(r_0, d) \in [j\pi/2 + \delta, (j + 1)\pi/2 - \delta], \quad \text{or} \\ &\phi(r_0, d) \in [(j + 1)\pi/2 - \delta, (j + 1)\pi/2]. \end{aligned} \tag{66}$$

Suppose j is odd. If $\phi(r_0, d) \in [j\pi/2, j\pi/2 + \delta]$ then by Lemma 4.1 and (61) there exists $r_1 \in (r_0, r_0 + 4\delta] \subset (r_0, r_0 + T_1/(8k)]$ such that $\phi(r_1, d) = j\pi/2 + \delta$.

By Lemma 4.2 and (61) there is $r_2 \in (r_1, r_1 + 2\pi/(m\omega(\rho_0, \delta))] \subset [r_1, r_1 + T_1/8k]$ such that $\phi(r_2, d) = (j + 1)\pi/2 - \delta$.

By Lemma 4.3, if $p \geq 2$, there exists $r_3 \in [r_2, r_2 + \delta/(8k)]$ such that $\phi(r_3, d) = (j + 1)\pi/2$. On the other hand, if $p \leq 2$, from Lemma 4.3 for $r \geq r_2$ and $\phi(r, d) \leq (j + 1)\pi/2$ we have $\phi'(r, d)\phi^{p-2}(r, d) \geq 8k$. Integration on $[r_2, r]$ and (61) give

$$\begin{aligned} 8k(p - 1)(r - r_2) &\leq \left(\frac{(j + 1)\pi}{2}\right)^{p-1} - \left(\frac{(j + 1)\pi}{2} - \delta\right)^{p-1} \\ &\leq 2\left(\frac{2}{(j + 1)\pi}\right)^{2-p} \delta. \end{aligned} \tag{67}$$

Therefore

$$\begin{aligned} r - r_2 &\leq 2\left(\frac{2}{(j + 1)\pi}\right)^{2-p} \frac{\delta}{8k(p - 1)} \\ &< \frac{T_1}{8k}. \end{aligned} \tag{68}$$

Hence there exists $r_3 \in [r_2, r_2 + T_1/(8k)]$ such that

$$r_3 \in [r_0, r_0 + 3T_1/(8k)] \subset [r_0, r_0 + T_1/(2k)] \quad \text{and} \quad \phi(r_3, d) = (j + 1)\pi/2. \tag{69}$$

If $\phi(r_0, d) \in [j\pi/2 + \delta, (j + 1)\pi/2 - \delta]$, then placing r_0 in the role of r_1 we see that there exists $r_3 \in [(j + 1)\pi/2 - \delta, (j + 1)\pi/2]$ that satisfies (69). Similarly if $\phi(r_0, d) \in [(j + 1)\pi/2 - \delta, (j + 1)\pi/2]$, placing r_0 in the role of r_2 above we find r_3 satisfying (69).

If j in (66) is an even positive integer and $\phi(r_0, d) \in [j\pi/2, j\pi/2 + \delta]$ applying Lemma 4.3 we see that there is $r_1 \in [r_0, r_0 + T_1/(8k)]$ such that $\phi(r_1, d) = j\pi/2 + \delta$. Then applying Lemma 4.2 it follows that there exists $r_2 \in [r_1, r_1 + 2\pi/(m\omega(\rho_0, \delta))]$ such that $\phi(r_2, d) = (j + 1)\pi/2 - \delta$. Finally, applying Lemma 4.1 there exists $r_3 \in [r_2, r_2 + T_1/(8k)]$ that satisfies (69). That is (69) is satisfied for both j even and j odd. Thus $\phi(r, d) - \phi(t, d) \geq \pi/2$ if $r - t \geq T_1/(2k)$, which implies

$$\phi(T_1, d) \geq \phi(T_1/2) + \frac{k\pi}{2} > \frac{k\pi}{2}. \tag{70}$$

This proves the proposition. □

By Proposition 1 given any positive integer k there exists d_k such that if $d \geq d_k$ then $\phi(T_1, d) > k\pi/2$. Since $\phi(r, d)$ is a continuous function, by the intermediate value theorem there exists $\hat{r} \in (0, T_1)$ so that $\phi(\hat{r}, d) = k\pi/2$. By part (ii) of Remark 4,

$$\phi(X, d) \geq \phi(\hat{r}) \geq k\pi/2.$$

Thus, we have proved:

Proposition 2. $\lim_{d\to+\infty} \phi(X, d) = +\infty$.

Now we are ready to prove Theorem 1.1.

5. Proof of Theorem 1.1.

Let $u(r, d)$ be the solution to problem

$$\begin{cases} \left(r^{N-1}\big|u'(r)\big|^{p-2} u'(r)\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, \quad 0 < r \leq X, \\ \qquad u'(0) = 0, \quad u(0) = d. \end{cases} \tag{71}$$

Let us define $a := u(X, d)$ and let $v(r, d)$ be the solution to problem

$$\begin{cases} \left(r^{N-1}\big|v'\big|^{p-2} v'\right)' + r^{N-1}W(r)g\big(v(r)\big) = 0, \qquad X < r < 1, \\ \qquad v(X) = a, \quad v'(X) = \hat{\zeta}, \end{cases} \tag{72}$$

where $\hat{\zeta} := \hat{\zeta}(a)$ is given by Theorem 2.7. Note that $v(1, d) = 0$. By Proposition 2 and the continuous dependence of $\phi(X, d)$ on d, there exists K such that if $k \geq K$ then there exist positive real numbers d_k and $\hat{d}_k$ such that

$$d_k < \hat{d}_k, \quad \phi(X, d_k) = k\pi, \quad \text{and} \quad \phi(X, \hat{d}_k) = k\pi + \pi/2. \tag{73}$$

Without loss of generality we may assume k to be even. The case k odd follows similarly. Since k is even, $u(X, d_k) > 0$ and $u'(X, d_k) = 0$. Therefore $\hat{\zeta}(u(X, d_k)) < 0 = u'(X, d_k)$. Also, $u'(X, \hat{d}_k) < 0$ and $u(X, \hat{d}_k) = 0$. Hence,

$$\hat{\zeta}(u(X, \hat{d}_k)) = 0 > u'(X, \hat{d}_k).$$

Thus, by the intermediate value theorem there exists $\bar{d}_k \in (d_k, \hat{d}_k)$ such that $u'(X, \bar{d}_k) = \hat{\zeta}(u(X, \bar{d}_k))$. Let $U_k(r)$ be the function defined by

$$U_k(r) = \begin{cases} u(r, \bar{d}_k) & r \in [0, X], \\ v(r, \bar{d}_k) & r \in [X, 1], \end{cases}$$

where v is given by (72). Since $u(X, d) = v(X, d)$, $u'(X, d) = v'(X, d)$, and $v(1, \bar{d}_k) = 0$, U_k is a radial solution to (1). Thus, the sequence $\{U_k(r)\}_k$ gives us infinitely many radially symmetric solutions to problem (1), which concludes the proof of Theorem 1.1. ☑

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