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Infinitely Many Stability Switches in a Problem with Sublinear Oscillatory Boundary Conditions

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Abstract We consider the elliptic equation $-\Delta u + u = 0$ with nonlinear boundary condition $\frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u)$, where $\frac{g(\lambda,x,s)}{s} \to 0$, as $|s| \to \infty$ and g is oscillatory. We provide sufficient conditions on g for the existence of unbounded sequences of stable solutions, unstable solutions, and turning points, even in the absence of resonant solutions.

Keywords Resonance · Stability · Instability · Multiplicity · Bifurcation from infinity · Sublinear oscillating boundary conditions · Turning points

Mathematics Subject Classification 35B32 · 35B34 · 35B35 · 58J55 · 35J25 · 35J60 · 35J65

1 Introduction

In this paper we consider solutions to the elliptic problem with nonlinear boundary conditions

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$. Roughly speaking, we assume that the nonlinearity at the boundary satisfies $\lim_{|s|\to\infty} \frac{g(\lambda,x,s)}{s} = 0$, and g is oscillatory. Our goal is to give conditions on the sublinear oscillatory term g that guarantee the existence of unbounded sequences of stable solutions, unstable solutions and turning

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points, even in the absence of resonant solutions. For $N = 1$ the problem (1.1) becomes a 2×2 system; in Sect. 4 we work out this case in detail.

Let $\{\sigma_i\}_{i=1}^{\infty}$ denote the sequence of *Steklov* eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta\Phi + \Phi = 0, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} = \sigma\Phi, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The Steklov eigenvalues form an increasing sequence of real numbers, $\{\sigma_i\}_{i=1}^{\infty}$. Each eigenvalue has finite multiplicity. The first eigenvalue σ_1 is simple and, due to Hopf's Lemma, we may assume its eigenfunction Φ_1 to be strictly positive in $\overline{\Omega}$. The eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(\partial\Omega)$ and we take $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$, see [4], [5, Chap. 3].

Throughout this paper we assume:

(H1) $g : \mathbb{R} \times \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. $g = g(\lambda, x, s)$ is measurable in $x \in \partial\Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$), and g is twice differentiable with respect to s. Moreover, there exist $r > N - 1$, $G_1 \in L^r(\partial\Omega)$ and continuous functions $\Lambda : \mathbb{R} \to \mathbb{R}^+$, and $U : \mathbb{R} \to \mathbb{R}^+$, satisfying

$$\begin{cases} \|g(\lambda, x, s)| \le \Lambda(\lambda) G_1(x) U(s), & \forall (\lambda, x, s) \in \mathbb{R} \times \partial\Omega \times \mathbb{R}, \\ \limsup\limits_{|s|\to\infty} \dfrac{U(s)}{|s|^\alpha} < +\infty & \text{for some} \quad \alpha < 1. \end{cases}$$

(H2) The partial derivative $g_s(\lambda, \cdot, \cdot) \in C(\partial\Omega \times \mathbb{R})$, where $g_s := \frac{\partial g}{\partial s}$, and there exist $\rho < 1$ and $F_1 \in L^r(\partial\Omega)$, such that

$$\frac{|g(\lambda, x, s) - s g_s(\lambda, x, s)|}{|s|^\rho} \le F_1(x), \quad \text{as } \lambda \to \sigma_1 \tag{1.3}$$

for $x \in \partial\Omega$ and $s \gg 1$ sufficiently large.

(H3) The second partial derivative $g_{ss}(\lambda, \cdot, \cdot) \in C(\partial\Omega \times \mathbb{R})$ is such that

$$\sup_{|s|\ge M} \left\| \frac{g_{ss}(\lambda, \cdot, s)}{|s|^{\rho-\alpha-1}} \right\|_{L^\infty(\partial\Omega)} \to 0 \quad \text{as } M \to \infty \quad \text{and } \lambda \to \sigma_1. \tag{1.4}$$

Observe that the exponents α, ρ may be negative, since we are interested at the behavior of g as $s \to \infty$.

As stated in [1, Theorem 3.4], due to (H1) there exists a connected set of positive weak solutions of (1.1). We denote it by $\mathcal{D}^+ \subset \mathbb{R} \times C(\bar{\Omega})$, and recall that for $(\lambda, u_\lambda) \in \mathcal{D}^+$

$$u_\lambda = s\Phi_1 + w_\lambda, \quad \text{with } w_\lambda = o(|s|) \quad \text{and} \quad |\sigma_1 - \lambda| = o(1) \quad \text{as } |s| \to \infty. \tag{1.5}$$

The set $\mathcal{D}^+$ is known as a *branch bifurcating from infinity* in the sense of Rabinowitz, see [1,13]. By a bootstrap argument, it can be proved that $u \in C^\nu(\overline{\Omega})$ for some $\nu > 0$, see [1, Proposition 2.3].

In Theorem 2, we provide sufficient conditions on g for the existence of unbounded sequences of *stable* solutions, *unstable* solutions, and *turning points*.

For $(\lambda, u_\lambda) \in \mathcal{D}^+$ we say that u_λ is a *stable* solution if there exists a neighborhood of u_λ in $C(\bar{\Omega})$ such that for initial data u_0 in that neighborhood the solution to the parabolic problem

$$\begin{cases} u_t - \Delta u + u = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0, x) = u_0(x), & \text{in } \Omega. \end{cases} \tag{1.6}$$

converges to u_λ as $t \to +\infty$. On the other hand we say that u_λ is *unstable* if any neighborhood of u_λ contains initial data u_0 for which the solution to (1.6) leaves that neighborhood in finite time. That is, asymptotic stability in the Lyapunov sense.

Definition 1 A solution (λ^*, u^*) of (1.1) in the branch of solutions $\mathcal{D}^+ \subset \mathbb{R} \times C(\bar{\Omega})$ is called a **turning point** if there is a neighborhood W of (λ^*, u^*) in $\mathbb{R} \times C(\bar{\Omega})$ such that, either $W \cap \mathcal{D}^+ \subset [\lambda^*, \infty) \times C(\bar{\Omega})$ or $W \cap \mathcal{D}^+ \subset (-\infty, \lambda^*] \times C(\bar{\Omega})$.

Our main result is the following Theorem.

Theorem 2 *Assume the nonlinearity g satisfies hypotheses* (H1)*,* (H2) *and* (H3)*. Assume also that*

$$\lim_{\substack{\lambda\to\sigma_1\\ s\to+\infty}} \int_{\partial\Omega} \left| \frac{g(\lambda, x, s\Phi_1) - s\Phi_1 g_s(\lambda, x, s\Phi_1)}{|s|^\rho} - \frac{g(\sigma_1, x, s\Phi_1) - s\Phi_1 g_s(\sigma_1, x, s\Phi_1)}{|s|^\rho} \right| \Phi_1 = 0. \tag{1.7}$$

Let $F : \mathbb{R} \times C(\bar{\Omega}) \to \mathbb{R}$ be defined by

$$F(\lambda, u) := \int_{\partial\Omega} \frac{ug(\lambda, \cdot, u) - u^2 g_s(\lambda, \cdot, u)}{|u|^{1+\rho}} \Phi_1^{1+\rho}. \tag{1.8}$$

If there exist sequences $\{s_n\}$, $\{s_n'\}$ converging to $+\infty$, such that

$$\lim_{n\to+\infty} F(\sigma_1, s_n'\Phi_1) < 0 < \lim_{n\to+\infty} F(\sigma_1, s_n\Phi_1), \tag{1.9}$$

then

(i) *There exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{D}^+$ of stable solutions to* (1.1) *and a sequence $\{(\lambda_n', u_n')\} \subset \mathcal{D}^+$ of unstable solutions such that $(\lambda_n, \|u_n\|_{L^\infty(\partial\Omega)}) \to (\sigma_1, \infty)$ and $(\lambda_n', \|u_n'\|_{L^\infty(\partial\Omega)}) \to (\sigma_1, \infty)$ as $n \to \infty$.*

(ii) *There exist a sequence $\{(\lambda_n^*, u_n^*)\} \subset \mathcal{D}^+$ of turning points such that $(\lambda_n^*, \|u_n^*\|_{L^\infty(\partial\Omega)}) \to (\sigma_1, \infty)$ as $n \to \infty$.*

Our main result, Theorem 2 above, is exemplified by

$$g(x, s) := s^\alpha \left[\sin\left(\left| \frac{s}{\Phi_1(x)} \right|^\beta \right) + C \right] \quad \text{with } \alpha < 1,\ \beta > 0,\ C \in \mathbb{R}, \quad \text{for } s \gg 1. \tag{1.10}$$

In fact we have:

Corollary 3 *Assume that g is given by* (1.10)*. If*

$$\beta > 0 \quad \textit{and} \quad \alpha + \beta < 1,$$

then, $\forall C \in \mathbb{R}$, the unbounded branch of positive solutions of (1.1) *contains a sequence of stable solutions, a sequence of unstable solutions and a sequence of turning points.*

The proof of this Corollary follows directly from Theorem 2.

In Figs. 1 and 2 we plot the bifurcation diagram for g as above. Figure 3 sketches the changes of stability of solutions.

In addition to the example provided in (1.10), a wide class of examples of nonlinearities satisfying the hypotheses of Theorem 2 may be obtained as follows. Without loss of generality we may assume that $\frac{1}{2} \leq \Phi_1(x) \leq 1$ for all $x \in \partial\Omega$. Let $\{t_j\}$ be a sequence of positive numbers with $t_{j+1} \geq 4t_j$. Let $h : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be an oscillatory differentiable function satisfying the following hypotheses:

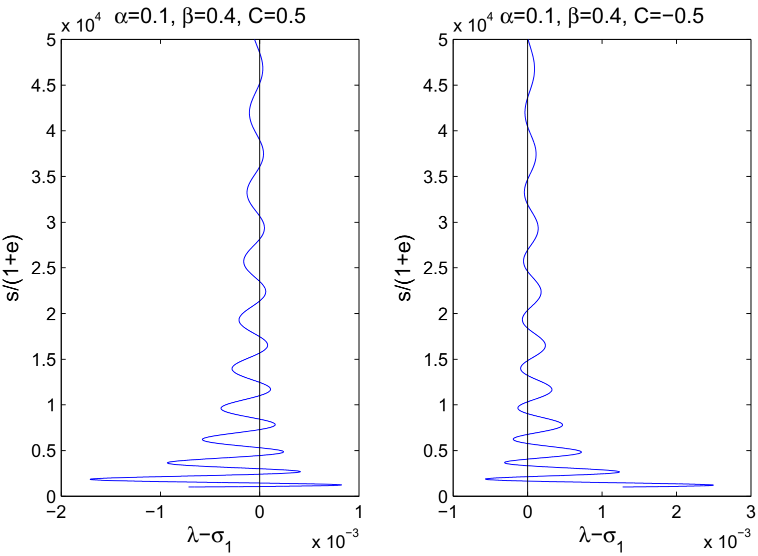

Fig. 1 Two bifurcation diagrams having infinitely many sub-critical solutions ($\lambda < \sigma_1$), super-critical solutions ($\lambda > \sigma_1$), stable solutions, unstable solutions, turning points and resonant solutions ($\lambda = \sigma_1$)

(A1) $\forall \lambda \in \mathbb{R}$,

$$\int_0^{t_1} h(\lambda, t)\, dt = 0, \quad \int_{t_j}^{t_{j+1}} h(\lambda, t)\, dt = 0, \quad \forall j \geq 1, \quad \left| \int_0^s h(\lambda, t)\, dt \right| \leq \frac{1}{s}, \quad \forall s \gg 1.$$

(A2)

$$|h(\lambda, s)| \leq \frac{1}{s^2}, \quad \forall s \gg 1, \ \lambda \in \mathbb{R}.$$

(A3)

$$|h_s(\lambda, s)| \leq \frac{1}{s^3}, \quad \forall s \gg 1, \ \lambda \in \mathbb{R}.$$

(A4) Let $h(\lambda, s) = h(\lambda, t_{2n}) = \frac{1}{t_{2n}^2} > 0$ for all $s \in [t_{2n}, 2t_{2n}]$, $n \geq 1$, and $\lambda \in \mathbb{R}$.

(A5) Let $h(\lambda, s) = h(\lambda, t_{2n+1}) = -\frac{1}{t_{2n+1}^2} < 0$ for all $s \in \left[t_{2n+1}, 2t_{2n+1}\right]$, $n \geq 1$, and $\lambda \in \mathbb{R}$.

Let

$$g(\lambda, s) = s \int_0^s h(\lambda, t)\, dt. \tag{1.11}$$

It is readily seen that (A1)–(A3) imply that

$$\frac{|g(\lambda, s)|}{|s|^\alpha} = \left| s^{1-\alpha} \int_0^s h(\lambda, t)\, dt \right| \leq s^{-\alpha}, \quad \text{for } s \gg 1, \ \lambda \in \mathbb{R},$$
$$\frac{|sg(\lambda, s) - s^2 g_s(\lambda, s)|}{|s|^{1+\rho}} = \left| s^{2-\rho} h(\lambda, s) \right| \leq s^{-\rho}, \quad \text{for } s \gg 1, \ \lambda \in \mathbb{R},$$

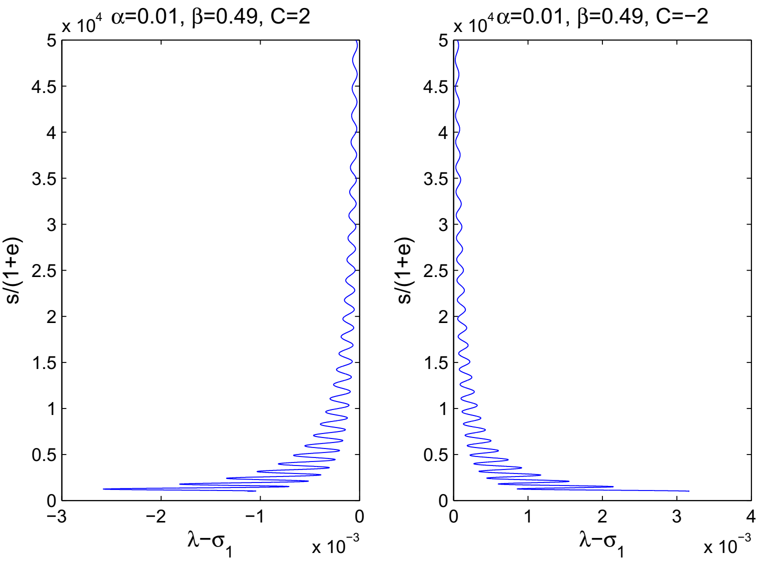

Fig. 2 Two bifurcation diagrams of stable and unstable solutions, on the *left* all of them are subcritical, on the *right* all of them are supercritical, and none is resonant

and

$$\limsup_{s\to\infty} \frac{|g_{ss}(\lambda, s)|}{|s|^{\rho-\alpha-1}} \le \limsup_{s\to\infty} \left(2|s|^{\alpha-\rho+1}|h| + |s|^{\alpha-\rho+2}|h_s|\right) \le \limsup_{s\to\infty} 3|s|^{\alpha-\rho-1} = 0,$$

uniformly for $\lambda \in \mathbb{R}$. Therefore (H1)–(H3) hold with $\rho \in (0, 1)$, and any $\alpha \in [0, 1)$ satisfying $\alpha < 1 + \rho$.

From definition (1.8),

$$F(\lambda, s\Phi_1) := -s^{2-\rho} \int_{\partial\Omega} h(\lambda, s\Phi_1)\, \Phi_1^3 = -s^2 \int_{\partial\Omega} h(\lambda, s\Phi_1)\, \Phi_1^3. \tag{1.12}$$

Finally choosing $s_n = 2t_{2n+1}$ and $s_n' = 2t_{2n}$, (A4)–(A5) imply that

$$F(\lambda, s_n\Phi_1) := (2t_{2n+1})^2 \int_{\partial\Omega} \frac{1}{t_{2n+1}^2}\, \Phi_1^3 = 4 \int_{\partial\Omega} \Phi_1^3, \tag{1.13}$$

and

$$F(\lambda, s_n'\Phi_1) := -(2t_{2n})^2 \int_{\partial\Omega} \frac{1}{t_{2n}^2}\, \Phi_1^3 = -4 \int_{\partial\Omega} \Phi_1^3, \tag{1.14}$$

therefore (1.9) holds, and consequently all the conditions in Theorem 2 hold.

Our result is sharp in that if condition (1.9) fails, all solutions in $\mathcal{D}^+$ may be either stable or unstable for s big enough, see [2, Theorem 3.4]. **Our result proves the existence of infinitely many turning points, even in the absence of resonant solutions, (i.e. solutions for $\boldsymbol{\lambda = \sigma_1}$)**, see Fig. 2. There it can be seen that the unbounded sequence of turning points given by Theorem 2 can be either *subcritical* (i.e. for values of the parameter $\lambda < \sigma_1$), see Fig. 2 left, or *supercritical* (i.e. for $\lambda > \sigma_1$), see Fig. 2 right, or may have a sequence of subcritical solutions as well as a sequence of supercritical solutions, see Fig. 1. Let us mention that, in this last case, by connectedness of $\mathcal{D}^+$, the branch contains infinitely many *resonant* solutions, see Fig. 1.

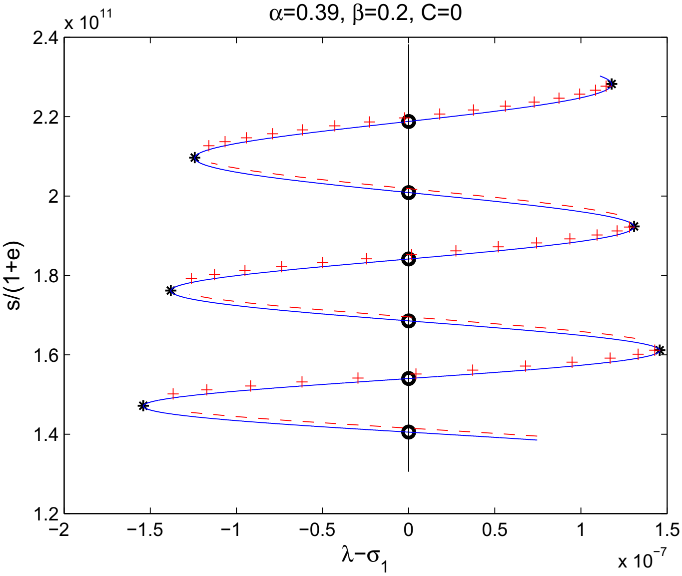

Fig. 3 Bifurcation diagram and sketch of the stability of solutions, $+$ for stable solutions and $-$ for unstable solutions. The *symbol* $*$ marks turning points and $\circ$ resonant solutions

The main difference with [3] is the possibility of existence of a branch of exclusively subcritical (or exclusively supercritical) solutions, see Figs. 1 versus 2, specifically the resonant solutions only appear in Fig. 1. Precisely the main ingredient for the proof of the existence of infinitely many turning points in [3] was the existence of infinitely many subcritical and supercritical solutions in a connected branch and consequently of infinitely many resonant solutions.

Related results for the case of a nonlinear reaction in Ω and homogeneous Dirichlet boundary conditions were established in [6–8,12]. In [8] the authors work in the unit ball $B \subset \mathbb{R}^N$, with a nonlinear term given by $\lambda u + \sin(u)$. They proved that when $\lambda = \lambda_1$, the first eigenvalue with Dirichlet boundary conditions, the problem has infinitely many solutions for $1 \leq N \leq 5$ and at most finitely many solutions for $N \geq 6$. This case is a limit case $\alpha = 0$, $\beta = 1$, not covered in this work. To the best of our knowledge, the role of the dimension has not been observed, in the case of nonlinear boundary conditions. Similar oscillatory phenomena, sometimes known as *snaking bifurcation*, can be observed in higher-order PDE, see [11,14]. We refer the reader to [9,10] for related problems with nonlinear boundary conditions.

This paper is organized as follows. In Sect. 2 we collect some essentially known results on Lyapunov stability. Section 3 contains the proof of our main result Theorem 2, giving sufficient conditions for having stable and unstable solutions. Finally Sect. 4 presents the one dimensional case.

2 Lyapunov Function and Stability

For λ fixed we consider

$$I(u)=\frac{1}{2}\int_{\Omega}\left(|\nabla u|^2+u^2\right)-\frac{\lambda}{2}\int_{\partial\Omega}u^2-\int_{\partial\Omega}G(\lambda,\cdot,u)$$

where $G(\lambda,x,s):=\int_0^s g(\lambda,x,t)\,dt$. An elementary calculation shows that if u is a solution to the parabolic equation (1.6) then $\frac{d}{dt}I(u(t))=-\int_{\Omega}u_t^2\,dx\leq 0$, i.e., I is a Lyapunov function for the parabolic problem (1.6).

Moreover, if u_λ is a solution to (1.1), then it is a critical point for I. Furthermore, u_λ is stable if the quadratic form

$$Q_{u_\lambda}(v)=\int_{\Omega}\left(|\nabla v|^2+v^2\right)-\int_{\partial\Omega}\left(\lambda v^2+g_s(\lambda,\cdot,u_\lambda)v^2\right) \tag{2.1}$$

is positive definite. On the other hand if Q_{u_λ} is negative definite in one direction then u_λ is unstable. Thus we have:

Lemma 4 *If $\mu_1\equiv\mu_1(\lambda,u_\lambda)$ denotes the principal eigenvalue of the linearized equation*

$$\begin{cases}-\Delta\varphi_1+\varphi_1=0, & \text{in } \Omega\\ \frac{\partial\varphi_1}{\partial n}=\mu_1\varphi_1+g_s(\lambda,x,u_\lambda)\varphi_1, & \text{on } \partial\Omega\end{cases} \tag{2.2}$$

then u_λ is stable, if $\mu_1>\lambda$. Also u_λ is unstable if $\mu_1<\lambda$.

Proof Suppose $\mu_1>\lambda$. The variational characterization of μ_1 states that

$$\mu_1:=\inf_{u\in H^1(\Omega)}\frac{\int_{\Omega}\left(|\nabla u|^2+u^2\right)-\int_{\partial\Omega}g_s(\lambda,\cdot,u_\lambda)u^2}{\int_{\partial\Omega}u^2}. \tag{2.3}$$

Therefore, for any $u\in H^1(\Omega)-\{0\}$, we have

$$\begin{aligned}0&\leq\int_{\Omega}\left(|\nabla u|^2+u^2\right)-\int_{\partial\Omega}\left(\mu_1u^2+g_s(\lambda,\cdot,u_\lambda)u^2\right)\\&<\int_{\Omega}\left(|\nabla u|^2+u^2\right)-\int_{\partial\Omega}\left(\lambda u^2+g_s(\lambda,\cdot,u_\lambda)u^2\right).\end{aligned} \tag{2.4}$$

Hence Q_{u_λ} is positive definite and u_λ is stable.

On the other hand, if $\mu_1<\lambda$, letting φ_1 denote the eigenfunction corresponding to the eigenvalue μ_1, one obtains

$$\begin{aligned}0&=\int_{\Omega}\left(|\nabla\varphi_1|^2+\varphi_1^2\right)-\int_{\partial\Omega}\left(\mu_1\varphi_1^2+g_s(\lambda,\cdot,u_\lambda)\varphi_1^2\right)\\&>\int_{\Omega}\left(|\nabla\varphi_1|^2+\varphi_1^2\right)-\int_{\partial\Omega}\left(\lambda\varphi_1^2+g_s(\lambda,\cdot,u_\lambda)\varphi_1^2\right).\end{aligned} \tag{2.5}$$

Thus Q_{u_λ} is negative definite in the direction of φ_1, which proves that u_λ is unstable. □

3 Auxiliary Lemmas and Proof of Our Main Result

Let $\mu_1\equiv\mu_1(\lambda,u_\lambda)$ denote the principal eigenvalue of (2.2), and $\varphi_1\equiv\varphi_1(\lambda,u_\lambda)$ denote the corresponding eigenfunction normalized it the $L^\infty(\partial\Omega)$ norm. Let us call them the *boundary*

Steklov eigenvalue and the *boundary Steklov eigenfunction* respectively. In the first place we note that even if $\alpha \neq \rho$, (where ρ has been defined in (H.2)), then the boundary Steklov eigenvalue $\mu_1 \to \sigma_1$ and the boundary Steklov eigenfunction $\varphi_1 \to \Phi_1$ as $\lambda \to \sigma_1$ and $\|u\|_{L^\infty(\partial\Omega)} \to \infty$. The following Lemma is a rewriting of Lemma 3.2 in [2]. Its proof is the same. The only restriction is that $\rho < 1$. We have the following result.

Lemma 5 *Assume* g *satisfies hypotheses* (H1) *and* (H2)*, and assume that* (λ_n, u_n) *is a sequence of solutions of* (1.1)*, satisfying* $\lim_{n\to\infty}\lambda_n = \sigma_1$ *and* $\lim_{n\to\infty}\|u_n\|_{L^\infty(\partial\Omega)} = \infty$.

Let us denote by $\mu_{1,n} = \mu_1(\lambda_n, u_n)$*,* $\varphi_{1,n} = \varphi_1(\lambda_n, u_n)$*, the first eigenvalue and eigenfunction defined by* (2.2)*. Then,* $\mu_{1,n}$*,* $\varphi_{1,n}$*, satisfy*

$$\begin{aligned}
&\mu_{1,n} \to \sigma_1 \quad \textit{as} \quad \lambda_n \to \sigma_1 \quad \textit{and} \quad \|u_n\|_{L^\infty(\partial\Omega)} \to \infty,\\
&\varphi_{1,n} \to \Phi_1 \quad \textit{in } H^1(\Omega)\cap C^\nu(\overline{\Omega}) \quad \textit{as} \quad \lambda_n \to \sigma_1 \quad \textit{and} \quad \|u_n\|_{L^\infty(\partial\Omega)} \to \infty,
\end{aligned} \tag{3.1}$$

for some $\nu \in (0, 1)$.

Remark 6 To adapt the proof of the first part of Lemma 3.2 in [2], notice that (H1) and (H2) imply

$$\frac{|g_s(\lambda, x, s)|}{|s|^{\gamma-1}} \le |s|^{\rho-\gamma}F_1(x) + C|s|^{\alpha-\gamma}G_1(x), \quad \text{as } \lambda \to \sigma_1, \quad \text{for } s \gg 1$$

for some positive constant C, and where $\gamma = \max\{\rho, \alpha\} < 1$. Hence $\frac{|g_s(\lambda,x,s)|}{|s|^{\gamma-1}} \le D_1(x)$ with $D_1 \in L^r(\partial\Omega)$ (where $r > N-1$), for s big enough, $x \in \partial\Omega$ and $\lambda \to \sigma_1$.

In order to analyze the changes in stability, we consider

$$\underline{\mathbf{F}}_+ := \int_{\partial\Omega} \liminf_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{sg(\lambda,\cdot,s) - s^2 g_s(\lambda,\cdot,s)}{|s|^{1+\rho}}\,\Phi_1^{1+\rho},$$

where $\rho < 1$. Replacing lim inf by lim sup we define the number $\overline{\mathbf{F}}_+$. Assume $\alpha = \rho$, if

$$\underline{\mathbf{F}}_+ > 0, \quad \text{then any solution in } \mathcal{D}^+ \text{ is stable and subcritical,}$$

see [2, Theorem 3.4], and if

$$\overline{\mathbf{F}}_+ < 0, \quad \text{then any solution in } \mathcal{D}^+ \text{ is unstable and supercritical,}$$

see [2, Theorem 3.5]. In this paper we consider nonlinearities for which

$$\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+,$$

Unlike the case $\alpha = \rho$, our assumption $\alpha \neq \rho$ allows for the existence of sequences of stable supercritical solutions and unstable subcritical solutions, which contravenes the above situations, see Theorem 2.

In order to determine whether a sequence of solutions (λ_n, u_n) is stable or unstable, we use Lemma 4. Let $\mu_{1,n} = \mu_1(\lambda_n, u_n)$ denote the first eigenvalue in (2.2) for $(\lambda, u_\lambda) = (\lambda_n, u_n)$. If $\mu_{1,n} > \lambda_n$ then u_n is stable. Roughly speaking, the following Lemma shows us that if $F(\lambda_n, u_n) > 0$, then $\mu_{1,n} - \lambda_n > 0$. Consequently, to study the stability, one must check the signs of

$$\liminf_{n\to\infty} F(\lambda_n, u_n) \quad \text{and} \quad \limsup_{n\to\infty} F(\lambda_n, u_n), \tag{3.2}$$

where F is defined by (1.8). This is done in Lemma 7. The following technical Lemma, gives us the rate at which $\lambda_n - \mu_{1,n}$ goes to 0, as $\lambda_n \to \sigma_1$. Next Lemma is essentially Lemma 3.3 in [2] rewritten for a different rate, we omit the proof.

Lemma 7 *Assume the nonlinearity g satisfies hypotheses* (H1) *and* (H2).

If (λ_n, u_n) *is a sequence of solutions of* (1.1), $u_n > 0$, *satisfying*

$$\lim_{n\to\infty} \lambda_n = \sigma_1, \quad \textit{and} \quad \lim_{n\to\infty} \|u_n\|_{L^\infty(\partial\Omega)} = \infty, \tag{3.3}$$

denoting by $\mu_{1,n} = \mu_1(\lambda_n, u_n)$, *the first eigenvalue in* (2.2), *then*

$$\begin{aligned}\frac{\underline{\boldsymbol{F}}_+}{\int_{\partial\Omega}\Phi_1^2} &\leq \frac{1}{\int_{\partial\Omega}\Phi_1^2} \liminf_{n\to\infty} F(\lambda_n, u_n)\\ &\leq \liminf_{n\to\infty} \frac{\mu_{1,n}-\lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}} \leq \limsup_{n\to\infty} \frac{\mu_{1,n}-\lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}}\\ &\leq \frac{1}{\int_{\partial\Omega}\Phi_1^2} \limsup_{n\to\infty} F(\lambda_n, u_n) \leq \frac{\overline{\mathbf{F}}_+}{\int_{\partial\Omega}\Phi_1^2}\end{aligned} \tag{3.4}$$

In order to calculate the limits in (3.4), we take advantage of [3, Proposition 3.2], where it is proved that if g is such that

$$|g(\lambda, x, s)| = O\left(|s|^\alpha\right) \text{ as } |s| \to \infty \text{ for some } \alpha < 1,$$

then, the solutions in $\mathcal{D}^\pm$, can be described as

$$u_n = s_n\Phi_1 + w_n, \quad \text{where} \quad \int_{\partial\Omega} w_n\Phi_1 = 0 \quad \text{and} \quad w_n = O(|s_n|^\alpha) \text{ as } n \to \infty.$$

We unveil the signs in (3.2) by looking at the signs of $\liminf_{n\to\infty} F(\sigma_1, s_n\Phi_1)$ and $\limsup_{n\to\infty} F(\sigma_1, s_n'\Phi_1)$, for some sequences $\{s_n\}$ and $\{s_n'\}$, using the following lemma.

The following technical lemma, a slight variant of [3, Lemma 3.3], allows us to unveil the signs in (3.2) from

$$-\infty < \lim_{n\to+\infty} F(\sigma_1, s_n'\Phi_1) < 0 < \lim_{n\to+\infty} F(\sigma_1, s_n\Phi_1) < \infty. \tag{3.5}$$

With these tools, in Theorem 2, we obtain the existence of unbounded sequences of stable and unstable solutions of (1.1) in $\mathcal{D}^+$.

Lemma 8 *Assume that g satisfies hypotheses* (H1), (H2), (H3) *and* (1.7).

If $\lambda_n \to \sigma_1$, $s_n \uparrow \infty$ *and there exists a constant* C *such that* $\|w_n\|_{L^\infty(\partial\Omega)} \leq C|s_n|^\alpha$ *for all* $n \to \infty$, *then*

$$\liminf_{n\to+\infty} F(\lambda_n, s_n\Phi_1 + w_n) \geq \liminf_{n\to+\infty} F(\sigma_1, s_n\Phi_1),$$

where F *is given by* (1.8). *Similarly*

$$\limsup_{n\to+\infty} F(\lambda_n, s_n\Phi_1 + w_n) \leq \limsup_{n\to+\infty} F(\sigma_1, s_n\Phi_1).$$

Proof For short, let us denote by $h = g - sg_s$. For all $(\lambda, s) \approx (\sigma_1, +\infty)$ and for any $w \in L^\infty(\partial\Omega)$ such that $\frac{1}{2}\Phi_1 > \frac{|w|}{s}$, we have (with a constant C that may change from line to line)

$$\begin{aligned}\int_{\partial\Omega} |h(\lambda, \cdot, s\Phi_1 + w) - h(\lambda, \cdot, s\Phi_1)|\Phi_1 &\leq C\|w\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \left| \int_0^1 h_s(\lambda, \cdot, s\Phi_1 + \tau w)\, d\tau \right|\\ &\leq C\|w\|_{L^\infty(\partial\Omega)} \sup_{\tau\in[0,1]} \|h_s(\lambda, \cdot, s\Phi_1 + \tau w)\|_{L^\infty(\partial\Omega)}\end{aligned}$$

Taking into account hypothesis (H3) and whenever $\|w\|_{L^\infty(\partial\Omega)} = O(|s|^\alpha)$, we deduce that

$$\int_{\partial\Omega} \frac{|h(\lambda,\cdot,s\Phi_1+w)-h(\lambda,\cdot,s\Phi_1)|}{|s|^\rho}\,\Phi_1 \le C \sup_{|s|\ge M}\left\|\frac{h_s(\lambda,\cdot,s)}{|s|^{\rho-\alpha}}\right\|_{L^\infty(\partial\Omega)} \longrightarrow 0 \quad (3.6)$$

as $\lambda\to\sigma_1,\ M\to+\infty$.

Consequently, for $\|w_n\|_{L^\infty(\partial\Omega)} = O(|s_n|^\alpha)$

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{\partial\Omega}\frac{s_n h(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n|^{1+\rho}}\,\Phi_1\\
&\ge \lim_{\substack{\lambda\to\sigma_1\\ s\to+\infty}}\int_{\partial\Omega}\frac{sh(\lambda,\cdot,s\Phi_1+w)-sh(\lambda,\cdot,s\Phi_1)}{|s|^{1+\rho}}\,\Phi_1+\liminf_{n\to+\infty}\int_{\partial\Omega}\frac{s_n h(\lambda_n,\cdot,s_n\Phi_1)}{|s_n|^{1+\rho}}\,\Phi_1\\
&=\liminf_{n\to+\infty}\int_{\partial\Omega}\frac{s_n h(\lambda_n,\cdot,s_n\Phi_1)}{|s_n|^{1+\rho}}\,\Phi_1\\
&\ge \lim_{n\to+\infty}\int_{\partial\Omega}\frac{s_n h(\lambda_n,\cdot,s_n\Phi_1)-s_n h(\sigma_1,\cdot,s_n\Phi_1)}{|s_n|^{1+\rho}}\,\Phi_1+\liminf_{n\to+\infty}\int_{\partial\Omega}\frac{s_n h(\sigma_1,\cdot,s_n\Phi_1)}{|s_n|^{1+\rho}}\,\Phi_1\\
&=\liminf_{n\to+\infty}\int_{\partial\Omega}\frac{s_n h(\sigma_1,\cdot,s_n\Phi_1)}{|s_n|^{1+\rho}}\,\Phi_1,
\end{aligned}$$

where we used firstly (3.6) and secondly hypothesis (1.7).

Now note that the left hand side above can be written as

$$\frac{s_n h(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n|^{1+\rho}}\,\Phi_1 = \frac{(s_n\Phi_1+w_n)h(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n\Phi_1+w_n|^{1+\rho}}\left|\Phi_1+\frac{w_n}{s_n}\right|^\rho\Phi_1.$$

Then, (H2) and the fact that $\Phi_1 + w_n/s_n \to \Phi_1$ in $L^\infty(\partial\Omega)$ conclude the proof. □

We are now ready to prove our main result, which states the existence of unbounded sequences of stable solutions, unbounded sequences of unstable solutions and also unbounded sequences of turning points.

Proof of Theorem 2 (i) To prove the result, we show that due to (1.9) we can find two unbounded sequences of solutions $\{(\lambda_n,u_n)\}$, $\{(\lambda'_n,u'_n)\}$, with λ_n,λ'_n close enough to σ_1, such that $\mu_{1,n} := \mu_1(\lambda_n,u_n) > \lambda_n$ and $\mu'_{1,n} := \mu_1(\lambda'_n,u'_n) < \lambda'_n$, respectively and then we use Lemma 4 to characterize the stability of the solutions. Below we focus on the stable case since the unstable one is analogous.

Let us now consider $\mathcal{D}^+$, the unbounded connected set of positive solutions of (1.1) known as a branch bifurcating from infinity, see [1, Theorem 3.4]. Since the projection of the unbounded branch of positive solutions on span$[\Phi_1]$, is an interval $[s_0,\infty)$, for any $s\in[s_0,\infty)$ there is a solution of (1.1), $u = s\Phi_1 + w$, see (1.5). We choose $(\lambda_n,u_n)\to(\sigma_1,\infty)$ on this branch such that

$$P(u_n) := \frac{\int_{\partial\Omega} u_n\Phi_1}{\int_{\partial\Omega}\Phi_1^2} = s_n, \quad (3.7)$$

with s_n as in (1.9).

By Lemma 7 we have

$$\liminf_{n\to\infty} \frac{\mu_{1,n}-\lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}} \geq \frac{1}{\int_{\partial\Omega}\Phi_1^2} \liminf_{n\to\infty} F(\lambda_n, u_n) \tag{3.8}$$

Due to (3.7), we may write $u_n = s_n\Phi_1 + w_n$. From [3, Proposition 3.2] and hypotheses (H2), we obtain that $w_n = O(|s_n|^\alpha)$. Applying Lemma 8 and since (1.9) hold, we infer

$$\liminf_{n\to\infty} F(\lambda_n, s_n\Phi_1 + w_n) \geq \liminf_{n\to+\infty} F(\sigma_1, s_n\Phi_1) > 0 \tag{3.9}$$

The inequalities (3.8)–(3.9) imply that $\mu_{1,n} > \lambda_n$ for λ_n close enough to σ_1. Likewise, it can be proved that $\mu'_{1,n} < \lambda'_n$ for λ'_n close enough to σ_1, ending this part of the proof.

(ii) Next, we will prove that here exists a sequence $\{(\lambda_n^*, u_n^*)\} \subset \mathcal{D}^+$ of turning points such that $(\lambda_n^*, \|u_n^*\|_{L^\infty(\partial\Omega)}) \to (\sigma_1, \infty)$ as $n\to\infty$. To achieve this part of the proof, we use Leray-Schauder degree theory. Let

$$K_n := \{(\lambda, u) \in \mathcal{D}^+ : P(u) = s \text{ and } s_n \leq s \leq s'_n\}.$$

For each $n\in\mathbb{N}$, K_n is a compact set in $\mathbb{R}\times C(\bar\Omega)$, see for instance [3, Proof of Theorem 3.4]. For each $n\in\mathbb{N}$ fixed, let $\lambda_{min} := \min\{\lambda : (\lambda, u)\in K_n\}$, and likewise λ_{max}. Assume on the contrary that K_n contains no turning point. In other words, assume that for each $\lambda\in[\lambda_{min}, \lambda_{max}]$ there exist a unique solution $u_\lambda \in K_n$.

For any $b\in L^q(\partial\Omega)$, $q\geq 1$, there exists a unique solution of

$$\begin{cases} -\Delta v + v = 0, & \text{in } \Omega \\ \frac{\partial v}{\partial n} = b, & \text{on } \partial\Omega. \end{cases}$$

Moreover $\|v\|_{W^{1,p}(\Omega)} \leq C\|b\|_{L^q(\partial\Omega)}$, with $p = q\frac{N}{N-1}$. We denote it by $T(b) = v$ and

$$S(b) := \gamma T(b), \quad \text{where } \gamma : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega) \text{ is the trace operator.}$$

The operator S is known as the *Neumann-to-Dirichlet operator*. If $q > N-1$, then the mapping S maps $L^q(\partial\Omega)$ into $C^\tau(\partial\Omega)$ for some $\tau\in(0,1)$, and is continuous and compact, see for instance [1, Lemma 2.1].

Let $H : [\lambda_{min}, \lambda_{max}] \times C(\partial\Omega) \to C(\partial\Omega)$ be the homotopy defined by

$$H(\lambda, u) := \lambda Su + S(g(\lambda, \cdot, u)).$$

Hence, the fixed points of $H(\lambda,\cdot)$ are the solutions to (1.1). Let $\varepsilon > 0$, writing $u = s\Phi_1 + w$, and taking into account that $\|w\|_{L^\infty(\partial\Omega)} = O(|s|^\alpha)$ with $\alpha < 1$, we obtain $\|u - s\Phi_1\|_{L^\infty(\partial\Omega)} \leq \varepsilon s$ for any s big enough.

Now consider the Leray-Schauder degree of $I - H(\lambda,\cdot)$ with respect to zero, in the set

$$\mathcal{O} := \bigcup_{s\in[s_n, s'_n]} \{u \in C(\bar\Omega) : \|u - s\Phi_1\|_{L^\infty(\partial\Omega)} \leq 2\varepsilon s\}.$$

From the homotopy invariance property, $\deg_{LS}(I - H(\lambda,\cdot), \mathcal{O}, 0)$ is well defined and independent of λ for $\lambda\in[\lambda_{min}, \lambda_{max}]$. In particular

$$\deg_{LS}(I - H(\lambda_n,\cdot), \mathcal{O}, 0) = \deg_{LS}(I - H(\lambda'_n,\cdot), \mathcal{O}, 0). \tag{3.10}$$

Since $\lambda_n < \mu_{1,n}$ by part (i), the linearized operator $I - \lambda_n S - S[g_s(\lambda_n, x, u_n)\cdot]$ is invertible and consequently u_n is an isolated fixed point. Therefore the fixed point index is well defined and moreover

$$\mathrm{i}\,(H(\lambda_n,\cdot), u_n) = \deg_{LS}(I - \lambda_n S - S[g_s(\lambda_n, x, u_n)\cdot], \mathcal{O}, 0) = (-1)^{m(\lambda_n)} = 1$$

where $m(\lambda_n)$ is sum of the algebraic multiplicities of the eigenvalues of the linearization strictly smaller than λ_n and $m(\lambda_n) = 0$ if the linearization has no eigenvalues $\mu_{i,n}$ of this kind.

Moreover, from the hypothesis that u_n is the only solution in K_n for the value of the parameter $\lambda = \lambda_n$, we deduce $\deg_{LS}(I - H(\lambda_n, \cdot), \mathcal{O}, 0) = \mathrm{i}\,(H(\lambda_n, \cdot), u_n)$.

On the other hand

$$\mathrm{i}\,(H(\lambda_n', \cdot), u_n') = \deg_{LS}(I - \lambda_n' S - S[g_s(\lambda_n', x, u_n')\cdot], \mathcal{O}, 0) = -1$$

and likewise $\deg_{LS}(I - H(\lambda_n', \cdot), \mathcal{O}, 0) = \mathrm{i}\,(H(\lambda_n', \cdot), u_n') = -1$ which contradicts (3.10) and the proof is achieved. □

4 The Case $N = 1$

Letting $\Omega = (0, 1)$, we may rewrite equation (1.1) as

$$\begin{cases} -u_{xx} + u = 0, & \text{in } (0,1) \\ -u_x(0) = \lambda u + g(\lambda, 0, u(0)), \\ u_x(1) = \lambda u + g(\lambda, 1, u(1)). \end{cases} \tag{4.1}$$

The general solution to this differential equation is $u(x) = ae^x + be^{-x}$. Therefore, it is a solution to (4.1) if (λ, a, b) satisfies

$$\begin{pmatrix} -(1+\lambda) & (1-\lambda) \\ (1-\lambda)e & -(1+\lambda)e^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} g(\lambda, 0, a+b) \\ g(\lambda, 1, ae + be^{-1}) \end{pmatrix}.$$

In this case, we only have two Steklov eigenvalues σ_1 and σ_2. They are given by the values $\lambda = \sigma_1$, $\lambda = \sigma_2$ for which the following matrix has zero determinant:

$$\begin{pmatrix} -(1+\lambda) & (1-\lambda) \\ (1-\lambda)e & -(1+\lambda)e^{-1} \end{pmatrix}.$$

These two values are given by

$$\sigma_1 = \frac{e-1}{e+1} < \sigma_2 = \frac{1}{\sigma_1} = \frac{e+1}{e-1}.$$

The eigenfunctions Φ_1 and Φ_2 for this problem are given by

$$\Phi_1(x) = \frac{e^x + e^{1-x}}{1+e}, \quad \Phi_2(x) = \frac{e^x - e^{1-x}}{1-e}.$$

Observe that $\Phi_1(0) = \Phi_1(1) = 1$ and $\Phi_2(0) = 1 = -\Phi_2(1)$.

Choose $g(\lambda, x, s) = s^\alpha \sin(s^\beta)$ for any $\alpha < 1$, $\beta > 0$. For any $\lambda \neq \sigma_1, \sigma_2$, the function $u = ae^x + be^{-x}$ is a solution to (4.1) if (λ, a, b) satisfies

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -(1+\lambda) & (1-\lambda) \\ (1-\lambda)e & -(1+\lambda)e^{-1} \end{pmatrix}^{-1} \begin{pmatrix} (a+b)^\alpha \sin((a+b)^\beta) \\ (ae+be^{-1})^\alpha \sin((ae+be^{-1})^\beta) \end{pmatrix}.$$

The sublinearity of g as $s \to \infty$ allows us to apply fixed-point arguments in $\mathbb{R}^2$ guaranteeing the existence of at least one solution for any $\lambda \neq \sigma_1, \sigma_2$.

Restricting the analysis to symmetric solutions $u_s(x) = s(e^x + e^{1-x})$, with $s \in \mathbb{R}$, it is easy to prove that $u_s(x)$ is a solution if and only if λ satisfies

$$\lambda(s) = \sigma_1 - \frac{g(s(e+1))}{s(e+1)} = \sigma_1 - \frac{\sin\left[[s(e+1)]^\beta\right]}{[s(e+1)]^{1-\alpha}}, \quad s > 0. \tag{4.2}$$

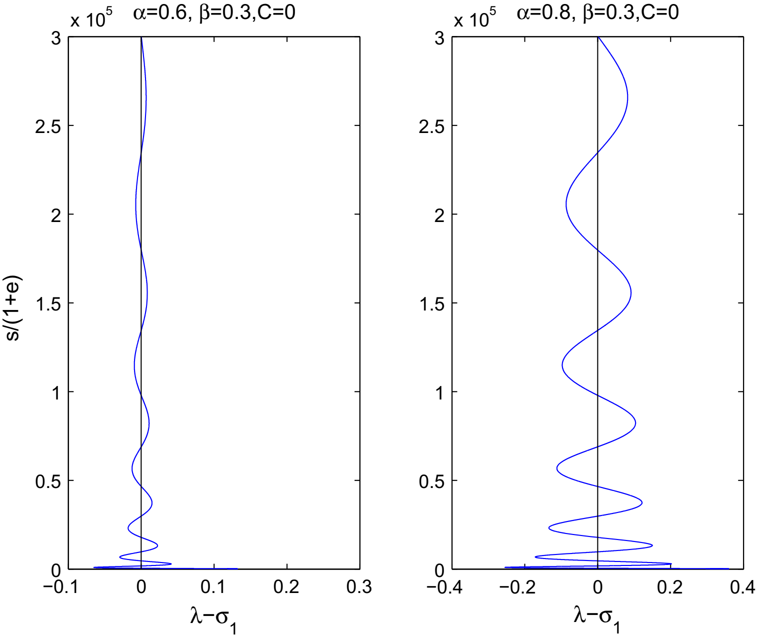

Fig. 4 A bifurcation diagram of changing stability solutions, on the *left* $\alpha+\beta<1$, and on the *right* $\alpha+\beta>1$ and in both cases $\lambda\to\sigma_1$

Therefore, $(\lambda(s), u_s)$ is an unbounded branch of solutions of (4.1) satisfying $(\lambda(s), \|u_s\|_\infty)\to(\sigma_1,\infty)$ as $s\to\infty$, see Fig. 4.

Next we apply Lemma 4 to elucidate the stability of the bifurcated solutions $\{(\lambda(s), u_s)\}$. The eigenvalue of the linearized equation, see (2.2), is given by

$$\mu_1\big((\lambda(s), u_s)\big) := \sigma_1 - \alpha\,\frac{\sin\left[[s(e+1)]^\beta\right]}{[s(e+1)]^{1-\alpha}} - [s(e+1)]^{\alpha+\beta-1}\cos\left[[s(e+1)]^\beta\right].$$

If

$$[s(e+1)]^\beta = \begin{cases}(2k+1)\pi\\ \frac{(2k+1)\pi}{2}\\ 2k\pi\end{cases},\quad \text{then}\quad \mu_1\big(\lambda(s), u_s\big)-\lambda(s)\begin{cases}>0\\ =0\\ <0.\end{cases}$$

Letting

$$u_{2k+1}(x) := \frac{[(2k+1)\pi]^{1/\beta}}{e+1}(e^x+e^{1-x})\quad\text{for any}\quad k\in\mathbb{Z},$$

we see that (σ_1, u_{2k+1}) is an unbounded sequence of stable solutions. Likewise, (σ_1, u_{2k}) is an unbounded sequence of unstable solutions where

$$u_{2k}(x) := \frac{(2k\pi)^{1/\beta}}{e+1}(e^x+e^{1-x})\quad\text{for any}\quad k\in\mathbb{Z}.$$

Moreover, defining

$$\lambda_k^* := \frac{e-1}{e+1} - \frac{(-1)^k\,\alpha}{[(k+1/2)\pi]^{1-\alpha}},\quad u_k^*(x) := \frac{[(2k+1)\pi]^{1/\beta}}{2(e+1)}(e^x+e^{1-x}),$$

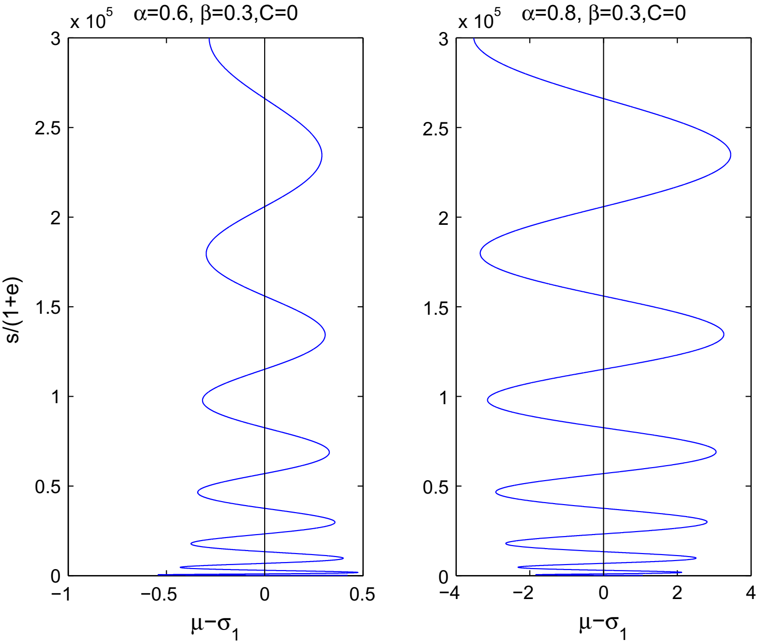

Fig. 5 The difference between $\mu - \sigma_1$. On the *left* $\alpha + \beta < 1$, and $\mu \to \sigma_1$, on the *right* $\alpha + \beta > 1$ and $\mu \not\to \sigma_1$

(λ_k^*, u_k^*) is an unbounded sequence of turning points. The branch bifurcating from infinity contains stable and unstable solutions, and there is an unbounded sequence of turning points. See Figs. 1, 2 and 3 for a bifurcation diagram when $N = 1$. In that case, there is no restriction on the size of β, see Fig. 4.

If $\alpha + \beta \geq 1$ then $\mu_1\big(\lambda(s), \cdot, u_s\big) \not\to \sigma_1$ as $s \to \infty$. On the other hand, the eigenvalue of the linearized equation satisfies $\mu_1\big(\lambda(s), \cdot, u_s\big) \to \sigma_1$ as $s \to \infty$, whenever $\alpha + \beta < 1$, see Fig. 5.

Moreover, if $\alpha + \beta < 1$,

$$\begin{aligned} \underline{\mathbf{F}}_+ &:= \int_{\partial\Omega} \liminf_{s\to+\infty} \frac{sg - s^2 g_s}{|s|^{1+\alpha+\beta}} \Phi^{1+\alpha+\beta} \\ &= \int_{\partial\Omega} \liminf_{s\to+\infty} -\beta\cos(s^\beta)\ \Phi^{1+\alpha+\beta} = -\beta \int_{\partial\Omega} \Phi^{1+\alpha+\beta}, \\ \overline{\mathbf{F}}_+ &:= \int_{\partial\Omega} \limsup_{s\to+\infty} \frac{sg - s^2 g_s}{|s|^{1+\alpha+\beta}} \Phi^{1+\alpha+\beta} \\ &= \int_{\partial\Omega} \limsup_{s\to+\infty} -\beta\cos(s^\beta)\ \Phi^{1+\alpha+\beta} = \beta \int_{\partial\Omega} \Phi^{1+\alpha+\beta} \end{aligned}$$

i.e. $\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+$.

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