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REGULAR SOLUTIONS TO ELLIPTIC EQUATIONS

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Dedicated to our admired friend John W. Neuberger

Abstract. A review of results and techniques on the existence of regular radial solutions to second order elliptic boundary value problems driven by linear and quasilinear operators is presented. Of particular interest are results where the solvability of a given elliptic problem can be analyzed by the relationship between the spectrum of the operator and the behavior of the nonlinearity near infinity and at zero. Energy arguments and Pohozaev type identities are used extensively in that analysis. An appendix with a proof of the contraction mapping principle best suited for using continuous dependence to ordinary differential equations on initial conditions is presented. Another appendix on the phase plane analysis as needed to take advantage of initial conditions is also included. For studies on singular solutions the reader is referred to Ardila et al., Milan J. Math (2014) and references therein.

1. Introduction

Motivated by the solvability of boundary value problems such as

$$-u''(t) = g(t, u(t)) \text{ for } t \in (0, \pi), \quad u(0) = 0,\ u(\pi) = 0, \tag{1.1}$$

extensive understanding of the solutions of equations of the type

$$-\Delta u(x) = g(x, u(x)) \text{ for } x \in \Omega, \quad u(x) = 0 \text{ for } x \in \partial\Omega, \tag{1.2}$$

has been achieved when $\Omega \subseteq \mathbb{R}^N$ is either a ball centered at the origin, an annulus (the region between two concentric balls in $\mathbb{R}^N$) centered at the origin, or $\mathbb{R}^N$, and g is radial in its first variable, i.e., $g(x, u) = g(y, u)$ if $\|x\| = \|y\|$. In (1.2), Δ denotes the Laplace operator $\partial^2_{x_1} + \cdots + \partial^2_{x_N}$. Here we present a summary of results and techniques involved in the analysis of such equations.

For a general bounded region W there is a sequence $0 < \lambda_1 < \lambda_2 \le \cdots \to +\infty$ and corresponding functions $\varphi_1, \varphi_2, \dots$ such that

$$-\Delta\varphi_i(x) = \lambda_i\varphi_i(x) \text{ for } x \in W, \quad \varphi(x) = 0 \text{ for } x \in \partial W. \tag{1.3}$$

The set $\sigma(-\Delta) = \{\lambda_i : i = 1, 2, \dots\}$ is known as the spectrum of $-\Delta$ subject to the boundary condition in (1.2). The eigenvalue λ_1 is simple, all others have

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finite multiplicity. When W is a ball or an annulus, $\sigma(-\Delta)$ contains a subsequence $0<\rho_1=\lambda_1<\rho_2<\dots\to+\infty$ with all the ρ_i's having a radial eigenfunction. The eigenvalues ρ_i are simple in the space of radial functions. The general theme is that the solvability of (1.2) is determined, to a large extent, by the relation between the range of $g(x,u)/u$ and the spectrum $\sigma(-\Delta)$. Similarly, when Ω is a ball or an annulus and g is radial in x, the existence of radial solutions to (1.2) is largely determined by the relation between $g(x,u)/u$ and the set $\sigma_{\mathrm{rad}}(-\Delta)=\{\rho_i : i=1,2,\dots\}$.

This motivates classifying the nonlinearity g according to the following behaviors:

Definition 1.1 (Classification of nonlinearities)**.** We say that

- g is asymptotically linear if $\lim_{|u|\to+\infty} g(x,u)/u\in\mathbb{R}$.
- g is resonant if $\lim_{|u|\to+\infty} g(x,u)/u\in\sigma(-\Delta)$.
- g is a jumping nonlinearity if $\lim_{u\to+\infty} g(x,u)/u, \lim_{u\to-\infty} g(x,u)/u\in\mathbb{R}$ with $\lim_{u\to+\infty} g(x,u)/u\neq\lim_{u\to-\infty} g(x,u)/u$.
- g is superlinear if $\lim_{u\to+\infty} g(x,u)/u=+\infty$.
- g is superlinear with subcritical growth if $\lim_{|u|\to+\infty}\frac{g(x,u)}{|u|^{p-1}u}\in\mathbb{R}$ with $1<p<\frac{N+2}{N-2}$.
- g is superlinear with critical growth if $\lim_{|u|\to+\infty}\frac{g(x,u)}{|u|^{p-1}u}\in\mathbb{R}$ with $p=\frac{N+2}{N-2}$.
- g is superlinear with supercritical growth if $\lim_{|u|\to+\infty}\frac{g(x,u)}{|u|^{p-1}u}\in\mathbb{R}$ with $p>\frac{N+2}{N-2}$.
- g is sub-super critical if $\lim_{u\to+\infty}\frac{g(x,u)}{u^p}\in\mathbb{R}$ and $\lim_{u\to-\infty}\frac{g(x,u)}{|u|^{q-1}u}\in\mathbb{R}$, with $1<p<\frac{N+2}{N-2}<q<\infty$.

All limits are assumed to be uniform with respect to x.

Writing the Laplace operator in spherical coordinates one sees that the regular (classical) radial solutions of the partial differential equation (1.2) are solutions to the ordinary differential equation

$$-u''(r)-\frac{N-1}{r}u'(r)=g(r,u(r)),\quad r\in I,\ Bu=0, \tag{1.4}$$

where $I=(0,b)$ when Ω is the ball of radius b centered at the origin and $Bu=u'(0)=u(b)$, $I=(a,b)$ with $0<a<b$ when Ω is an annulus and $B(u)=u(a)=u(b)$, and $I=(0,+\infty)$ when $\Omega=\mathbb{R}^N$ and $B(u)=u'(0)=u(\infty)$. When Ω is an annulus the differential equation in (1.4) is a regular ordinary differential equation, when Ω is a ball it is a singular ordinary differential equation with a singularity at 0 and when $\Omega=\mathbb{R}^N$, by transforming $(0,\infty)$ into a bounded interval we can view the differential equation in (1.4) as a singular ordinary differential equation with singularities at both ends of the interval. In order to simplify the presentation we will assume that $g(r,u)=g(u)$ with g differentiable. See [12] for recent results on the case $g(r,u)=g(u)W(r)$ and W is sign-changing.

Under additional hypothesis on g, see [30], the positive solutions to (1.4) are the only positive solutions to (1.2). For sign-changing solutions this is not valid. For example, when Ω is a ball or annulus and g is superlinear and *odd-like*, (1.2) may have infinitely many non-radial solutions [1, 2, 9]. See also [3, 11].

Under adequate conditions on N and g, the second order differential equation in (1.4) has solutions in the sense of distributions that blow up at the origin

($\lim_{r\to 0+} u(r) = +\infty$). The reader is referred to [6] for a study on such solutions. Finally we remark that the results here presented extend to more general boundary conditions such as $\partial u/\partial\eta = 0$, $\alpha u + \beta\partial u/\partial\eta = 0$, and other nonlinear boundary conditions.

2. The shooting method and phase plane analysis

Much of the understanding of solutions to (1.4) is based on the analysis of the initial value problem

$$-u''(r) - \frac{N-1}{r}u'(r) = g(u)\ r \in I, \quad u(0) = d,\ u'(0) = 0, \tag{2.1}$$

when Ω is a ball or $\mathbb{R}^N$ and

$$-u''(r) - \frac{N-1}{r}u'(r) = g(u)\ r \in I, \quad u(a) = 0,\ u'(a) = d, \tag{2.2}$$

when Ω is an annulus. Arguments based on the Contraction Mapping Principle (see Appendix A) show that if g is locally Lipschitz and g' satisfies adequate growth conditions then (2.1) and (2.2) have unique solutions that are defined on $[a, b]$ and depend continuously on the initial value d. This is not the case if $g(r, u) = W(r)g(u)$ and W is sign-changing. For such cases, the solutions to (2.1) may blow-up where W is negative (see [12]).

For any constant d, a solution of (2.1) that satisfies $u(b) = 0$ is a solution of (1.4) when Ω is the ball of radius b centered at the origin. Similarly, a solution of (2.2) that satisfies $u(b) = 0$ is a solution of (1.4) when Ω is the annulus of inner radius a and outer radius b. This observation motivates the *shooting method* based on studying the behavior of solutions to (2.1) in the (u, u') plane as d varies. To indicate the dependence on d we let $u(t; d)$ denote a solution of (2.1), but when the role of d is not essential we will continue to use u or $u(r)$ to denote the solution. We outline the key steps and refer to [17, 19] for more details.

Suppose that g is superlinear (see Definition 1.1) and let u be the solution of (2.1). Applying the integrating factor r^{N-1} to (2.1) yields

$$-r^{N-1}u'(r) = \int_0^r s^{N-1}g(u(s))\,ds.$$

Let $k \in (0, 1)$. For a given $d > 0$ let $r_0 > 0$ be the first value such that $u(r_0; d) = kd$ and $u(r; d) > kd$ for $r \in (0, r_0)$. If in addition g has subcritical, critical, or supercritical growth, for $0 < r \le r_0$ it follows that

$$-u'(r) = O(rd^p). \tag{2.3}$$

Integrating from 0 to r_0 one sees that $r_0 = O(d^{\frac{1-p}{2}})$. Combining this estimate with the energy

$$E(r) = \frac{(u'(r))^2}{2} + G(u(r)), \tag{2.4}$$

where $G(u) = \int_0^u g(s)\,ds$, and Pohozaev energy

$$\begin{aligned} P(r) &= r^{N-1}\Big(rE(r) + \frac{N-2}{2}u(r)u'(r)\Big) \\ &= \int_0^r s^{N-1}\Big(NG(u(s)) - \frac{N-2}{2}u(s)g(u(s))\Big)ds, \end{aligned} \tag{2.5}$$

we can gain insight into the behavior of the solutions to (2.1). For example, using $r_0 = O(d^{\frac{1-p}{2}})$ and assuming $\delta u^p \le g(u) \le u^p$ for some $\delta \in (0,1]$, for d large, there exists positive constants K, M such that

$$r^N E(r) \ge K r_0^N E(r_0) \ge K r_0^N \left(G(kd)\right) \ge M d^{N\left(\frac{1-p}{2}\right)} d^{p+1}.$$

In the subcritical growth case, $1 < p < \frac{N+2}{N-2}$ and so the exponent on d will be positive in which case $E(r) \to \infty$ as $d \to \infty$. From these estimates if we consider phase plane dynamics for $(u(r;d), u'(r;d))$ and define

$$\rho^2(r;d) = (u(r;d))^2 + (u'(r;d))^2$$

then $\rho^2(r;d) \to \infty$ as $d \to \infty$, uniformly for $r \in [0, R]$. Therefore, there exists a unique continuous argument function $\theta(r;d)$ such that

$$u(r;d) = \rho(r;d)\cos\theta(r;d) \tag{2.6}$$

$$u'(r;d) = -\rho(r;d)\sin\theta(r;d) \tag{2.7}$$

with $\theta(0;d) = 0$ (see Appendix B). A direct calculation shows

$$\theta'(r;d) = \sin^2\theta(r;d) + \frac{\cos\theta(r;d)}{\rho(r;d)}\Big(\frac{N-1}{r}u' + g(u)\Big).$$

Using this one can show that for any fixed $R > 0$ the argument $\theta(R;d) \to \infty$ as $d \to \infty$. Since θ is continuous one can use the intermediate value theorem to construct an infinite sequence of values $d_1 < \cdots < d_k < \cdots \to \infty$ such that $\theta(R;d_k) = k\pi + \frac{\pi}{2}$ (hence $u(R;d_k) = 0$), which will be a radial solution of (1.4) on the ball of radius R, establishing infinitely many solutions for the ball [17]. A similar analysis can be used to establish infinitely many radial solutions on the ball for sub-super critical nonlinearities, see [15].

The Pohozaev energy (2.5) utilized above is obtained by multiplying (2.1) by $\mu(r) = \phi(r)u + \psi(r)u'$ for some to be determined coefficients ϕ, ψ and integrating by parts:

$$\begin{aligned} &-\phi u u' - \frac{\psi}{2}(u')^2\Big|_\alpha^\beta + \int_\alpha^\beta \Big(\phi' - \frac{N-1}{r}\phi\Big)uu' \\ &+ \Big(\phi + \frac{\psi'}{2} - \frac{N-1}{r}\psi\Big)(u')^2\,dr \\ &= \int_\alpha^\beta \mu(r)g(u)\,dr. \end{aligned} \tag{2.8}$$

Choosing $\phi = r^{N-1}$ will eliminate the uu' integral. Similarly, for this choice of ϕ the function $\psi = \frac{2}{N-2}r^N$ will eliminate the $(u')^2$ integral, reducing (2.8) to

$$-r^{N-1}uu' - \frac{r^N}{N-2}(u')^2\Big|_\alpha^\beta = \int_\alpha^\beta r^{N-1}ug(u) + \frac{2}{N-2}r^N u'g(u)\,dr. \tag{2.9}$$

Integrating the $u'g(u) = (G(u))'$ term and multiplying by $-\frac{N-2}{2}$ this simplifies to

$$r^{N-1}\Big(rE(u) + \frac{N-2}{2}uu'\Big)\Big|_\alpha^\beta = \int_\alpha^\beta \Big(NG(u) - \frac{N-2}{2}ug(u)\Big)r^{N-1}\,dr. \tag{2.10}$$

In particular, with $\alpha = 0$ and $\beta = r_0$ we have

$$P(r_0) = \int_0^{r_0} \Big(NG(u) - \frac{N-2}{2}ug(u)\Big)r^{N-1}\,dr. \tag{2.11}$$

For example, if $f(u) = |u|^{p-1}u$ then the integrand is negative for $p > \frac{N+2}{N-2}$ hence the integral is negative, but we showed earlier that $P(r_0) > 0$, a contradiction, so it follows from the Pohozaev energy that there are no radial solutions of (2.1) when $p > \frac{N+2}{N-2}$.

3. Bifurcation analysis

Suppose now that $ug(u) > 0$ for $|u| \neq 0$, and $g'(0) > 0$. For $\lambda > 0$ consider the equation

$$-u''(r) - \frac{N-1}{r}u'(r) = \lambda g(u)\; r \in [0,1), \quad u'(0) = 0,\; u(1) = 0. \tag{3.1}$$

For $g(u) = u$ and $u(0) \neq 0$, the solutions to (3.1) are the radial eigenfunctions and corresponding eigenvalues to (1.3) when Ω is the unit ball centered at the origin. We will denote such eigenvalues by $\rho_1 < \rho_2 < \dots < \rho_k < \dots \to +\infty$. It turns out that such eigenvalues are simple in the space of radial functions. That is, for such eigenvalues, all the radial eigenfunctions are linearly independent. Also, $\rho_1 = \lambda_1$.

Since $g(0) = 0$, $u_0 \equiv 0$ is a solution to (3.1) for any $\lambda > 0$ (the trivial branch). We are interested in nontrivial branches of solutions that bifurcate from the trivial branch at some value of λ. If $\hat{\lambda}g'(0) \notin \sigma(-\Delta)$, arguments based on the implicit function theorem show that there exists $\epsilon > 0$ such that u_0 is the only solution to (3.1) for $|\lambda - \hat{\lambda}| < \epsilon$ and $|u| < \epsilon$. This begs the question: What if $\hat{\lambda}g'(0) \in \sigma(-\Delta)$? The answer is given by the Crandall-Rabinowitz Theorem [23, Theorem 1.7] which captures the nature of the bifurcation that occurs. Let

$$Y = \big\{y : [0,1] \to \mathbb{R} : y \text{ is continuous and } \int_0^1 y(r)\varphi_k(r)r^{N-1}dr = 0\big\}.$$

The Crandall-Rabinowitz theorem says that since the eigenvalue $\rho_k = \hat{\lambda}g'(0)$ is simple there exists $\epsilon > 0$ and continuous functions $v : (-\epsilon,\epsilon) \to Y$ and $L : (-\epsilon,\epsilon) \to \mathbb{R}$ such that $v(0) = L(0) = v'(0) = L'(0) = 0$ and all nontrivial solutions to (3.1) in a neighborhood of $(\hat{\lambda}, 0) = (\frac{\rho_k}{g'(0)}, 0)$ are of the form

$$\lambda(t) = \frac{\rho_k}{g'(0)} + L(t) \quad \text{and} \quad u(t) = t\varphi_k + v(t). \tag{3.2}$$

General global bifurcation results (e.g., [24]) imply that (3.1) has two connected sets of solutions $S_{k,1}$ and $S_{k,2}$ where $S_{k,1}$ contains $\{(\lambda(t), u(t)) : t \in (0,\epsilon)\}$ and $S_{k,2}$ contains $\{(\lambda(t), u(t)) : t \in (-\epsilon, 0)\}$. Moreover, such branches of solutions are either unbounded or contain two points of the form $(\rho_k, 0)$ in their closure. In our case, since $g(0) = 0$, by the uniqueness of solutions to initial value problems, if (λ, u) is a solution to (3.1) and $u(x) = 0$ then $u'(x) \neq 0$. This implies that if S is a connected set of solutions to (3.1) the number of zeros for nontrivial solutions along the branch is constant and $u(0)$ does not change sign. Thus we may assume that $u(0) > 0$ for $(\lambda, u) \in S_{k,1}$ and $u(0) < 0$ for $(\lambda, u) \in S_{k,2}$. In particular, it follows that $S_{k,j} \cap S_{n,m} = \emptyset$ if $(k,j) \neq (n,m)$. Hence no branch can have two points of the form $(\rho_k, 0)$ in its closure and so all continua $S_{k,j}$ are unbounded. With more precise information about the nonlinearity g one can refine the behavior of these unbounded continua to establish existence and multiplicity results. For example, Figures 1 and 2 illustrate the solution branches for the superlinear subcritical case and the asymptotically linear case, which we discuss in the next section.

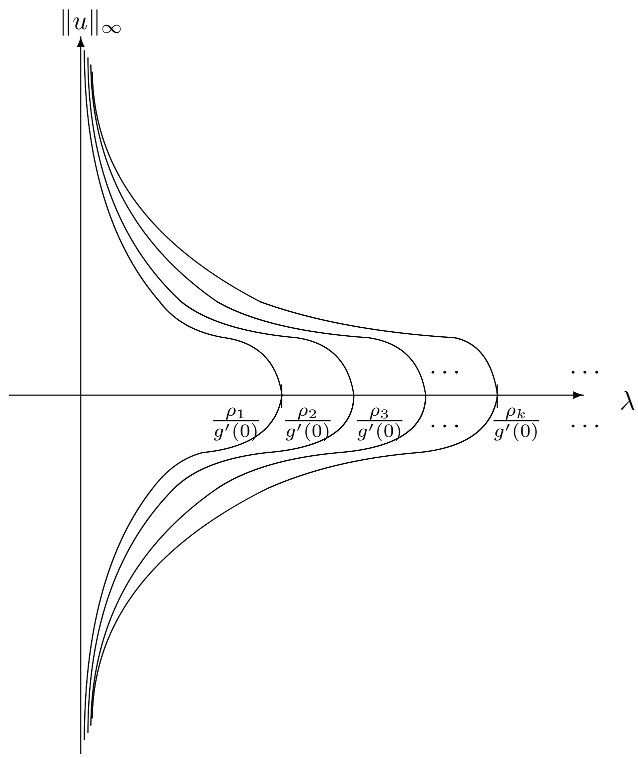

FIGURE 1. Bifurcation curves for superlinear g with subcritical growth.

4. Asymptotically linear nonlinearities

For the sake of simplicity in the notations we write:

$$g'(+\infty) = \lim_{u \to +\infty} \frac{g(u)}{u}, \quad g'(-\infty) = \lim_{u \to -\infty} \frac{g(u)}{u}, \quad g'(\infty) = \lim_{|u| \to +\infty} \frac{g(x,u)}{u}, \quad (4.1)$$

when they exist.

Let us illustrate how to combine the above bifurcation arguments to prove the following theorem.

Theorem 4.1. *If* $0 < g'(0) \in (\rho_j, \rho_{j+1})$, $0 < g'(\infty) \in (\rho_k, \rho_{k+1})$, $ug(u) > 0$ *for* $u \neq 0$, *then* (1.4) *has* $2|k-j|+1$ *solutions.*

Proof. Without loss of generality we may assume that $j \leq k$. If $j = k$, taking $u = 0$ we have the conclusion of the theorem. Thus we assume $j < k$. As stated in Section 3, for each $i = 1, 2, \dots$, there exist two unbounded connected sets of solutions $S_{i,1}$ and $S_{i,2}$ bifurcating from $(\rho_i/g'(0), 0)$. Taking a sequence $(\rho_n, u_n) \in S_{i,\tau}$, for $\tau \in \{1, 2\}$, with $\lim_{n\to+\infty} \|u_n\| = +\infty$ one sees that $\{\rho_n\}$ converges to $\rho_i/g'(\infty)$.

Since $\rho_i/g'(0) > 1 > \rho_i/g'(\infty)$ for $i = j+1, \dots, k$, the bifurcation curve $S_{i,1}$ contains elements of the form (λ, u) with λ close to $\lambda_i/g'(0) > 1$ and u close to 0. Since $S_{i,1}$ is unbounded it contains elements of the form (λ, u) with λ close to $\rho_i/g'(\infty) < 1$. By the connectedness of $S_{i,1}$ there must exist $(\lambda, u_{i,1}) \in S_{i,1}$ with $\lambda = 1$, in which case $u_{i,1}$ is a solution to (1.4). Similarly, one has a solution $u_{i,2}$

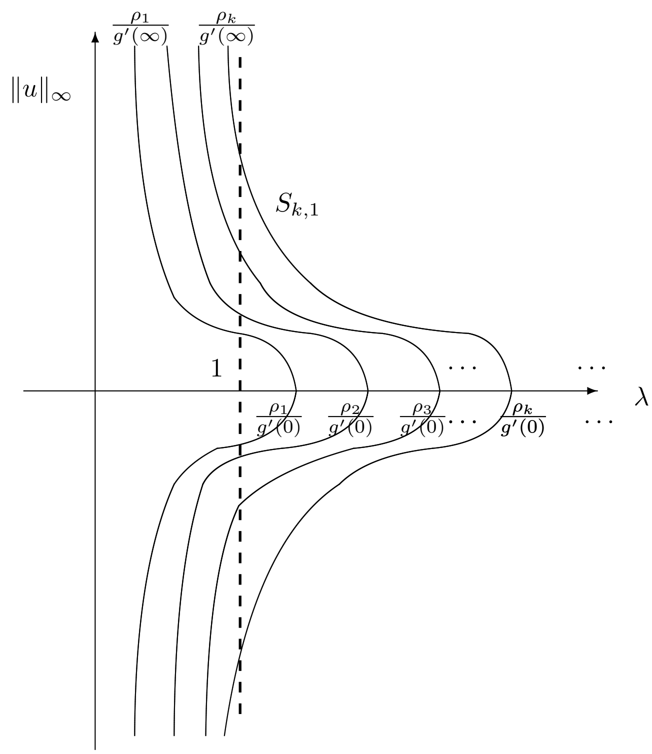

FIGURE 2. Bifurcation curves for asymptotically linear g, $g'(0) < \lambda_1 < \rho_k < g'(\infty)$.

to (1.4) in $S_{i,2}$. Hence $0, u_{k+1,1}, u_{k+1,2}, \dots, u_{j,1}, u_{j,2}$ are $2(j-k)+1$ solutions to (1.2), which proves the theorem. $\square$

Figure 2 provides a graphical description of the foregoing proof.

In the absence of the hypothesis $ug'(u) > 0$ for $u \neq 0$ additional solutions may arise. For example, the following result is established in [10]:

Theorem 4.2. *If $g'(0) = g'(\infty) \in (\rho_j, +\infty)$, and $g(z) = 0$ for some $z > 0$ then (1.4) has $4j - 1$ solutions.*

Figure 3 suggests the proof of Theorem 4.2.

5. Jumping nonlinearities

In [5], for any bounded region in $\mathbb{R}^N$, the solvability of the boundary value problem

$$-\Delta u(x) = g(u(x)) + v(x) + c\varphi_1(x) \text{ for } x \in \Omega, \quad u(x) = 0 \text{ for } x \in \partial\Omega, \tag{5.1}$$

was considered assuming that g is a jumping nonlinearity with g convex and

$$g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2, \tag{5.2}$$

where $\lambda_1 < \lambda_2$ are the smallest eigenvalues of the Laplacian in Ω subject to the boundary condition in (5.2), φ_1 a positive eigenfunction corresponding to the eigenvalue λ_1, $v \in L^2(\Omega)$ with $\int_\Omega v(x)\varphi_1(x)dx = 0$, and $c \in \mathbb{R}$. The following was proven in [5]:

For each $v \in L^2(\Omega)$ there exists a real number $D(v)$ such that:

$$\begin{aligned}&\text{(a) if } c > D(v) \text{ then (5.1) has no solution,}\\&\text{(b) if } c = D(v) \text{ then (5.1) has exactly one solution; and}\\&\text{(c) if } c < D(v) \text{ then (5.1) has exactly two solutions.}\end{aligned} \tag{5.3}$$

Such a precise result attracted a great deal of interest motivating questions such as: What if the interval $(g'(-\infty), g'(+\infty))$ contains eigenvalues other than λ_1? Among other applications, equation (5.1) arises in the design of suspension bridges, see [35]. A reasonable conjecture is that if $(g'(-\infty), g'(+\infty))$ contains k eigenvalues then for c sufficiently large and negative the problem (5.1) has $2k$ solutions. Moreover, if $g'(-\infty) \in \mathbb{R}$ and $g'(+\infty) = +\infty$ then for large negative values of c the equation has a large number of solutions. These conjectures are known as the Lazer-McKenna conjecture and the literature is rich in contributions to this ample field. See [35, 36].

When we restrict ourselves to the case where Ω is the unit ball in $\mathbb{R}^N$ and the solutions are radial, using bifurcation arguments one can show the following [15]:

Theorem 5.1. *Suppose*

$$g'(-\infty) < \rho_1([j/2]+1)^2 < \rho_k < g'(+\infty) < \rho_{k+1}. \tag{5.4}$$

Then for t negative and of sufficiently large magnitude the problem (1.2) *has at least* $2(k-j)$ *radial solutions, of which* $k-j$ *satisfy* $u(0) > 0$.

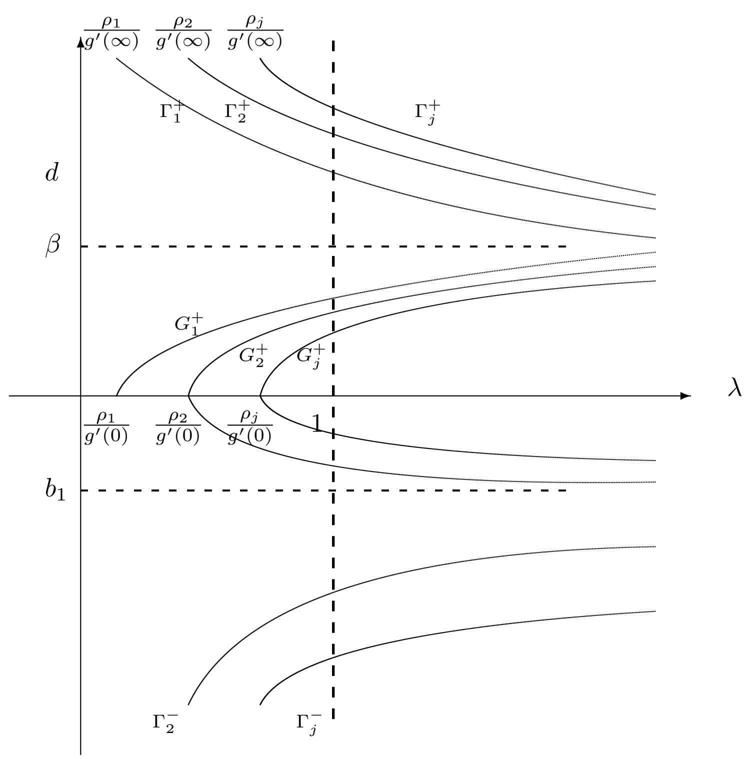

FIGURE 3. Bifurcation curves for g asymptotically linear having a zero at $\beta > 0$.

6. Sub-super critical problems

Motivated by the results in [17], the existence of radial solutions to (1.2) on the unit ball for problems with

$$\lim_{u\to+\infty}\frac{g(x,u)}{u^p}\in\mathbb{R},\quad \lim_{u\to-\infty}\frac{g(x,u)}{-|u|^q}\in\mathbb{R}\quad \text{with } 1<p<\frac{N+2}{N-2}<q<\infty \tag{6.1}$$

was considered in [19] and the following result was established.

Theorem 6.1. *If* (6.1) *is satisfied, then equation* (1.4) *has infinitely many solutions.*

The proof of Theorem 6.1 relies on proving that the energy E satisfies

$$\lim_{d\to+\infty}E(r,d)=+\infty, \tag{6.2}$$

uniformly for $r\in[0,1]$. This is achieved by proving that the Pohozaev energy P satisfies $\lim_{d\to+\infty}P(r,d)=+\infty$ uniformly on compact subsets of $(0,1]$. The main idea behind this proof is that for $d>0$ large, if $x_i<x_{i+1}<x_{i+2}$ are three consecutive zeros of u with $u>0$ on (x_i,x_{i+1}) and $u<0$ on (x_{i+1},x_{i+2}) then

$$\int_{x_i}^{x_{i+1}}s^{N-1}\Gamma(u(s))ds>-\int_{x_{i+1}}^{x_{i+2}}s^{N-1}\Gamma(u(s))ds, \tag{6.3}$$

where

$$\Gamma(u)=NG(u)-\frac{N-2}{2}ug(u). \tag{6.4}$$

Note that $1<p<\frac{N+2}{N-2}<q<\infty$ implies that $\Gamma(u)>0$ for $u>0$ large and $\Gamma(u)<0$ for $u<0$ large.

7. Shooting from singularity to singularity and Laplace-Beltrami problems

In this section we consider an elliptic equation on a differentiable manifold that leads to a singular ordinary differential equation like (1.4) on a bounded interval where the coefficients of $u'(r)$ are singular at both ends of the interval.

Let $M\subset\mathbb{R}^N$, $N\ge 3$, be a compact connected hypersurface of revolution of class C^2 without boundary that intersects its axis of revolution. Without loss of generality we may assume that $\{(0,\dots,0,z);z\in\mathbb{R}\}$ is the axis of revolution and that $P_\pm=(0,\dots,0,\pm1)\in M$. Let $d:M\times M\to[0,\infty)$ denote the *geodesic distance* in M and $a=\max\{d(P_{-1},x);x\in M\}=d(P_-,P_+)$, see [20]. Hence there exist differentiable functions $G,z:[0,a]\to[0,\infty)$ such that:

$$\begin{gathered}G(t)=0\ \text{ if and only if }\ t\in\{0,a\},\\ M=\{(\theta,z(r)):\theta=G(r),r\in[0,a]\},\\ G'(0)=-G'(a)=1,\ z(0)=-1,\ \text{and } z(a)=1.\end{gathered}\tag{7.1}$$

Since M is compact (see [31]) there exists $\epsilon_2>0$ such that

$$U:=\{x\in\mathbb{R}^N:\min\{|x-y|;y\in M\}<\epsilon_2\}=\{x+s\eta(x):x\in M,s\in(-\epsilon_2,\epsilon_2)\}, \tag{7.2}$$

where $\eta:M\to\mathbb{R}^N$ is a differentiable function such that $\eta(x)$ is normal to M at x and $|\eta(x)|=1$ for all $x\in M$ (see [31]). Moreover, $x_1+s_1\eta(x_1)=x_2+s_2\eta(x_2)$ if and only if $x_1=x_2$ and $s_1=s_2$. For $u\in C^2(M,\mathbb{R})$, let $\tilde u:U\to\mathbb{R}$ be defined by $\tilde{u}(x + s\,\eta(x)) = u(x)$, for $x \in M$. The Laplace-Beltrami operator of M at u, denoted $\Delta_M u$, is defined as the restriction to M of the $\mathbb{R}^N$-Laplacian of $\tilde{u}$.

A function $u : M \to \mathbb{R}$ is invariant under rotations of M about its axis of symmetry if and only if $u(x_1, \dots, x_{N-1}, x_N) = u(y_1, \dots, y_{N-1}, x_N)$ for $x_1^2 + \dots + x_{N-1}^2 = y_1^2 + \dots + y_{N-1}^2$. In the Appendix of [20] it is shown that a rotationally symmetric function $u : M \to \mathbb{R}$ is a solution of

$$\Delta_M u + f(u) = q(r) \quad \text{on } M \tag{7.3}$$

if and only if

$$v(r) = u(0, G(r), z(r)) = u(x_1, \dots, x_{N-2}, x_{N-1}, z(r)) \tag{7.4}$$

with $x_1^2 + \dots x_{N-2}^2 = G^2(r)$ satisfies

$$v''(r) + \Big(\frac{(N-2)G'(r)}{G(r)}\Big)v'(r) + f(v(r)) = q(r), \tag{7.5}$$

subject to

$$v(0) \in \mathbb{R}, \quad v(a) \in \mathbb{R}, \quad v'(0) = v'(a) = 0. \tag{7.6}$$

Using this equivalence, in [20] the following result is proven.

Theorem 7.1. *If f is differentiable and there exist $p_1, p_2 \in (1, (N+1)/(N-3))$ such that*

$$\lim_{u\to-\infty} \frac{f(u)}{|u|^{p_1-1}u} \in (0,\infty) \quad \textit{and} \quad \lim_{u\to+\infty} \frac{f(u)}{|u|^{p_2-1}u} \in (0,\infty), \tag{7.7}$$

then equation (7.3) *has infinitely many rotationally symmetric solutions.*

The version of this result for the unit sphere in $\mathbb{R}^N$ and solutions symmetric about the equator was proven in [14]. The additional symmetry reduces the problem to an equation in $[0, a/2]$ which is treated shooting from the singular point 0 to $a/2$, which is a regular point. Equation (7.5) is singular at both 0 and a, so one is shooting from singularity to singularity.

The proof of Theorem 7.1 is carried out shooting from the singular point 0 to the singular point a by considering (7.5) subject to the initial conditions

$$v(0) = d, \quad v'(0) = 0 \tag{7.8}$$

and aiming for d such that

$$\lim_{r\to a} v(r) \in \mathbb{R} \quad \text{and} \quad \lim_{r\to a} v'(r) = 0. \tag{7.9}$$

In summary, the proof of Theorem 7.1 follows from the following steps, see [20].

1. For any $d \in \mathbb{R}$ the solution $v := v_d$ to (7.5) is defined in $[0, a)$.
2. $\lim_{|d|\to\infty}(v^2(r) + (v'(r))^2) = +\infty$ uniformly for $r \in [0, d)$.
3. For each d the solution to (7.5)-(7.8) has finitely many zeros.
4. The number of zeros of v tends to $+\infty$ as $|d|$ tends to $+\infty$.
5. If v is a bounded solution to (7.5)-(7.8) then v satisfies (7.9), that is, u defined by (7.4) is a solution to (7.3).
6. There exists $D > 0$ such that $\{d : |d| \geq D \text{ and } \lim_{r\to a} v_d(r) = \pm\infty\}$ is open.

8. Quasilinear equations

Many of the above results extend to equations of the form

$$-\operatorname{div}(\psi(|\nabla u|)\nabla u)(x) = g(x, u(x)) \quad \text{for } x \in \Omega, \quad u(x) = 0 \text{ for } x \in \partial\Omega. \tag{8.1}$$

Prominent among such equations are *p-Laplace equations* and *k-Hessian equations*. The p-Laplace operator, denoted Δ_p, is given by $\psi(y) = y^{p-2}$, i.e., $\Delta_p u(x) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Note that $\Delta_2 = \Delta$. When u is radial this becomes

$$\Delta_p u(x) = (|u_r^{p-2}|u_r)_r(r). \tag{8.2}$$

To describe k-Hessian equation, recall that the Hessian of a function u at x is given by the matrix

$$D^2(u)(x) = \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}(x)\Big). \tag{8.3}$$

The k-Hessian of u at x, denoted $S_k u(x)$, is given by the k^{th}-symmetric function of the eigenvalues of $D^2 u(x)$. Since the eigenvalues of $D^2 u(x)$ define the sectional curvatures of the graph of u at x and the graph of u is invariant under rotations when u is a radial function, we have $S_k u(x) = S_k u(\|x\|, 0, \dots, 0)$. A direct calculation shows that

$$D^2(u)(r, 0, \dots, 0) = \begin{pmatrix} u_{rr} & 0 & 0 & \dots & 0 \\ 0 & \frac{u_r}{r} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \frac{u_r}{r} \end{pmatrix} (r, 0, \dots, 0). \tag{8.4}$$

Hence, for a radial function u,

$$\begin{gathered} S_1 u(x) = u_{rr}(r) + \frac{N-1}{r} u_r(r) = \Delta u(x) \\ S_k u(x) = \binom{N-1}{k-1}\Big(\frac{u_r(r)}{r}\Big)^{k-1} u_{rr}(r) + \binom{N-1}{k}\Big(\frac{u_r(r)}{r}\Big)^{k} \quad k = 2, \dots, N-1 \\ S_N u(x) = \Big(\frac{u_r(r)}{r}\Big)^{N-1} u_{rr}(r), \end{gathered} \tag{8.5}$$

where $\|x\| = r$. The operator S_N is known as the Monge-Ampere operator. As pointed out in [33], for radial functions the p-Laplacian operator and the k-Hessian operators are particular cases of operators of the form

$$Q(u)(r) = r^{-\gamma}(r^{\alpha}|u_r(r)|^{\beta} u_r(r))_r. \tag{8.6}$$

For a recent result on the existence of radial solutions for equations such as (8.1) we refer the reader to [13] where the following extension of Theorem 6.1 is proven.

Theorem 8.1. *If*

$$g(s) := \begin{cases} |s|^{q_1}, & s \geq 0 \\ -|s|^{q_2}, & s < 0, \end{cases} \tag{8.7}$$

with

$$p - 1 < q_1 < p^* - 1 < q_2 < +\infty, \tag{8.8}$$

where $p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent, then (8.1) *has infinitely many radial solutions.*

9. Appendix A: The contraction mapping principle

A fundamental tool in proving the existence of solutions to differential equations with continuous dependence on parameters such as initial conditions, boundary conditions, or forcing terms, is the following theorem known as the Contraction Mapping Principle. The *Inverse Function Theorem* and *Implicit Function Theorem* are important consequences of the Contraction Mapping Principle.

Theorem 9.1 (Contraction Mapping Principle). *Let (X, d) be a complete metric space and (Y, δ) a metric space. If $f : X \times Y \to X$ is continuous and there exists a real number $a \in [0, 1)$ such that*

$$d(f(x_1, y), f(x_2, y)) \le a\, d(x_1, x_2) \text{ for all } \quad x_1, x_2 \in X, y \in Y, \tag{9.1}$$

then there exists a continuous function $\varphi : Y \to X$ such that $f(\varphi(y), y) = \varphi(y)$.

Proof. Let $\bar{x} \in X$ and $y \in Y$. Consider the sequence in X defined by $x_1 = f(\bar{x}, y)$, $x_2 = f(x_1, y)$, …, $x_{k+1} = f(x_k, y)$. We show that $\{x_n\}$ is a Cauchy sequence. Let k be a positive integer. By (9.1), we have

$$\begin{aligned} d(x_{k+1}, x_k) &= d(f(x_k, y), f(x_{k-1}, y)) \\ &\le ad(x_{k-1}, x_{k-2}) \le \dots \\ &\le a^k d(x_1, \bar{x}). \end{aligned} \tag{9.2}$$

Hence, for $m > n$,

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m-1}) + \dots + d(x_{n+1}, x_n) \\ &\le a^n\, d(x_1, \bar{x}) \left(a^{m-n} + \dots + 1\right) \\ &\le \frac{a^n}{1-a}\, d(x_1, \bar{x}), \end{aligned} \tag{9.3}$$

which implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $\varphi(y) \in X$ such that $\lim_{n\to\infty} x_n = \varphi(y)$. By the continuity of f we have

$$f(\varphi(y), y) = \lim_{n\to\infty} f(x_n, y) = \lim_{n\to\infty} x_{n+1} = \varphi(y), \tag{9.4}$$

hence $\phi(y)$ is a fixed point of $f(\cdot, y)$.

Suppose now that $f(z, y) = z$ and $f(w, y) = w$. From (9.1), we have that $d(z, w) = d(f(z, y), f(w, y)) \le a\, d(z, w)$. Hence $(1 - a)\, d(z, w) \le 0$ which proves that $z = w$. Thus $\varphi(y)$ is unique and so the function $\varphi : Y \to X$ is well-defined.

To show that $\varphi : Y \to X$ is continuous, suppose $\lim_{n\to\infty} y_n = \hat{y}$. Then

$$\begin{aligned} d(\varphi(y_n), \varphi(y)) &= d(f(\varphi(y_n), y_n), f(\varphi(y), y)) \\ &\le d(f(\varphi(y_n), y_n), f(\varphi(y), y_n)) + d(f(\varphi(y), y_n), f(\varphi(y), y)) \\ &\le ad(\varphi(y_n), \varphi(y)) + d(f(\varphi(y), y_n), f(\varphi(y), y)). \end{aligned} \tag{9.5}$$

Hence $d(\varphi(y_n), \varphi(y)) \le d(f(\varphi(y), y_n), f(\varphi(y), y))/(1-a)$. Since f is continuous, the right hand side in the last inequality converges to 0. Hence $\lim_{n\to+\infty} \varphi(y_n) = \varphi(y)$ proving the continuity of φ and completing the proof. ☐

10. Appendix B: Existence of the argument function

Let $x : [0, T] \to \mathbb{R}$ and $y : [0, T] \to \mathbb{R}$ be continuous functions such that

$$\rho(t) = \sqrt{(x(t))^2 + (y(t))^2} > 0 \quad \text{for all } t \in [0, T].$$

Lemma 10.1. *There exists a continuous function $\theta : [0, T] \to \mathbb{R}$ such that*

$$x(t) = \rho(t)\cos(\theta(t)) \quad \text{and} \quad y(t) = \rho(t)\sin(\theta(t)) \tag{10.1}$$

for all $t \in [0, T]$.

Proof. Without loss of generality we assume $x(0) > 0$. Let $\epsilon > 0$ such that $x(t) > 0$ for all $t \in [0, \epsilon]$. For $t \in [0, \epsilon]$, let $\theta_0(t) = \tan^{-1}(y(t)/x(t))$. Then $\cos\theta_0(t) = x(t)/\rho(t)$ and $\sin\theta_0(t) = y(t)/\rho(t)$ and so θ_0 satisfies (10.1) on $[0, \epsilon]$. Let $S = \big\{t \in [\epsilon, T] :$ there exists continuous $\theta : [0, t] \to \mathbb{R}$ such that θ satisfies (10.1) and coincides with θ_0 on $[0, \epsilon]\big\}$, and let $\tau = \sup S$. By definition we know $\tau \geq \epsilon$. Let us see that $\tau = T$ and hence θ_0 is defined on $[0, T]$. Suppose $\tau < T$. If $x(\tau) \neq 0$, there exists $\delta > 0$ such that if $|t - \tau| < \delta$ then $|y(t)/\rho(t)| \leq (1 + |y(\tau)/\rho(\tau)|)/2$. Let k be an integer such that $\theta(\tau) \in (\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi)$ (recall $x(\tau) \neq 0$). Now we define for $t \in (\tau, \tau + \delta)$,

$$\theta(t) = \sin_k^{-1}(y(t)/\rho(t))$$

where $\sin_k^{-1}$ is the inverse of $\sin x$ on the interval $(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi)$. Since $\sin_k^{-1}$ is continuous, we have a contradiction to the definition of τ, and so $\tau = T$, proving the lemma. $\square$

References

[1] H. Aduén, A. Castro; Infinitely many nonradial solutions to a superlinear Dirichlet problem, *Proc. Amer. Math. Soc.*, 131 (2003), no. 3, 835-843.

[2] H. Aduén, A. Castro, J. Cossio; Uniqueness of large radial solutions and existence of nonradial solutions for a superlinear Dirichlet problem in annuli, *J. Math. Anal. Appl.*, 337 (2008), no. 1, 348-359.

[3] A. Aftalion, F. Pacella; Qualitative properties of nodal solutions of semilinear elliptic equations in radially symmetric domains. *C. R. Math. Acad. Sci. Paris*, 339 (2004), no. 5, 339-344.

[4] J. A. Aguilar, I. Peral; Positive radial solutions of quasilinear equations involving supercritical growth, *Nonlinear Differential Equations Appl.*, 5 (3): 309–331, 1998.

[5] A. Ambrosetti, G. Prodi; On the inversion of some differentiable mappings with singularities between Banach spaces, *Ann. Mat. Pura Appl.*, 93(1972), 231–247.

[6] V. Ardila, J. F. Caicedo, A. Castro; Existence of non-degenerate continua of singular radial solutions for several classes of semilinear elliptic problems, *Milan J. Math.*, Vol. 82 (2014) 313—330.

[7] A. Bahri, J. M. Coron; On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.*, 41 (3): 253–294, 1988.

[8] G. Bognár, P. Drábek; The p-Laplacian equation with superlinear and supercritical growth, multiplicity of radial solutions, *Nonlinear Analysis*, 60 (4): 719-728, 2005.

[9] A. Castro, M. Clapp; Upper estimates for the energy of solutions of nonhomogeneous boundary value problems, *Proc. Amer. Math. Soc.*, 134 (2006), no. 1, 167–175.

[10] A. Castro, J. Cossio; Multiple radial solutions for semilinear Dirichlet problem in a ball, *Rev. Colombiana de Mat.*, 101 (1) (1993), 15–24.

[11] A. Castro, J. Cossio; Construction of a radial solution to a superlinear Dirichlet problem that changes sign exactly once. Contributions to nonlinear analysis, 149–160, *Progr. Nonlinear Differential Equations Appl.*, 66, Birkhäuser, Basel, 2006.

[12] A. Castro, J. Cossio, S. Herrón, C. Vélez; Infinitely many radial solutions for a p-Laplacian problem with indefinite weight. *Discrete Contin. Dyn. Syst.*, 41 (2021), no. 10, 4805–4821.

[13] A. Castro, J. Cossio, S. Herrón, R. Pardo, C. Vélez; Infinitely many radial solutions for a sub-super critical p-Laplacian problem, *Ann. Mat. Pura Appl.* (4) 199 (2020), no. 2, 737–766.

[14] A. Castro, E.M. Fischer; Infinitely many rotationally symmetric solutions to a class of semilinear Laplace-Beltrami equations on spheres, *Canadian Mathematical Bulletin,* Vol 58 (2015), no. 4, 723–729.

[15] A. Castro, S. Gadam; The Lazer McKenna conjecture for radial solutions in the R^N ball, *Electron. J. Differential Equations*, 1993 (1993) no. 101, 1–6.

[16] A. Castro, H. Kuiper; On the number of radially symmetric solutions to Dirichlet problems with jumping nonlinearities of superlinear order, *Trans. Amer. Math. Soc.*, 351 (1999), no. 5, 191–1945.
[17] A. Castro, A. Kurepa; Infinitely many radially symmetric solutions to a superlinear Dirichlet problem in a ball, *Proc. Amer. Math. Soc.*, 101 (1): 57–64, 1987.
[18] A. Castro, A. Kurepa; Energy analysis of nonlinear singular differential equations and applications, *Revista Colombiana de Matemáticas*, Vol XXI (1987), 155–166.
[19] A. Castro, J. Kwon, C.M. Tan; Infinitely many radial solutions for a sub-super critical Dirichlet boundary value problem in a ball, *Electronic Journal of Differential Equations*, 2007 (200) No. 111, 1–10.
[20] A. Castro, I. Ventura; Shooting from singularity to singularity and a semilinear Laplace-Beltrami equation, *Bollettino della Unione Matematica Italiana*, https://doi.org/10.1007/s40574-019-00191-y.
[21] J. Cossio, S. Herrón; Existence of radial solutions for an asymptotically linear p-Laplacian problem, *J. Math. Anal. Appl.*, 345 (2008), 583–592.
[22] J. Cossio, S. Herrón, C. Vélez; Infinitely many radial solutions for a p-Laplacian problem p-superlinear at the origin, *J. Math. Anal. Appl.*, 376 (2011), 741–749.
[23] M. G. Crandall, P. H. Rabinowitz; Bifurcation from simple eigenvalues, *J. Functional Analysis*, 8 (1971), 321–340.
[24] K. Deimling; *Nonlinear functional analysis*, Courier Corporation, 2010
[25] E. DiBenedetto; $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.*, 7-8 (1983), 827–850.
[26] J. Dolbeault, I. Flores; Geometry of phase space and solutions of semilinear elliptic equations in a ball, *Trans. Amer. Math. Soc.*, 359 (2007), no. 9, 4073–4087.
[27] A. El Hachimi, F. De Thelin; Infinitely many radially symmetric solutions for a quasilinear elliptic problem in a ball, *J. Differential Equations*, 128 (19196), 78–102.
[28] I. Flores, M. Franca; Phase plane analysis for radial solutions to supercritical quasilinear elliptic equations in a ball. *Nonlinear Anal.* 125 : 128–149, 2015.
[29] M. García-Huidobro, R. Manásevich, K. Schmitt; Positive radial solutions of quasilinear elliptic partial differential equations on a ball, *Nonlinear Analysis*, 35 (1999) (2), 175-190.
[30] B. Gidas, W. M. Ni, L. Nirenberg; Symmetry and related properties via the maximum principle, *Comm. Math. Phys.*, 68 (1979), no. 3, 209-243.
[31] V. Guillemin, A. Pollak; *Differential Topology*, AMS Chelsea Publishing, 2014.
[32] S. Herrón, E. Lopera; Signed radial solutions for a weighted p-superlinear Problem, *Electronic Journal of Differential Equations*, 2014 (2014), no. 24, 1-13.
[33] J. Jacobsen, K. Schmitt; The Liouville-Bratu-Gelfand problem for radial operators, *J. Differential Equations*, 184 (2002), no. 1, 283-298.
[34] J. Jacobsen, K. Schmitt; *Radial solutions of quasilinear elliptic differential equations,* Handbook of differential equations, 359-435, Elsevier/North-Holland, Amsterdam, 2004.
[35] A. C. Lazer, P. J. McKenna; Large amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, *SIAM Review*, 32 (1990), 537-578.
[36] A. C. Lazer, P. J. McKenna; On a conjecture related to the number of solution of a nonlinear Dirichlet problem, *Proc. Roy. Soc. Edin.* 95A (1983), 275-283.
[37] S. I. Pohozaev; On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR*, 165 (1965), 36-39.
[38] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations,* CBMS Regional Conference Series in Mathematics, 65. 1986.

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