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Stationary Distribution of Recombination on 4x4 Grid Graph as it Relates to Gerrymandering

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Abstract

A gerrymandered political districting plan is used to benefit a group seeking to elect more of their own officials into office. This practice happens at the city, county and state level. A gerrymandered plan can be strategically designed based on partisanship, race, and other factors. Gerrymandering poses a contradiction to the idea of "one person, one vote" ruled by the United States Supreme Court case *Reynolds v. Sims* (1964) because it values one demographic's votes more than another's, thus creating an unfair advantage and compromising American democracy. To prevent the practice of gerrymandering, we must know how to detect a gerrymandered plan. We can use math to quantify districting plans to test if they were gerrymandered. To do this, a widely used method is randomly sampling plans to get a baseline to test if the current or proposed plan is an outlier. If the plan is an outlier, then it can be argued that the plan was gerrymandered. Recombination is a Markov Chain and is one method to sample; however, the distribution it samples from is unknown and therefore presents a problem. In this thesis I examine the four-by-four grid graph and find the stationary distribution of Recombination. I analyze the 117 possible districting plans that arise from a four-by-four grid graph and what each stationary probability means. This new insight will be added to the collective work at understanding gerrymandering and how to mathematically detect and prevent it. Such analysis of Recombination Markov chains has been successfully used in court in Pennsylvania and North Carolina.

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Chapter 1

Introduction

Gerrymandering, the process of strategically designing political districts to win the majority of seats for a party's candidate, is detrimental to the "one person, one vote" framework, as ruled in the Supreme Court case *Reynolds v. Sims* (1964). The "one person one vote" concept also comes up in the context of Equal Protection in the United States Constitution under the 14th Amendment (1). Because detecting gerrymandering can be subtle, there are now substantial efforts to mathematically detect and quantify gerrymandered plans.

The case of Thomas Hofeller showed how large data can change the process of redistricting. After his death, tens of thousands of his files were made public and exposed the deep connection he had to many gerrymandered districting plans across the nation (13).

From here we begin to recognize the importance of finding, and subsequently using, data on districting plans to understand whether or not the plans are gerrymandered. Often when it comes to the discretion of a judge or panel of judges, the plans can appear reasonable. They can pass the visual appearance test, the district sizes are relatively even, and there's a proportional outcome. Yet even when all this remains true, a plan can still be gerrymandered.

A research group at Duke University called Quantifying Gerrymandering studies potentially gerrymandered districting plans to mathematically find unbiased evidence for or against gerrymandering. One case the group assessed was about the North Carolina general assembly districting plan. Through the use of Markov chains, specifically Flip walks, they concluded that the plan under-elects Democrats (30). Flip walks offer a mode of quantifying gerrymandered plans but can be slow and there is no proof of whether or not it is mixing fast. The researchers also used Flip walks on Maryland, a state that Republicans have argued use partisan gerrymandered plans to benefit Democrats, but their findings were inconclusive (25).

Other methods, such as the efficiency gap and significance without mixing have been used in states such as Wisconsin (8) and Pennsylvania (20) to prove gerrymandered districting plans. These methods do not always work across state lines as we see in Massachusetts (19).

Moon Duchin, an associate professor at Tufts University, was involved in studying Massachusetts districting plans for whether or not they are Democrat-favoring partisan gerrymandered plans with the use of Recombination, a Markov chain method (18). She found that the efficiency gap is not a useful tool in quantifying Massachusetts because the Republican population is spread evenly across the state and there is no way in which a districting plan can be drawn to have a Republican majority. She concluded that each state must be examined individually, there is not one way in which plans can be determined gerrymandered or not (19).

Recombination is a method of using Markov chains and spanning trees to establish a baseline for plan comparison. It is an alternative to Flip walks because it seems faster and involves fewer parameters. This method has faults: there is a lack of knowledge about how well it mixes and the stationary distribution is unknown in most cases. Thus, in this thesis we calculate the stationary distribution of Recombination for a 4×4 grid graph. This can be found because there are only 117 possible plans.

We find that out of the 117 plans, they can be categorized into 22 distinct groups based on their symmetry. Many of the plans are rotations and reflections of each other, therefore they have the same probability as one another. Out of the 22 groups, there are four probability ranges that the plans center around. The first is all of the plans with probability around 0.003 that do not have a square district. See Figure 5.1 as an example of the plan with the smallest probability. The second is all the plans with probability around 0.009 that have one square district, and the third are the plans with probability around 0.024 that have two square districts. The final and fourth group is the singular plan with all four districts as squares with probability about 0.196, see Figure 5.10. These findings support the idea that the stationary distribution is related to the number of spanning trees, and square districts have more spanning trees than the others. This was surprising to discover, and we hope to eventually extrapolate these methods to be used in the state level.

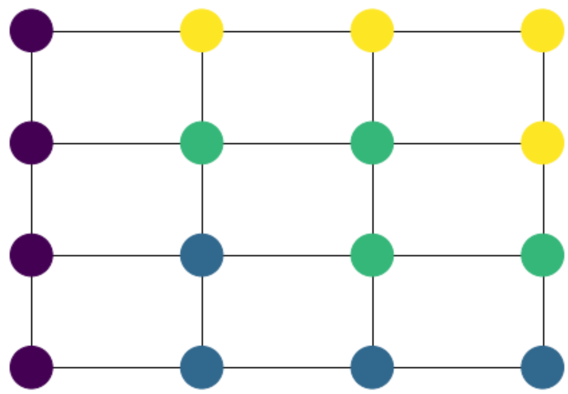


Figure 1.1 Smallest probability in all 22 distinct weights, probability = 0.0030503098, 8 rotations/reflections

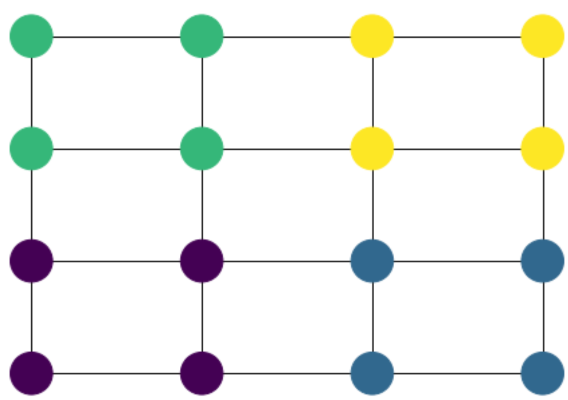


Figure 1.2 Largest probability in all 22 distinct weights, probability = 0.1966606746, 1 rotation/reflection

Chapter 2

Gerrymandering, Voting, and the Use of Data in Redistricting in the United States

Arguably gerrymandering has existed since the beginning of the districting process in the United States. Gerrymandering is the act of politicians altering districting plans to ensure a majority of that states' or cities's seat to benefit their party. For the purposes of this discussion, we will focus on state-level gerrymandering, though examples from cities will arise. The phrase "gerrymander" dates back to 1812 "by those who felt aggrieved by what they saw as the unfair abuse of the districting process." (Gerrymandering and the Construction, 200). The term originated from Governor Elbridge Gerry who signed a bill that created oddly shaped districts that gave his party an advantage in the Massachusetts State House election. An artist drew claws, a head, and wings on Gerry's home district and declared that it looked like a salamander. Thus, the name "Gerry-mander"(4) came into popular vernacular. Arguably the first gerrymander, or "henrymander"(4) as it is now referred to, occurred in Virginia's first congressional elections in 1789 when Patrick Henry managed to secure his election over James Madison.

Districting plans can be gerrymandered to "pack" or "crack" a certain population to limit the power of their votes. Packing a district means that one population or demographic that votes similarly is put into one district which diminishes the power of their votes overall. Though that population wins in one district by a vast majority, their votes lose in the other districts because there is not a large enough population to get a majority vote. Cracking means dispersing a population into many districts, making it nearly impossible for that population to win the majority of votes in any of the districts. Packing and cracking aim at generating wasted votes for the opponent and thus reducing opponents' number of seats. These are two main tactics to create a gerrymandered plan, but not the only ones. According to Maryland Governor Larry Hogan, gerrymandering is "one of the biggest problems we have in America." (23).

There are now substantial efforts to upend continued gerrymandering tactics that remain in use today. However, detecting a gerrymandered districting plan can be incredibly difficult. Bizarre shapes and disproportional outcomes are often misleading indicators of a gerrymandered plan. An example of a bizarre shape is Illinois's "earmuff" (32) district, or the fourth Congressional district, that is shaped like an earmuff. Though appearing gerrymandered because of its odd shape, the district was court mandated to group the Latino population together. Basically, unusual shapes do not necessarily constitute gerrymandering. Similarly, through examination of the Massachusetts case discussed in Chapter 3, Duchin et. al. (19) demonstrate that the existence of benign and structural obstructions to secure representation do not solely rely on the vote count but also how the votes are distributed around the state.

The failure of accountability towards these new districting plans leads us to this research. This project is a puzzle piece in the larger goal of being able to mathematically identify gerrymandered plans. Focusing on an algorithm that is deeply discussed later in this paper, we were able to help understand a tool that is used to quantify gerrymandering, specifically in a four-by-four grid graph. Though this example is minute compared to a real world scenario, understanding the smaller problems must be solved before we can be sure of the full picture.

2.1 Historical Gerrymandering

The party that holds a Congressional majority usually has the power to determine districting plans in the state. The politicians affiliated with the majority-holding party have strong incentives to influence districting plans to ensure seats via votes. In the early 19th century, states were required to choose when and how to redistrict and when a new party captured control of the state government and the "probability of a redistricting event spiked." (21) The goal was, and still is to this day, to maximize the number of seats their party could win on Election Day. For example, in the April 1892 volume, the Atlantic Monthly wrote that Democratic voters in Kansas "have no more hope of being represented in Congress at Washington than if they had no vote at all." (21) This was due to gerrymandering. Since districting plans fundamentally alter the course of public policy and have lasting effects on American politics, it is critical to ensure an equal opportunity for all votes to have equal weight in an election.

Due to minorities tending to vote Democrat, in 1995 the Supreme Court ruled in *Miller v. Johnson* that race can continue to be a factor in drawing districts as long as the primary reasoning behind the districting is partisanship rather than race (21). Race can be a factor in designing a districting plan but cannot be the predominant factor. Since minority populations are predominantly Democratic voters, implicit racial gerrymandering can still be used. This means that, although racial gerrymandering is illegal by the Voting Rights Act of 1965 passed by Congress with the intention of reversing the systematic disenfranchisement of African Americans primarily in the South, racial gerrymandering can be masked by partisan gerrymandering (21). Before the Voting Rights Act, parties were more likely than their current-day parties to draw districting plans with competitive congressional districts in order to guarantee a boost in the number of seats they held (21).

2.2 Voting Patterns in the United States

Gerrymandered districting plans are based off of demographic data about who votes, where they vote, and who they vote for. By using this data, those who draw districting plans can do so in such a way that packs and cracks certain groups to disable their voting power and maximize seats for the given political party. Before diving into how to identify a gerrymandered plan, we must have a deeper understanding of voting in the United States.

We analyze the example of the 2012 and 2016 presidential elections compared to previous elections. Though this data is dated, as we approach the 2020 presidential election it becomes increasingly important to know voter demographics. Voter demographics grant us the knowledge of who votes and what to expect in the upcoming election. An estimated 93 million eligible citizens did not vote in the 2012 presidential election (22) which is equivalent to 57.2 percent of eligible citizens voting, a decrease from the 2008 presidential election that saw 62.3 percent of eligible citizens voting (22). Hawaii had the lowest overall state turnout of eligible voters with the count being at 43.6 percent. According to Ronald Brownstein at *The Atlantic*, in 2016, the Census calculated that nearly two-thirds of eligible white voters cast a ballot, African American turnout dropped to 59 percent, and Latino turnout remained at around 48 percent (9). These voting patterns extend beyond race to age. About 46 percent of eligible voters under 30 turned out which was far below the involvement of those 45 and older. Similarly, voting patterns extend to education levels. According to calculations done by Michael McDonald, a University of Florida political scientist, about 45 percent of eligible white voters without a college degree voted (9). Those creating gerrymandered plans can use this data to place voters in certain districts based on their demographic's voting patterns.

Knowing the growth possibility of a previously underutilized demographic can benefit those desiring gerrymandered districting plans because they can plan accordingly for the size of districts. Among voting patterns across race, age and education level there is also a question of sex. When women were granted the right to vote, the Census Bureau had yet to track voter turnout which means there is no data on the pace at which women began practicing their right to vote (34). The Census Bureau began tracking voter turnout in 1964. Luckily, in 1920 federal census data and local voting records tracked Chicagoans. That data shows that 46 percent of women voted in the presidential election compared to 75 percent of men. And although not on the presidential election level, in the 1923 Chicago mayoral election, 35 percent of eligible women voted compared to 63 percent of men (34). Two main factors played into the lack of eligible women voters: "disbelief in woman's voting" (34) and "objections of husband."(34) Compared to 17.2 percent of men, 32.8 percent of the female respondents in the study claimed their main reason for not voting was because of general reluctance to engage with politics or to the particular election that year (34). Fast forwarding 60 years, in 1980, 59.4 percent of eligible women voted in the presidential election compared to 59.1 percent of eligible men (34). This drastic difference illustrates the capacity of women voters and the ability to captivate a previously underutilized group of potential voters. Furthermore, since the 1980 presidential election, the proportion of eligible female voters has exceeded the proportion of eligible male voters (6). Nowadays, women tend to vote in higher numbers than men. Part of this could be due in part because women constitute more than half of the population in the United States. In recent elections, women cast almost ten million more votes than men (6). Although there is no difference between a ballot cast by a woman than by a man, it is essential to understand demographic voting trends over time in order to understand how districts can be constructed when redistricting.

According to Brownstein at *The Atlantic*, the 2020 presidential election voter turnout is estimated to exceed previous voting records from past decades, if not the past century (9). The new generation of eligible voters are potentially producing the most diverse electorate in American history. The Democratic voter-targeting firm Catalist projected that about 156 million people may vote in the fall, a 17 million person increase from the 2016 presidential election. With this surge of new and old voters alike, the demographic data becomes essential to understanding the outcome of the election. Michael McDonald, a political scientist at University of Florida, estimates that upwards of two-thirds of eligible voters may vote this fall. If this ends up being the case, the 2020 presidential election would be the highest presidential-year voter turnout since 1908, when 65.7 percent of eligible Americans voted. With this increase in voter turnout comes a new level of responsibility. The stakes for gerrymandered plans are high because a presidential election brings the possibility of a surge in turnout. With a surge in turnout comes more informed demographic voting trends and thus higher stakes in how the data is used to redistrict.

2.3 Counting Population in the United States

There are many ways to count the population of a state in order to calculate how to draw district lines. This is important because districts should have equal populations, as ruled by the Supreme Court case in *Westberry v. Sanders* (1964). The current population calculator is Total Population (TPOP) which was determined by the Supreme Court. TPOP, as the name suggests, counts the total population of the state regardless of age or citizenship. Between TPOP and Citizen Voting Age Population (CVAP) lies Total Voting Age Population (TVAP or VAP). TVAP disregards citizenship but counts for age. People living in the United States who are of eligible voting age are counted under TVAP (2).

One population counting method that is not in practice is the Citizen Voting Age Population (CVAP). CVAP counts the number of citizens eligible to vote as means of calculating population. This strategy effectively disregards underage folks as well as folks who are not legally citizens. The total CVAP is used by a state to create the ideal district population for each district. Then, using this information, each district's variance from the ideal district population would calculate both the least and most populous district and would also compute the total percentage deviation for a redistricting plan as a whole (2). By using the total CVAP, there is an account for the prisoners in the state. This practice is often referred to as "prison adjustment"(2) where the prisoner count goes towards where the person lived before incarceration and not towards the district in which the prison resides. The practice of prisoner adjustment is generally believed to be favorable to Democrats because the count does not inflate the district's population to that of those who cannot vote due to incarceration. However, this depends on the locations of the prisons and is still tied to total population (2).

Additionally, in order to achieve an accurate calculation of CVAP, there must be a citizenship on the 2020 Decennial Census questionnaire (2). The census is important for understanding how to draw a districting plan because it determines who gets counted within the population. The suggestion to add a citizen question on the census by President Trump was quickly dismantled. Unsurprisingly, switching to the use of CVAP "would be advantageous to Republicans and Non-Hispanic Whites." (2) Thus, at least in the near future, CVAP will not be the new standard when redistricting.

Since the United States uses TPOP, to jump to CVAP as the new standard "would be a high leap" (2). Although the manner in which population is counted to establish a new standard for redistricting is important, the party who controls the actual line-drawing process in most instances, possesses a huge advantage that outweighs almost all other factors influencing the redistricting process (2). In general, it matters more who redistricts and not how.

2.4 Automatic Voting Registration (AVR)

In order to ensure a standard for redistricting such as CVAP, TVAP, or TPOP works, there must be voters themselves. Many states are brainstorming and implementing registration efforts. Knowing the constituents who vote helps design districting plans and the more people who vote leads to a deeper and better understanding of voting patterns. One way to increase the number of voters is Automatic Voter Registration (AVR). According to Nathaniel Rakich at FiveThirtyEight, around 16 states plus the District of Columbia have enacted (though in several cases, not yet implemented) some version of AVR (33). AVR creates an opt-out system rather than an opt-in system that is in place in most states. The state, often through the Department of

Motor Vehicles (DMV), automatically registers eligible citizens when they first interact with a government agency.

Evidence exists that AVR works, although it is unknown how many people registered themselves or updated their registrations on their own. In general, “people who were registered through AVR do vote – but not necessarily at the same rate as those who register themselves.”(33) In D.C. in 2018, between 42 and 54 percent of people registered through AVR voted. Comparatively, between 46 and 76 percent of those who did not register through AVR cast ballots (33). The latter data does not account for those whose registrations may have been updated by AVR.

Oregon serves as an example of AVR’s possibilities. On March 16, 2015, the Governor Kate Brown historically signed the nation’s first AVR law which resulted in a jump from 73 percent of eligible voters registered in 2014 up to 90 percent by the 2018 election. That leap was the “most voters per capita per day of any jurisdiction” (33) in the researchers’ sample. The caveat is that people are not offered an opt out possibility at the time of their transaction but instead have 21 days to return a notice to cancel their registration. In contrast, most other states ask people if they want to opt out at the time of the transaction itself. The lack of offering an opt out of registration may lead to someone being unaware of their registration status. In Oregon, voting registration records are public and include home addresses unless the person filled out an application form to keep their information confidential. This can potentially lead to dangerous situations for victims of domestic violence or stalking (33).

Other areas in the United States such as D.C., Rhode Island, and California have utilized AVR differently than Oregon. Councilmember Charles Allen introduced AVR to D.C. and attributes the high turnout rate due to AVR to the fact that before an election each district sends individualized postcards to each registered voter as a reminder to vote. In Rhode Island, the DMV employee asks the customer if they want to opt out of AVR. In theory, this makes it more likely for those who decide not to opt out to know that they are registered and, in turn, more likely to vote when in comparison to Oregon. In the case of California, the state was woefully unprepared for utilizing AVR. Employees were not trained and the outdated computer system the department described as “a 40-year-old dinosaur" (33) was not equipped to handle AVR. California exposed itself to hackers, glitches, and failure to register people in time to vote, resulting in a mess which serves as a cautionary tale of how AVR can create serious problems when government agencies are not competent.

With the positive intention of increasing those who vote comes with downsides. One of which is that AVR assumes that the main reason people do not vote is because they are not registered, which ignores other possible explanations. Costs and benefits must be weighed, but it is very likely that AVR leads to higher turnout for people whose information is made up to date, which in turn makes it easier to contact them and urge them to vote. Thus, the increase in people who vote, the higher the impact of elections and the more thorough understanding of voting patterns (33).

2.5 Thomas Hofeller: The Power of Data

To understand the drastic implications of gerrymandering, we look to Thomas Hofeller as a prominent example. His story illustrates how grand a scale the problem of gerrymandering. Hofeller utilized citizen population and voting data to his advantage and to the advantage of the Republic party (GOP). Hofeller recently passed away and the legacy of his actions remains pertinent to understanding gerrymandering.

2.5.1 Who he was

General

Thomas Hofeller, or as David Daley at *The New Yorker* called the "master of the modern gerrymander" (15), was a Republican Party operative known for his ability to create Republican-winning districting plans. During presentations to other GOP operatives and legislators, he often emphasized caution over protecting information stored and sent on computers. He stressed secure computer networks, never sending emails that should not be made public, keeping computers in private locations, and never leaving work exposed. In training sessions his PowerPoints would say "avoid recklessness" (15), "always be discreet" (15) and warned that "emails are the tool of the devil."(15) He argued that "Redistricting is one of the most profitable and business like investments that the GOP can make. Even if it results in only the gain or preservation of one or two additional congressional seats for 10 years, it is more than worth this investment."(17) A large number of white conservative Republicans and a small number of progressive minorities replaced white moderate Democrats in his gerrymandered plans.

Profit

Hofeller did not draw redistricting plans simply for the benefit of the Republican party, rather his talents came at a high price. According to a letter from 2014 reported by Daley at *The Intercept*, his fee started at $7,500 and he capped his price at $16,000, or $200 an hour for a maximum of eighty hours (17). Knowing that he could end up in court, he charged an additional $325 per hour for depositions and trial testimony. His time included project management and the production of tables and maps. Between the years 2011 and 2017, Hofeller billed the Republican party, including national Republican organizations, tens of thousands of dollars. The sheer amount of money that was spent on Hofeller's skills magnifies the extent in which people were, and are, willing to go to ensure a winning seat.

Travel

Hofeller travelled throughout the country and did not stay within one state or region of the country. As reported by Daley, more than two dozen of Hofeller's PowerPoint presentations indicate that he travelled across the nation throughout 2009, 2010, and 2011 emphasizing the importance of redistricting in conversations with state legislators (14). Though Hofeller may not have met directly with legislators, he met with party officials responsible for drawing the plans and guided them towards the legislative process. The lack of Hofeller's localization reveals how immense of an issue gerrymandering is and how widespread his actions were.

CVAP / Prison Gerrymandering

According to Daley at *The New Yorker*, Hofeller was a supporter of drawing districts using Citizen Voting Age Population (CVAP) (15). He was a part of the Republican effort to add a citizenship question to the 2020 census, a detail necessary for the use of CVAP. He encouraged the use of prison gerrymandering, an element of CVAP, which counts incarcerated persons who cannot vote. This allowed state legislative lines to be drawn based on the number of citizen voters, which Hofeller argued made it easier to pack Democrats and minorities into fewer districts, which in turn provided an advantage to Republicans.

Hofeller in Trump Era

Hofeller held confidence in the strength of his districting plans but e-mail exchanges in 2016 reported by Daley demonstrate that he was concerned about the repercussions of an anti-Trump wave that could shift state legislature to Democrats. In an e-mail to a consultant to California Senate Republicans in August 2016, Hofeller expressed frustration about Trump's hold over the Republican Party. However, he maintained his confidence that his maps would endure any transferal of power over to Democrats. He called Trump "only a product to this stupidity"(15) and wrote "Even in the coming political bloodbath we should still maintain majority control of the General Assembly."(15) He mocked Democrats and claimed they hold on to the "hope [that] the Obamista judiciary will come to their rescue." (15)

2.5.2 Death and Documents

As reported by *The Intercept*, in August 2018 after Hofeller passed away, all of this information was made public by Hofeller's daughter, Stephanie, when she discovered backups of over 70,000 of his files (14). She contacted Common Cause, a watchdog organization that works on voting rights, who in turn, subpoenaed her to provide them. A handful were made public, including files that led to the removal of President Trump's citizenship question on the 2020 census. Geographic Strategies, a consulting firm co-founded by Hoeller, then sued to prevent more files from entering the public sphere. In early November 2019, a state court in North Carolina ruled tens of thousands of Hofeller's files be made public. Included in those files was his work on maps as well as litigation in states including Texas, Missouri, Arizona, Virginia, Alabama, Massachusetts, Florida, and North Carolina, among others. The state court did not release all of the files; nearly 1,000 files will remain confidential that were specifically Hofeller's. Litigation continues over an additional 135,000 files Geographic Strategies claims as its own. Common Cause released a statement after the court ruling stating, "Now the truth can come out about all of Hofeller's shocking efforts to rig elections in almost every state."(14).

2.5.3 Content of Hard Drives

Contents of his files show that Hofeller was involved in more redistricting cycles than previously imagined. He was already known for limiting the impact of voters of color in North Carolina, Texas, Missouri, and Virginia, and new documents showed he participated in the 2010 redistricting cycle in Alabama, Florida and West Virginia (13). In a *New Yorker report*, his files suggest he was deeply involved in GOP mapmaking nationwide and could lead to more investigations over gerrymandered plans. Contents of his files show that dating back to 2011 Hofeller estimated the CVAP in North Carolina, Texas, and Arizona, among other states, even though drawing maps based off of this is illegal (15).

Some of Hofeller's memos explicitly stated the connection between race and redistricting, an illegal tactic when redistricting, within the Republican strategists at the highest level of the national party. The goal for Republican strategists was to exploit the creation of "majority-minority" seats. The strategy was to pack black voters into a limited number of seats and equate Democrats and minorities in the minds of white voters, especially in the South. Furthermore, some memos showed that top GOP leaders from many states recognized that when they gained control over state legislatures, they also gained control over redistricting. The memos specifically mentioned Michigan, Pennsylvania, Florida, North Carolina, Texas, Ohio, Wisconsin, and many more states. One memo noted that Democrats would "take the hit."(17)

2.5.4 Involvement

Florida

In 2010, a state constitutional amendment barred partisan gerrymandering in Florida. Yet, as *The New Yorker* reports, e-mails and files show that Hofeller communicated with and visited top GOP political operatives in Florida in 2011 (15). As reported by David Daley at *The Intercept*, Hofeller's files contained mapping software programmed with Florida residents' addresses and a spreadsheet from July 2011 named "Florida Minority Senate Data." (13) Though the rows are not labeled, Daley argues that they appear to show minority voting strength across a dozen state Senate districts. The operatives helped organize or draw state legislative and congressional maps that matched the districts that were later enacted. At a trial, the operatives unironically insisted that drawing the maps was merely a "hobby" (13). This did not convince the federal district court judge who, in 2016, concluded that the GOP conducted a stealth redistricting operation that snuck unconstitutionally partisan gerrymandered maps into the public process, a decision upheld by the United States Supreme Court which, in

June 2019, ruled that the congressional map was a partisan gerrymander.

North Carolina

North Carolina, a competitive purple state, has a history of gerrymandering. A purple state means that the state is divided in Republican and Democratic votes. The state's congressional districting map drawn in 2011 was thrown out for unconstitutional racial gerrymandering. North Carolina Republicans claimed that the maps discriminated based on partisanship, not race (15). But in states where minority voters are often Democratic voters, a partisan gerrymander can become a legal way to draw a racial gerrymander. It can be incredibly difficult to distinguish between the two. Thus, in 2016, Rep. David Lewis and the North Carolina legislature contacted Hofeller to redraw the congressional lines. As reported in October, 2019 by Daley at *The Intercept*, Hofeller drafted maps that would give Democrats only one or two seats (16). A review of the records and e-mails found in Hofeller's files raise questions about whether Hofeller unconstitutionally used race data to draw North Carolina's congressional districts.

However, Hofeller was involved with North Carolina years before he was contacted by Lewis. In February 2016, a panel of three federal district court judges struck down his 2011 map as a racial gerrymander in violation of the 14th Amendment's "one person one vote" protections (16). Hofeller was undeterred. Within a week after the ruling, Hofeller returned to drawing districting maps with the intention of portraying them as partisan gerrymandered (16).

Immediately following the ruling, Hofeller's hard drive was filled with possibilities for North Carolina districting plans. According to Daley at *The Intercept*, one of his maps labeled Plan 17A, created 11 GOP districts, guaranteed Democrats one district, and included one toss-up seat that remained 49.5 percent Republican (16). This plan packs more North Carolina Democrats into a single seat that would be, according to Hofeller's calculations, 72.1 percent Democrat and just 27.9 percent Republican. Thus, the Democratic vote is packed. A second map, labeled Plan ST-B, gave 10 seats to Republicans, two to Democrats, and one competitive district with again a 49.5 percent GOP population. All of his plans appear reasonable on their face. They hold counties together, appear contiguous, and score well on compactness tests, according to Daley (16). The maps pass both the eye test and state legal standards, while still providing Republicans as many as 10 or 11 reliable victories. Though none of Hofeller's aggressive plans were used, they indicate the extreme possibilities for redistricting with new-age technologies and mapmaking software.

As reported by *The New Yorker*, Hofeller's congressional districting plan was "perhaps one of the clearest and ugliest gerrymanders in North Carolina – or in the entire nation"(15), specifically the congressional-district line that severs the nation's largest historical black college. North Carolina A&T State University in Greensboro, a majority minority campus, was cut in half in such a precise way that it all but guarantees the college will be represented in Congress by two Republicans for years to come. North Carolina Republicans have long denied the intentionality of diluting black voting power by cracking the university into two districts which is unconstitutional racial gerrymandering. The map also cracks liberal Asheville in two, thus diluting the voting power of Democrats.

Hofeller's files prove what was so long thought of as unconstitutional racial gerrymandering by voting rights advocates but there was little proof of it. He created giant databases that detailed the racial makeup, voting patterns, and residence halls of more than a thousand North Carolina A&T students (15). He also tracked data from tens of thousands of college students across North Carolina concerning their race, voting patterns, and addresses. Some of his spreadsheets have more than fifty different fields for racial, gender, and geographic information, solely for the thousands of college students. He cross-referenced the students' information against the state driver's license files to determine whether these students likely possessed the proper identification to vote. North Carolina Republicans had recently passed one of the strictest voter I.D. laws in the country which rejected forms of identification often used by students, government employees, and racial minorities. Hofeller used the fact that college students and people of color tend to vote Democratic so by organizing them into ability to vote he was able to crack left-leaning communities.

His hard drive contained maps of Greensboro titled "Greensboro Master Race", "Greensboro - Pct Blk - City Only VAP," "Greensboro 45+ BVAP Compactness" and "Greensboro 50+ BVAP Compactness."(15) BVAP is the black voting-age population, meaning the number of eligible to vote African-Americans. Also found in Hofeller's hard drive was a map of North Carolina's 2017 state judicial gerrymander with the black voting-age population by district overlaid on it. Using an algorithm he had carefully curated, Hofeller believed he could accurately predict the outcome of any North Carolina race. He had created maps that he predicted would turnout a 11-2 or even 12-1 Republican map. In his response to the court ruling that his map was an unconstitutional racial gerrymandering, he turned around and drew an even more partisan map that could have elected an all-white delegation. He referred to the congressional map as a solution to the "problem we should be able to remedy." (14)

2.6 Reaction to Gerrymandering Allegations

Unsurprisingly, as reported in *The Intercept* by David Daley, many top Republicans deny in court and in public that gerrymandering gives them any advantage over Democrats (17). In 2017, Chris West, the spokesperson for former Republican Virginia Speaker of the House William Howell, claimed that "The problem is not district lines; the problem is weak candidates who run poor campaigns based on bad ideas," (17) suggesting Democratic candidates are weaker. Furthering this idea, also in 2017, Wisconsin state Rep. Kathleen Bernier, a Republican, told the *Wall Street Journal* "We have better candidates, better issues, and a better understanding of what our constituents want to do."(17) Additionally, top Republican strategists and political operatives have admitted to exploiting racial data and the Voting Rights Act in order to flip the South red and tilt electoral maps in their direction (17). However, gerrymandering is not partisan, though Hofeller may have amplified Republicans' data collection for gerrymandered plans. Massachusetts fell under scope for Democratic partisan gerrymander but as discussed later, was found not to be a gerrymander. Similarly, Maryland was argued to be a Democratic partisan gerrymander and the results are currently inconclusive. This illustrates the necessity of unbiased nonpartisan tools for detecting and quantifying gerrymandering.

Chapter 3

Mathematical Efforts to Combat Gerrymandering

Non-partisan voting rights advocates recently began using to use math to detect and quantify gerrymandering. The goal is to find a direct and nearly unbiased way to provide evidence of gerrymandered plans so that they can be brought to court and no longer be in use. Though there were some trial and errors with which mathematical methods to use, each method showed that the progress already made and the progress in which we need to work towards.

3.1 North Carolina: Markov chains and Flip Walks

In this section we discuss Hofeller's engagement with North Carolina redistricting and how it was brought to court, along with the math that supported the rulings. In both a state court ruling and the Supreme Court. A research group with Duke University used Markov chains with Flips to provide evidence that the plans were gerrymandered.

3.1.1 North Carolina State Court Ruling

In September 2019, a state court threw out the North Carolina congressional districting map drawn in 2016 for being an unconstitutional partisan gerrymander favoring Republicans (17). The state was carefully curated to adhere to a Republican dominance but that was upended. The House map drawn by Republicans, and now known to be highly involved with Hofeller, all but guaranteed the party's control of 10 of the state's 13 House districts (16). However, if Democrats continued to win about half of the statewide vote under a fair map, they could win at least six out of the state's 13 congressional seats, which is three more than they have right now (16). Those seats could increase the Democrat's majority in the House of Representatives in 2020 or potentially give Democrats control of the chamber.

In 2016, one of the map's drafters now-famously boasted that he had given Republicans a 10-to-3 edge in seats "because I do not believe it's possible to draw a map with 11 Republicans and two Democrats." (36) Mark Joseph Stern reports that upon presenting the plan, the legislature in charge of redistricting stated "I think electing Republicans is better than electing Democrats", "so I drew this map to help foster what I think is better for the country."(35)

The three-judge state court ruled that the map cannot be used in the 2020 election. Even if the 2020 primaries are delayed, the importance lies in a lawful election. The state panel stated that the map violated provisions in North Carolina's Constitution, such as freedom of speech and assembly and equal protection under the law, which protects citizens' "right to vote on equal terms."(35) The state panel also noted that the map violated the guarantee of free elections. This decision to invalidate the state's current maps wreaks havoc on the voting process just months before congressional candidates must be elected from districts that no longer exist. Reported by *The Intercept*, the most likely approach now is that the state legislatures will need to quickly draw and win court approval of new maps. The court "respectfully urges the General Assembly to adopt an expeditious process" of redistricting "that ensures full transparency and allows for bipartisan participation and consensus." (35) Michael Wines with the New York Times reports that in attempts to appear non-biased, party leaders publicly announced they would redraw the map using partisan rather than racial parameters (36).

3.1.2 Post State Court Ruling, United States Supreme Court Ruling

In January 2018, according to *The New York Times*, the US Supreme Court refused to invalidate this exact same gerrymander in *Rucho v. Common Cause* (28). The United States Supreme Court temporarily blocked the trial court's order requiring a revised congressional districting map in North Carolina. The state court's decision to invalidate this plan is a test case for the fight against partisan gerrymandering. Wines writes, "it has been a triumph for voting rights." (35) According to Adam Liptak and Alan Blinder with *The New York Times*, the United States Supreme Court has never struck down a voting map as an unconstitutional partisan gerrymander but has done so in regards to racial gerrymandering (28). J. Michael Bitzer, a political scientist at Catawba College in North Carolina, said the Supreme Court's order was an important, if perhaps temporary, win for Republicans in North Carolina (28). The Democrat lawyers' brief stated that "the Republican contingent of the legislature wants to enjoy the fruits of their grossly unconstitutional actions for yet another election cycle."(28) The state legislature ultimately settled on the 10-3 districting map. According to *The Intercept*, in 2018 Democrats won their districts with 69.9, 75.1, and 73.1 percent of the vote, while every Republican running a contested race landed safely in the 50s (16). Though Democrats won more total votes, the map spread Republican votes more effectively and efficiently, guaranteeing a Republican win.

3.1.3 Mathematics Analysis of 2017 North Carolina General Assembly Districting Plan

In September 2019, Jonathon Mattingly with the nonpartisan research group at Duke University called Quantifying Gerrymandering, published an article outlining the group's analysis of the 2017 North Carolina General Assembly redistricting plan (30). By representing the state as a graph and each node as a precinct, the group was able to mathematically quantify the districting plan.

Through the use of Markov chains to build a baseline to compare to, Mattingly's group used the common Flip walk, which randomly changes the district of of a single node at a time. This method is slow and offers no proof of whether or not it is mixing fast, meaning we sample from something close to intended distribution or not. However, one of their central findings was that the legislature's redistricting plan implemented a firewall to protect Republican majorities and supermajorities. Linked on their site, Quantifying Gerrymandering illustrates the range of Democratic seat counts shifted with the statewide fraction of Democratic votes under various shifts to historical elections through an animated bar-graph. The group emphasizes the United States Senate vote in 2016. When the statewide Democratic vote fraction is below 49 percent the enacted plan elects a typical number of Democrats when compared to the ensemble the group created. When the Democratic vote fraction increases to 50.5 percent and over 52 percent, almost all of the plans in the ensemble break the Republican supermajority. However, the enacted plan continues to elect fewer than 48 Democrats to the state House. Furthermore, as the Democratic vote fraction increases to 54.5 percent, the enacted plan elects fewer Democrats than the ensemble where nearly all of the plans in the ensemble predict a Democratic majority in the House. When the Democratic vote fraction surpasses 55 percent, plans in the ensemble illustrate a strong majority to the Democrats yet the Republicans retain their majority in the enacted plan. Consistently, as the Democratic vote fraction increases, the enacted plan elects fewer Democrats than the ensemble. Quantifying Gerrymandering strongly emphasizes the fact that "the story is NOT about proportional representation"(30), but rather about how the 2017 North Carolina General Assembly districting plan systematically under-elects Democrats "to a shocking degree."(30) The group continues with examples from the 2008 United States Senate votes, the 2012 Governor election, and the 2016 Lt. Governor votes all in North Carolina, all intentional partisan gerrymandering that cannot be explained away by natural packing due to geography. Mattingly and his co-researhcers end the article by saying "if you are worried about the state of our democracy, you should be."(30)

3.2 Maryland: Markov Chains and Flip Walks

Similar to Massachusetts, Maryland is highly contested for having gerrymandered districts in favor of Democrats. In an interview with Lulu Garcia-Navarro on NPR in January 2018, state Delegate Kirill Reznik, a Democrat, argued that he does not support Republican Governor Larry Hogan's proposals to change the process of redistricting (23). He argued that since he thinks there are more states that have a Republican redistricting problem than a Democratic one, before implementing an independent commission like California, Virginia must do so as well. His idea is that to be fair and equitable, a Republican-dominant state with a potential gerrymandering issue must also commit to creating an independent commission to redraw districts.

Mattingly, with Quantifying Gerrymandering, has informally conducted research using Flips on Maryland (25). He and his fellow researchers are inconclusive in their results (25). However in November 2018, a three-judge federal court panel ruled that after the 2010 Census the state unconstitutionally drew districts to benefit Democrats (7). Late last March in 2019, an attorney representing a group of Maryland Republicans urged the United States Supreme Court to end the practice of drawing sharply partisan congressional districts. Some of the justices appeared hesitant to rule in an area that for a long time in the domain of states. In the arguments, Maryland Attorney General Brian Frosh claimed he was not defending gerrymandering but rather wanted the Supreme Court to establish rules and standards when Maryland redraws the map. Former Maryland Secretary of State John Willis provided historical information to the state about the case and he posed the question, "whatever the result, the real question for the court is, 'Should the judiciary get engaged more than it has previously?'"(7) His question indicates the thin line separating national versus state judiciary and mirrored some of the justices' reactions to rule in this case. Representative David Trone, a Democrat who holds Maryland's 6th District seat, furthered the posed question when he disagreed with the lower court's ruling and said that a national solution to gerrymandering isn't just necessary, it is required.

3.3 Wisconsin: Efficiency Gap

Unconstitutional partisan gerrymandering can be extremely difficult to detect. To diminish this difficulty, Mira Bernstein and Moon Duchin, mathematics researchers, used a simple formula called the efficiency gap (EG) (8). Though not used in this thesis, the EG's goal is to detect and reject gerrymandered congressional and legislative maps that are aimed at keeping one party dominant over another. EG does not require multiple elections to pull data from, it can be computed based on a single election. If the result surpasses a certain threshold, then the districting plan is found to have a discriminatory partisan effect and is therefore determined gerrymandered. According to Bernstein and Duchin, the efficiency gap fundamentally measures whether the seat share S is close to $2V - 1/2$, where V is the vote share (8). The efficiency gap, $EG = 2V - S - 1/2$ arguably can identify a legally actionable gerrymander when its magnitude is greater than 8 percent. Bernstein and Duchin note that the EG formula counts wasted votes for the winning or losing side, such as votes lost in the impacts of packing and cracking, to be in excess of the 50 percent needed to win. To determine the difference between packed and cracked districts, the EG looks at which side, winning or losing, wasted votes were located. If majority of all of the wasted votes belong to the winning side, then the district is packed. If majority of the wasted votes belong to the losing side, then the district is competitive. Alternatively, if there are multiple adjacent districts where the majority of wasted votes belong to the losing side, then there's a possibility the districting plan is cracked. If the EG is nearly zero, then the plan is fair, meaning that both parties waste about an equal number of votes. Surprisingly, EG does not penalize packing, cracking, or peculiarly-shaped districts. EG is not a perfect solution to detecting gerrymandered plans; it has undesirable properties and limitations that make it not reliable. Although EG is attractive in its simple construction, especially to legal scholars, it is still in its beginning stages of statistical testing and modeling.

In November 2016 for the first time in 30 years, a federal court in Wisconsin overturned a legislative map as an unconstitutional partisan gerrymander (8). One of the main central focuses in the district court's ruling was the high efficiency gap in Wisconsin's 2012 to 2016 elections that favored Republicans. Due to the extremity of this case and the novelty of the EG data, this case is under appeal to the Supreme Court. The Supreme Court's ruling could determine a standardization for detecting gerrymandered plans in the future.

3.4 Pennsylvania: Significance Without Mixing

Pennsylvania, like the other states examined in this chapter, has a history of gerrymandering. According to the Voting Rights Data Institute at Tufts University, in February 2018, two districting plans for Pennsylvania were submitted; one by Governor Tom Wolf, a Democrat, and one by Speaker Michael Turzai and President Pro Tem Joe Scarnati, both Republicans (5). Wolf's plan was fair, constitutional, avoids cracking, packing, and unnecessary splitting of regions. On the contrary, Turzai and Scarnati's plan is an extreme outlier among redistricting plans, according to a detailed analysis and rigorous calculations by Moon Duchin, Association Professor of Mathematics at Tufts University (20). Through production of three billion maps that are at least as compact, preserve at least as many counties, and keep population deviation within the one percent threshold, Duchin claims that the Turzai-Scarnati plan is "overwhelmingly likely to have been drawn to increase partisan advantage." (20) She argues that there is less than a 0.1 percent chance that the plan was drawn in a non-partisan manner. Furthermore, she claims that in contrast, the GOV plan, Wolf's plan, "falls squarely within the ensemble of similar plans" (20) created using non-partisan criteria and therefore indicates that the plan does not favor Democrats.

Wes Pedgen has also used math to bring a Pennsylvanian gerrymandered districting plan to court. Although Markov chains have proven themselves to be an incredible asset to detecting and quantifying gerrymandered plans, in some applications it is unknown how long a chain must run to generate good samples. Often in practice, the required time is too long. The researchers Chikina, Frieze, and Pegden have crafted a new test, still using Markov chains (11). The test is not rigorous without good bounds on the mixing time it takes for the Markov chain to generate samples.

Their tests runs from taking a random walk from the presented state for any number of steps. They prove that an observation of the presented state is an e-outlier on the walk is significant under the null hypothesis that the state was chosen from a stationary distribution. The researchers assume nothing about the Markov chain beyond reversibility (11).

The researchers explain how smaller tests can be conducted at the district-level, where one can compare the difference between the mean and median votes. If the difference is unusually large, then the plan may be gerrymandered. Other ways of measuring district to district can be judged based on generally reasonable differences in statistical properties. The researchers tested their application on a rigorous detection of gerrymandering in Congressional districting. As described in Chikina, Frieze, Mattingly, and Pegden's paper "Separating Effect from Significance in Markov Chain Tests" (10), the math uses Markov chains to set a baseline with typical maps where the invariant distribution is concentrated. From there, the group argued that from there, one can rigorously assess the likelihood of choosing a particular map. The researchers found that in Pennsylvania is 2012, 48.77 percent of voters were cast for Republican representatives and 50.20 percent of voters were cast for Democrat representatives. The election resulted in 13 Republican representatives and 5 Democrat representatives. Thus, the state "spectacularly" (10) failed their test.

Deceiving groups disguised as non-partisan organizations are known to funnel money towards the creation of gerrymandered districting plans. Pennsylvanians Against Gerrymander (PAG) was formed in August 2019 by the Gober Group, an election law firm based in northern Virginia, according to Peter Maass and Lee Fang at *The Intercept* (29). Kathryn Murdoch, the daughter-in-law of Fox News founded Rupert Murdoch, is the co-chair and largest donor to PAG. Their website, as documented by Maass and Fang, states that PAG supports an independent commission to redraw federal district lines in Pennsylvania after the 2020 Census (29). An independent commission is a nonpartisan group who collectively create a districting plan. It is generally favored by advocates of voting reform and it is generally regarded as the ideal way to draw fair plans rather than letting party politicians control it. The rollout of the group was a silent one such that Pat Beaty, the legislative director of the nonpartisan Fair Districts PA, "laughed out loud" (29) when he read Unite America's description of PAG as "leading the legislative advocacy campaign" claiming that he had "never heard of these people."(29)

Contrary to public image, PAG is backed by GOP-inclined lobbyists who fully wish to gerrymander the state into Republican majority. On September 19, 2019, Unite America donated more than $5 million to PAG and three groups that advocate for fairer redistricting in Massachusetts, Alaska, and New York(29). Long, Nyquist & Associates and Maverick Strategies, two of the most powerful lobbying firms in Pennsylvania, and registered as working on behalf of PAG; yet, both firms are regarded as GOP-inclined. Additionally, both firms employ lobbyists who were closely linked to Republican engineered rigged election maps in Pennsylvania after the 2000 and 2010 censuses.(29)

In 2011, Long, from Long, Nyquist & Associates, helped create All Votes Matter, and organization designed to change the way the electoral votes in the state are awarded. The system would ensure a Republican would win the majority of Pennsylvania's electoral college votes even if President Barack Obama had won the state in his 2012 reelection. However, following a national outcry, the All Votes Matter bid failed. Also in 2011, Krystjan Callahan, the previous chief of staff for State Rep. Mike Turzai, was thanked for his work in what election analyst Sean Trende called the "Gerrymander of the Decade."(29) The map was so carefully curated that in the 7th district the city was only connected by a single steakhouse restaurant. The map placed Democrats at a disadvantage and locked 13 House seats for Republicans. In 2012, the map worked as intended. Now, Callahan is a lobbyist for PAG through his work with Maverick Strategies.

Pennsylvanians Against Gerrymandering is not alone in its effort to mask gerrymandering intentions. Apparently, Alan Philip, a contact on PAG webpage for lobbying registration form and website registration, is a Colorado-based Republican consultant who was involved in End Gerrymandering Now, a 2016 ballot measure attempt to reform Colorado's redistricting process. The mask was unveiled when *The Colorado Independent*, a left-leaning news outlet, wondered if the group was "actually a nefarious Trojan Horse plot to tint Colorado red?"(29) People were concerned that the group was a stealth attempt by the GOP to craft rules that favored Republican candidates by insufficiently prioritizing district lines that preserve communities of interest.

3.5 Massachusetts: Fairness vs. Proportionality

Massachusetts came under question when for the third Census cycle in a row, the state held nine to ten seats in the House of Representatives. The state is home to over 6.5 million people, which according to the 2010 Census is up from 6.3 million people in the 2000 Census. This accounts to about two to three percent of the United States' population. After the 2010 Census, the number of Congressional delegates allocated to Massachusetts dropped by one because the state's population did not keep pace with the nation's. According to Duchin et. al. in "Locating the Representational Baseline: Republicans in Massachusetts" (19), the fact that the state holds up to ten seats means that a district can be won with as little as 6 percent of the statewide vote. The underperformance of Republicans in Massachusetts elections is not owed to the lack of Republicans to field House candidates. Rather these trends can be explained by using math to examine the physical distribution of votes throughout the state. Though Republicans carry 30 to 35 percent of the votes statewide, they are so uniformly distributed that there is no way to succeed in a majority. They represent an extreme that is not common in many states in that the statewide percentage of Republicans is the same in each town and precinct rather than having certain towns or precincts having a higher percentage of Republican votes in the statewide count.

The researchers figured out this crucial information about Massachusetts by creating numerous districting plans and illustrating that to have Republicans win would be such an extreme outlier it is nearly impossible (19). The core of their analysis is a rigorous proof that regardless of the districting plan, certain actual observed voting patterns guarantee this idea of a lockout effect. Duchin et. al. assessed the numerical distribution of votes in 13 statewide elections and found that in five of them, "the number alone make it literally impossible to build a R-favoring collection of towns or precincts with enough population to be a Congressional district." (19) The researchers reached the conclusion that extreme representational outcomes, such as the case of Republicans in Massachusetts, are not always attributable to gerrymandering nor to how voters from either party are arranged in the state. To further their point, the researchers looked at the vote shares between George W. Bush and Al Gore. Bush received over 35 percent of the two-way vote share against Al Gore; yet Duchin et. al. concluded that it is “mathematically impossible to construct a collection of town, however scattered, with at least 10 percent of the population and where Bush received more collective votes than Gore.”(19)

As described in the Wisconsin case, the EG is often an incredible resource to detect a gerrymandered plan. Duchin et. al., note the special case of Massachusetts and how EG cannot be universally used (19). They found that in quintillions of possible 9-district plans, not a single plan has an EG below 11 percent in any of the five races in which it was nearly impossible to contrive a Republican-favoring plan. As mentioned previously, it was argued that an EG with magnitude above 8 percent can be used to flag a legally actionable gerrymander. Thus, the researchers conclude that it is essential to understand the subtleties of establishing a reasonable baseline to determine when gerrymandering has occurred (19).

In “A Computational Approach to Measuring Vote Elasticity and Competitiveness” DeFord et. al. also uses Recombination, a method of using Markov chains and spanning trees to establish a baseline to compare to, to address the Massachusetts case (18). Issues with Recombination are the lack of knowledge surrounding how well it mixes as well as not knowing what the stationary distribution is. In the next section, we find the stationary distribution for a previously unknown grid graph.

3.6 Virginia: Recombination

In June 2018, a District Court in Virginia ruled that 11 House of Delegates districts were racial gerrymanders. The three judge panel found that Black residents were isolated in packed districts, thus diluting their votes. Statements in the court confirmed that the 2011 Enacted plan was designed to have greater than or equal to 55 percent Black Voting Age Population (BVAP) in 11 districts. This high elevation of BVAP suppressed the BVAP level in 22 neighboring districts. According to the Voting Rights Act, the range of BVAP values from 37 percent to 55 percent (3). The 2011 Enacted plan was targeted to aim for districts above the 55 percent line.

The Metric Geometry and Gerrymandering Group (MGGG) published a report in November 2018 regarding this gerrymander and titled their report “Comparison of Districting Plans for the Virginia House of Delegates"(3). They decided to use this Markov chain method to construct a baseline for comparison in order to observe and quantify the tradeoff between elevated BVAP in part of the map and broader effects on the districting outcomes (3). Their Markov chain sampling process has population equality, contiguity, and compactness built into the steps, making it a potentially more viable option than the methods described in this chapter.

The researchers begin their algorithm starting from plans previously proposed for adoption and 100 neutral maps. They then performed chains of random alternations, collecting a large sample from the resulting maps as their collection of comparable plans. They created a plot to depict the $20,000$ steps from a Recombination Markov chain, illustrating the plans that do not exceed 60 percent BVAP. The plot diagram indicates evidence of where and how elevating the BVAP in the top 12 districts suppresses the BVAP in the remainder of the plans. Diluting the BVAP impacts not only the areas where it was already very low but also affects districts that were at or nearing the zone in which statistical analysis has indicated opportunities to elect more candidates of choice for the Black community (3).

MGGG found that elevating BVAP in the 2011 Enacted plan causes at least ten and up to 17 other districts to have suppressed BVAP levels, far below what would be expected from race-neutral redistricting. The 2011 Enacted plan has no districts at all in the crucial range of 37 to 55 percent BVAP, while neutral redistricting tells us to expect as many as ten (3).

Hundreds of thousands of race-neutral plans found by Markov chain techniques indicate that without sacrificing population balance, contiguity, or compactness, three additional districts are pushed over the BVAP level. These methods suggest that a substantial share of race neutral plans that comport with traditional districting principles would do so (3). Thus, Recombination is an important tool to use when considering how to detect and quantify gerrymandered plans.

Although Recombination was useful in Virginia, we do not know the stationary distribution. This thesis finds the stationary distribution for a 4×4 grid graph, which hopefully will be expanded to use at the state level.

Chapter 4

Pre-Existing Theory

Due to the recent development of math used to detect and quantify gerrymandering, there are many algorithms currently in use. Often these algorithms use Markov chains, but not exclusively. As research accumulates, each method shows its benefits as well as what it lacks. For the purposes of this thesis, we studied Recombination, a Markov chain. There are many advantages to using this method, however the problem is that the stationary distribution is unknown in most cases. This is bad because we don't know the baseline we're comparing to. We found the stationary distribution for the four-by-four grid-graph case.

4.1 Markov Chains, Transition Matrix, State Spaces

A Markov chain is a memoryless random process on a state space. A Markov chain is a process which moves among the elements of a set Ω by way of probability. At a position x, the next position is chosen according to a fixed probability distribution that depends only on x. The Markov chain is the sequence of random variables of states visited, e.g. X_{i} = state at step i. The memoryless transitions are based off a probability distribution. Memoryless means they lack the ability to produce context-dependent content because they cannot take into account previous states of being. A Markov chain can be visualized as a weighted graph that can have loops and/or walks, where the weights of the loops and walks are probabilities. Markov chains are common in algorithms designed to identify gerrymandered plans because they can quickly generate random samples from large spaces (27).

The probability distribution of state transitions is called the transition matrix. At each position within the transition matrix the entry at $(i, j) = Prob(i, j)$, meaning the probability of where to go if at state i. If the Markov chain has N possible states, then the matrix will be of size $N \times N$. Each row in the transition matrix must sum to one because each row represents its own probability distribution, thus making it a stochastic matrix.

A Markov chain also has an initial state vector, which is a $1 \times N$ matrix. Entry i of the vector describes the probability of the chain beginning at state i. Then, the probability of moving from state i to state j over M steps must be calculated.

In this thesis, the states are districting plans and there are 117 states, or possibilities for district plans in this case, meaning the matrix is 117×117, such that the entry (i, j) is the probability of transitioning from state i to state j. We calculate this value with code, linked in Chapter 5. Instead of calculating the entry (i, i), which is the probability of going from one districting plan to itself, we have calculated all other values and subtracted that sum from one such that the matrix remains stochastic.

4.2 Stationary Distribution

The stationary distribution of a Markov chain is probability distribution π, satisfying $\pi = \pi P$ where P is the transition matrix.

A Markov chain is irreducible if there is a sequence of valid transitions from any state to any other state, meaning for all $x, y \in \Omega$ there is a t such that $P^{\mathrm{t}}(x, y) > 0$. A Markov chain is aperiodic if for all $x \in \Omega$,

$$gcd\{t : P^{\mathrm{t}}(x, x) > 0\} = 1.$$

A Markov chain is ergodic if it is both irreducible and aperiodic. Any finite, ergodic Markov chain converges to a unique stationary distribution given by $x, y \in \Omega$,

$$\pi(y) = \lim_{t \to \infty} x_0 P^{\mathrm{t}}(x, y).$$

The stationary distribution is also modeled with the formula $\lim_{t \to \infty} x_0 P^{\mathrm{t}}$ when Markov chain is finite and ergodic.

In this thesis, we find π by solving $\pi P = \pi$, not by using the limit. As opposed to repeatedly raising a matrix to a power, we use linear algebra to solve for π, which gives us our stationary distribution.

4.3 States as Graphs

In nearly all mathematical methods of quantifying districting plans, states are represented as graphs where vertexes are geographical units such as census blocks or precincts. Often the nodes have population of the region they are present in and the districting plan is a balanced partition, meaning all districts within it are approximately equally sized. For this thesis, all nodes have equal populations and plans must be exactly balanced.

4.4 Recombination

The Markov chain we study in this thesis is Recombination. Recombination follows the structure that if in state i, the next transition is determined by the following algorithm:

1. Pick two random districts, uniformly from all pairs.
2. Take the union of those two districts.
3. Pick a random spanning tree, uniformly among all spanning trees.
4. Pick a uniformly random edge e of that spanning tree.
5. Use e to split the spanning tree.
6. If the two parts that are left are of the same size, make those parts into new districts. Otherwise, remain at plan i.

Each transition probability from i to j can be calculated exactly.

4.5 Kirchoff's Matrix-Tree Theorem

Kirchoff's Matrix-Tree Theorem is used to count the number of spanning trees in a given graph. This theorem is used to compute entries of P. To understand the theorem, we must define terms used in the theorem.

A spanning tree is a connected graph with no cycles. It is a spanning tree of a graph G if it spans G, that is, it includes every vertex of G, and is a subgraph of G if every edge in the tree belongs to G. A vertex set of G, with $V = \{v_1, \ldots, v_n\}$ is a subgraph of a graph is a spanning tree if it is a tree that contains every vertex in V. If the graph on n vertices with $V = \{v_1, \ldots, v_n\}$ then its graph Laplacian is an $n \times n$ matrix whose entries can have three cases. If $i = j$, then the entry is the degree of that vertex. If i does not equal j and the two vertices are connected, then the entry is -1. Otherwise, the entry is 0. Then, to find a minor of the Laplacian, one must take out any row and any column from the Laplacian. Kirchoff's Matrix-Tree Theorem states that the determinant of any minor of the Laplacian is the number of spanning trees of G. This gives us an efficient way to count the number of spanning trees, which we will use to calculate the transition probabilities of P.

Chapter 5

Results and Methods

5.1 Results

We exactly calculated π, the stationary distribution of Recombination, for the 4×4 grid into four districts. The probabilities we found are organized into four classes depending on the number of square districts in the plans. These results are interesting because square districts have a larger amount of spanning trees, thus increasing the probability. Our detailed results are as follows.

Due to reflection and rotation not affecting the probability, we were able to group the 117 districting plans into a total of 22 distinct weights. Each group has a range from one to eight districting plans within it, depending on its reflection and rotation abilities. In the case of the group having one plan, it means that when rotated or reflected, the plan remains the same. All values are in Table 5.1 to Table 5.4.

Out of the 22 distinct weights, we noticed that the values hovered around four values, the smallest grouping of distinct weights is around 0.003 and includes all the plans without a square district. We found that there are 14 values that fall within the 0.0030 to 0.0035 range. See Figure 5.1, which shows the districting plan with the smallest probability out of the 22 distinct weights. The distinct plan with the largest probability within this range has a spiral-like structure, as seen in Figure 5.2.

The second smallest grouping of distinct weights is within the 0.0090 to 0.0092 range, about three times larger of a probability than the smallest grouping. Each of these districting plans, of which there are four distinct districting plans, has exactly one square district in it. See Figure 5.3,

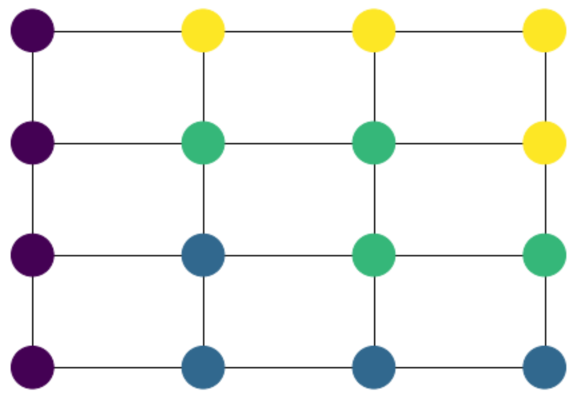


Figure 5.1 Smallest probability in all 22 distinct weights, probability = 0.0030503098, 8 rotations/reflections

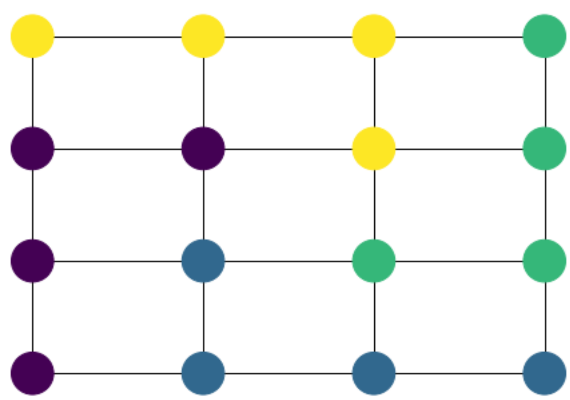


Figure 5.2 Largest probability of the smallest grouping of distinct weights without squares, probability = 0.0034370206, 2 rotations/reflections

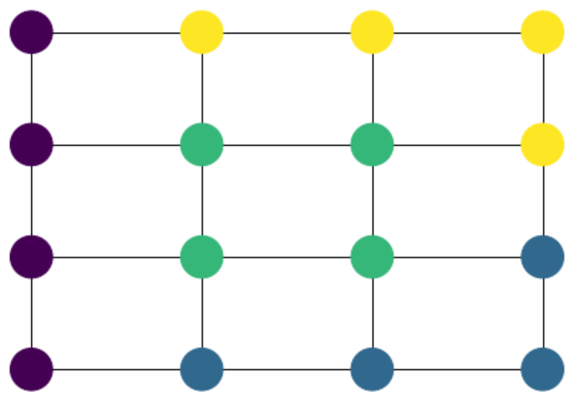


Figure 5.3 Smallest probability in second grouping of distinct weights with one square, probability = 0.0090170844, 4 rotations/reflections

Figure 5.4, Figure 5.5, and Figure 5.6 for reference of each of the four distinct weights within the second smallest grouping. Because the square graph has more spanning trees, it makes sense why this group of plans has a higher probability than the plans without squares. The grouping of districting plans that fall on the larger size with the probability value being 0.0092067382 could be larger than the others because the square district is in the corner of the districting plan. It's possible that this increases the probability because there is more room within the remaining area of the districting plan for the other districts to take a wider range of shapes. The grouping of districting plans that falls on the smaller size within this second grouping with the probability value 0.0090170844 has the square district in the middle of the plan. Thus, this inhibits the potential range of shapes the other districts can take.

The third grouping has three sets of districting plans with probabilities ranging from 0.024 to 0.025. The probabilities within this grouping are eight times larger than the smallest grouping. Every districting plans in this group has two square districts. See Figure 5.7, Figure 5.8, and Figure 5.9 for each of the three distinct ways to craft a plan with two square districts. Because there are fewer ways to draw districting plans with two squares in it, there are understandably fewer plans that hover around this range of probabilities.

The final grouping includes only one districting plan which is the plan with four squares; see Figure 5.10. The probability for this plan far exceeds the rest because of the fact that four square districts means more spanning trees. Because there is only one way to draw a districting plan in a 4x4

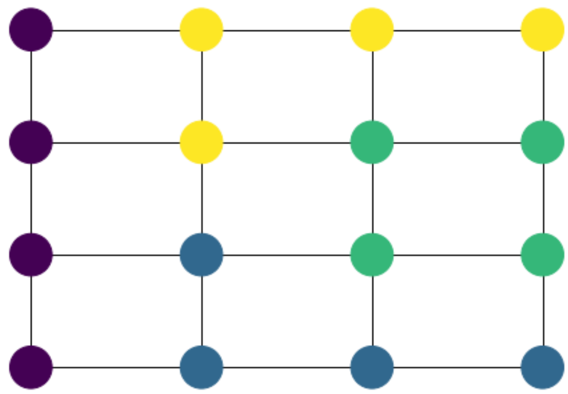


Figure 5.4 Second smallest probability in second grouping of distinct weights with one square, probability = 0.0091650711, 4 rotations/reflections

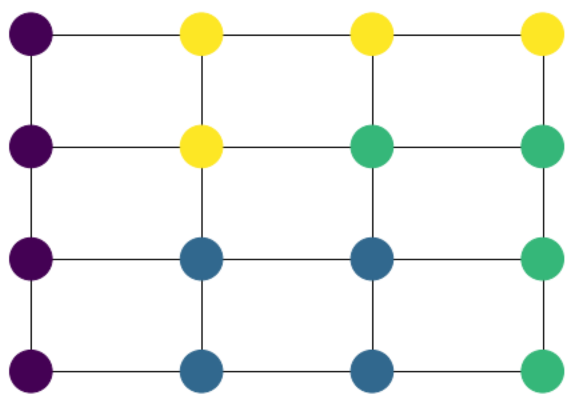


Figure 5.5 Third smallest probability in second grouping of distinct weights with one square, probability = 0.0091653883, 8 rotations/reflections

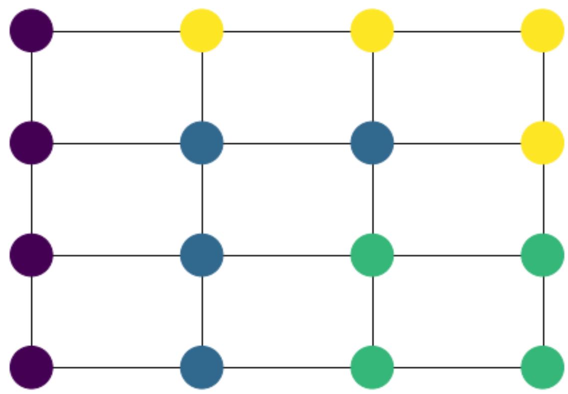


Figure 5.6 Largest probability in second grouping of distinct weights with one square, probability = 0.0092067382, 8 rotations/reflections

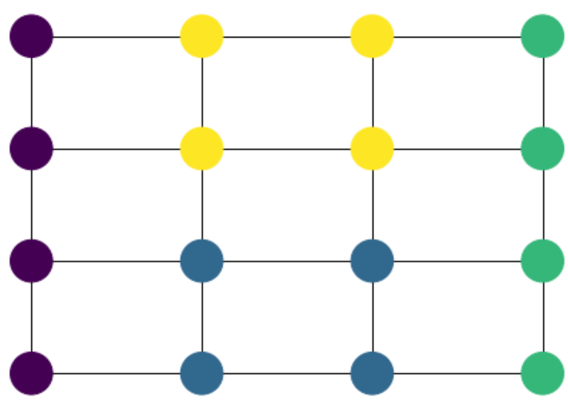


Figure 5.7 Smallest probability in third grouping of distinct weights with two squares, probability = 0.0244428477, 2 rotations/reflections

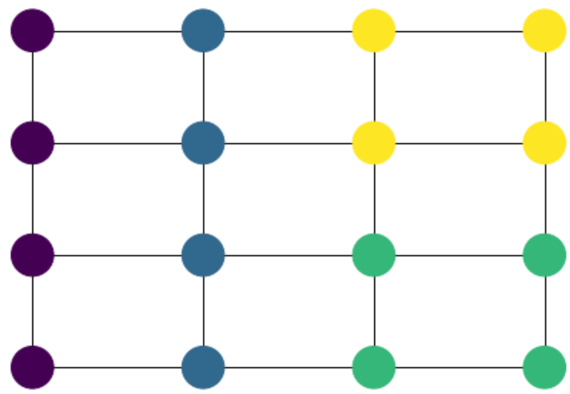


Figure 5.8 Second smallest probability in third grouping of distinct weights with two squares, probability = 0.0245788627, 4 rotations/reflections

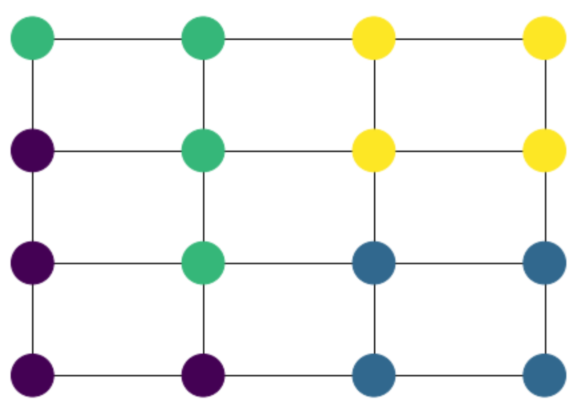


Figure 5.9 Largest probability in third grouping of distinct weights with two squares, probability = 0.0245844451, 8 rotations/reflections

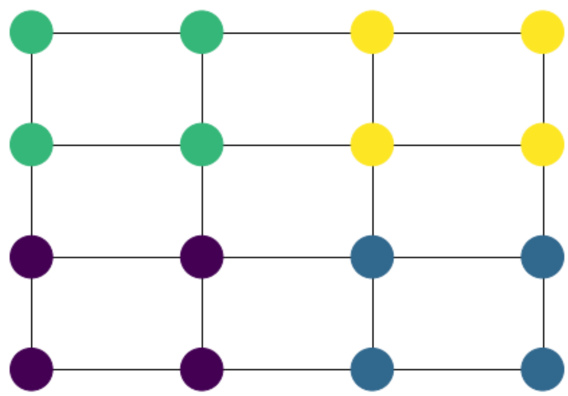


Figure 5.10 Largest probability in all 22 distinct weights, probability = 0.1966606746, 1 rotation/reflection

grid graph with four squares, there is only one districting plan with this probability. This largest probability districting plan is 64 times larger than the plans without any squares. This probability is also eight times larger than probabilities of plans with two square districts.

Based off of these findings, we can conclude that square districts carry more probability for the districting plan that a plan with no square districts. This confirms the belief that stationary distribution is related to the number of spanning trees, though clearly there is more at play here.

5.2 Methods

In using Recombination, we examined the $4x4$ case. We explicitly calculated P and used linear algebra to solve $\pi P = \pi$.

We started with a list of all possible plans and indexed them 1 through 117. We set up a table of 117×117 and set it entirely to zero.

For each pair (i, j) we found the common list which is the list of district or districts the plans in comparison have in common. Then, we examine if the two plans in question have two districts in common. The plans need to have two districts in common for the Recombination method to work. If they have three or four in common then they are the same plans. As previously described, we calculate these probabilities when two districts are in common by maintaining stochasticity.

The first step in Recombination is picking two random districts uniformly among the $\binom{4}{2}$ possible pairs. The probability of doing so is $\frac{1}{\binom{4}{2}}$. Skipping the second step of Recombination because it does not affect the probability of transition leads us to the third step of picking a random spanning tree. The probability of doing so is $\frac{1}{\text{number of spanning trees}}$. We found the number of spanning trees by using Kirchoff's Matrix-Tree Theorem as discussed in Chapter 4. Thus, the fraction becomes $\frac{1}{det}$ where det is the determinant of the Laplacian of the union of two districts. We find the determinant by first quantifying the vertices of districts not in common. Doing so is simpler in calculation than finding vertices of districts in common Once that is found, we take the union of all the vertices, make the Laplacian, and then find the determinant of the Laplacian.

The fourth step of Recombination is picking a random edge of the spanning tree. In a union of two 4 vertex districts graph there are eight vertices, thus there are seven edges. The probability of picking a random edge of the spanning tree is $\frac{1}{7}$. Finally, skipping the fifth step of removing that edge to separate two districts because it does not involve a calculation in the Recombination method, we reach the sixth step which is if the two parts are of the same size, then they are our new districts. The probability of this is the number of ways we could have gotten to the same plan via a specific spanning tree and edge where the probability $= \frac{1}{6} \times \frac{1}{\text{spanning tree}} \times \frac{1}{7}$.

However, we could have arrived at the same plan with different spanning trees and edges. To find the probability we multiple each of the steps together as so: $\frac{1}{det} \times \frac{1}{\binom{4}{2}} \times \frac{1}{7} \times$ ways to get same plan.

We are counting the number of spanning trees that, when cut, give the new districts. We found the number of ways to get to the same plan through a series of calculations. We first look at the number of edges between plan j and its common list, as described above. This means that given the list of edges in the common list, the number of edges between is the sum of edges that have a node in the common list and a node not in the common list. In the code we run through range four because there are four possible edges between the two districts. If the value is not in the common list and the district is a square, then we multiply the ways it could be the same plan by four because a square district has four spanning tree possibilities.

We fill in each probability in the matrix with the proper probability. For the probabilities of returning to the same districting plan, we maintain stochasticity. We do so by summing up the row and setting the diagonal to 1 minus the sum.

When solving the $\pi P = \pi$ we solved $(P^{\mathrm{T}} - I)\pi = 0$. Due to $P^{\mathrm{T}} - I$ not being full rank, we replaced the final column with 1's because we need a square matrix for the transposition to work. Entries in π must sum to 1. We solved the $(P - I)$ by summing the rows to maintain stochasticity, meaning they sum to 1. Using the transition matrix we found above, we set the 117th column to 1. We transpose the transition matrix. Then we set X to be 1×117 column vector of all zeros where the 117th entry is 1. We transpose X, and use a code on Python, np.linalg.solve, that uses linear algebra to solve for π. This gives us our stationary distribution.

In the following tables we truncated the values to 10 decimal places because when the values exceeded 10, we found numerical inconsistencies. Rotated and reflected districting plans must have the same probability distributions because of symmetry. Thus, differences in probabilities among the same plans reflect estimations in the code. Additionally, the numbered plans are according to our enumeration.

The code for these calculations and the list of plans can be found at Github with this link under the names "Listof4x4Plans" and "RecomStat-Dist4x4Grid.py":

https://github.com/sarah-cannon/RecombinationStationaryDistribution

Table 5.1 Table of Complete Findings: First Grouping of Smallest Probabilities.

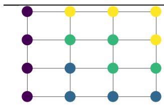
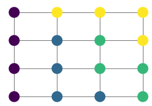
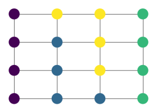
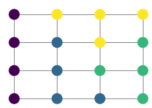
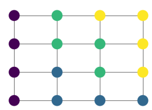
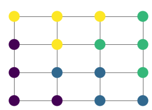
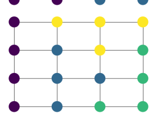
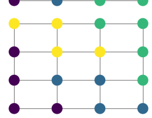
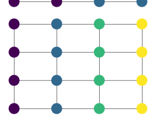
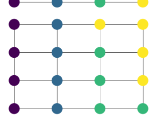
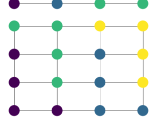
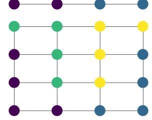
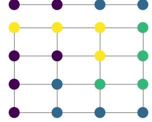
Plan	Number of Reflections/Rotations	Plans that are Reflections/Rotations	Probability
	8	12, 22, 23, 45, 65, 93, 94, 102	0.0030503098
	8	4, 20, 26, 64, 68, 87, 103, 108	0.0030505409
	4	6, 16, 106, 109	0.0030547519
	8	5, 18, 35, 36, 82, 83, 99, 107	0.0030559639
	8	11, 15, 24, 46, 63, 91, 95, 111	0.0030564604
	2	37, 80	0.0030566196
	8	27, 38, 40, 61, 69, 79, 81, 85	0.0030572753
	8	7, 17, 25, 39, 62, 84, 98, 110	0.0030606318
	4	28, 60, 70, 86	0.0030626204
	2	0, 116	0.0030667129
	8	1, 3, 29, 59, 74, 75, 114, 115	0.0030728275
	4	30, 57, 73, 78	0.0030765601
	4	33, 58, 72, 76	0.0031571178
	2	44, 88	0.0034370206

Table 5.2 Table of Complete Findings: Second Grouping of Probabilities

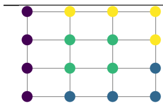
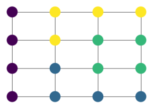
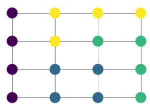
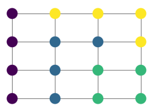
Plan	Number of Reflections/Rotations	Plans that are Reflections/Rotations	Probability
	4	21, 34, 66, 100	0.0090170844
	4	13, 41, 92, 97	0.0091650711
	8	9, 14, 42, 43, 89, 90, 96, 105	0.0091653883
	8	8, 19, 32, 47, 54, 67, 101, 112	0.0092067382

Table 5.3 Table of Complete Findings: Third Grouping of Smallest Probabilities

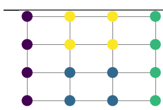
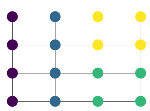
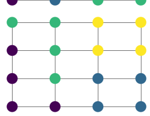
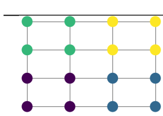
Plan	Number of Reflections/Rotations	Plans that are Reflections/Rotations	Probability
	2	10, 104	0.0244428477
	4	2, 48, 53, 113	0.0245788627
	8	31, 49, 51, 52, 55, 56, 71, 77	0.0245844451

Table 5.4 Table of Complete Findings: Fourth Grouping of Largest Probabilities

Plan	Number of Reflections/Rotations	Plans that are Reflections/Rotations	Probability
	1	50	0.1966606746

Chapter 6

Conclusions and Future Work

Gerrymandering targets those not holding majority power and violates the basis of America: democracy. Although America has a controversial history with democracy and lack of equality that was not discussed in this paper, the United States was founded on the idea of democracy and gerrymandered plans directly contradict this basis. When the individual's vote no longer matters, democracy is no longer in play. By disabling the American vote that was fought so hard for, the politicians behind gerrymandered districting plans fundamentally reject the country they claim to stand for. Their patriotism only suits themselves when they can hold power.

Thus, it is essential to detect and quantify gerrymandered plans in hopes that the plans will cease to be in use. There are many unique methods to detect and quantify plans, each with varied pros and cons. Recombination is a method with pros outweighing cons, except for the lack of baseline. This thesis found the baseline stationary distribution for a 4×4 grid graph, adding to the foundation of understanding how to use Recombination as a productive method.

There are many ways to begin considering detecting and quantifying gerrymandering. Though we only examined one way, there is another way closely linked to the way we chose that is a variant of Recombination also in use. The alternative way is instead of picking two random districts, we pick a random cut edge, where cut edge means the endpoints are in different districts. Future work includes finding the stationary distribution for a 4×4 grid graph using this alternative method within Recombination.

Furthermore, we hope that the stationary distributions of more small graphs will be found which will further this research beyond this thesis. Understanding the stationary distribution in small examples may help understand in the state level.We hope that this information found will contribute to the detection and quantification of gerrymandered plans in the most productive way possible.

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