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The Boundedness of the Hardy-Littlewood Maximal Function and the Strong Maximal Function on the Space BMO

Wenhao Zhang

CLAREMONT MCKENNA COLLEGE

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SUBMITTED TO
Professor Winston Ou
AND
Professor Chiu-Yen Kao
BY
Wenhao Zhang

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Abstract

In this thesis, we present the space BMO, the one-parameter Hardy-Littlewood maximal function, and the two-parameter strong maximal function. We use the John-Nirenberg inequality, the relation between Muckenhoupt weights and BMO, and the Coifman-Rochberg proposition on constructing A_1 weights with the Hardy-Littlewood maximal function to show the boundedness of the Hardy-Littlewood maximal function on BMO. The analogous statement for the strong maximal function is not yet understood. We begin our exploration of this problem by discussing an equivalence between the boundedness of the strong maximal function on rectangular BMO and the fact that the strong maximal function maps A_∞ weights into the A_1 class. We then extend a multiparameter counterexample to the Coifman-Rochberg proposition proposed by Soria (1987) and discuss the difficulties in modifying it into a A_∞ counterexample that would disapprove the boundedness of the strong maximal function.

CHAPTER 1

Introduction

Notation

$|E|$: Lebesgue measure of a set $\mathrm{E} \in \mathbb{R}^d$.
$B(x,r)$: open balls of radius r centered at $x \in \mathbb{R}^d$, i.e. $\{y \in \mathbb{R}^d \mid |x-y| < r, r > 0\}$.
$\alpha B(x,r) = B(x, \alpha r)$.
f_B: mean value of function f over ball B, i.e. $\frac{1}{|B|}\int_B f(x)\,dx$.
$w(E)$: $\int_E w(x)dx$, w is a nonnegative locally integrable function.
$L^p(\mathbb{R}^d)$: Banach Space of functions such that

$$\|f\|_{L^p} := \int\limits_{\mathbb{R}^d} |f^p|^{\frac{1}{p}}\,dx < \infty, \text{ where } 1 \leq p < \infty.$$

$L^\infty(\mathbb{R}^d)$: $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_\infty := \operatorname{ess\,sup} |f(x)| < \infty$.
p': conjugate opponent of $p : \dfrac{1}{p'} + \dfrac{1}{p} = 1\ (p > 1)$.
type (p,q): for operator T, $\|Tf\|_q \leq A\|f\|_p$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and A does not depend on f, i.e. T is bounded from L_p to L_q.

1.1. The space BMO

The Banach space of function of bounded mean oscillation (abbreviated as BMO) was first introduced by John in 1961, when he studied rotation and strain in elasticity theory.

Definition. *(BMO function) Let f be a locally integrable function on $\mathbb{R}^d$. f is said to be in the space BMO, if there exists a constant A such that*

$$\sup_B \frac{1}{|B|} \int\limits_B \left|f(x) - f_B\right|\,dx < A$$

where the supremum is taken over all balls B in $\mathbb{R}^d$. The smallest A that satisfies the property is denoted as $\|f\|_{BMO}$ or $\|f\|_*$, the norm of f in space BMO.

Note that constant functions have zero BMO norm where $f(x) = f_B$ for all $B \ni x$; in fact one considers the equivalence classes of functions modulo additive constants. Further, the definition is equivalent when we replace balls with cubes. Any bounded functions are in BMO but the converse is false (e.g., $\log(|x|)$ is in BMO. See [**12**] for details.) Precisely, for $f \in L_\infty$, using the triangle inequality, $\|f\|_* \leq 2\|f\|_\infty$, therefore $L_\infty \subset BMO$.

John and Nirenberg later proved the John-Nirenberg inequality, demonstrating that the percentage of any cube on which a BMO function differs by more than

λ from its mean decreases exponentially dependent on that constant. So BMO is also called the John-Nirenberg space.

Proposition 1. *(The John–Nirenberg Inequality) [**7**] If $f \in BMO$, and $\lambda > 0$, for all cubes Q in $\mathbb{R}^d$, there exist constants c_1, c_2 such that*

$$\left|\left\{x \in Q \mid \left|f - f_Q\right| > \lambda\right\}\right| \leq c_1 |Q| \, e^{\frac{-c_2 \lambda}{\|f\|_*}}$$

Here we omit the proof of the above proposition. However, the BMO function actually satisfies a stronger type of inequality that holds for every $p < \infty$, and a limiting version of its exponential integrability [**11**] [**5**]. The following corollary gives a more precise statement of this assertion.

Corollary 1.1.1. *If $f \in BMO$, then:*

(1) *For every $0 < p < \infty$:*

$$\|f\|_{*,p} \coloneqq \sup\{\frac{1}{|B|} \int\limits_B |f(x) - f_B|^p dx\}^{\frac{1}{p}} \leq C_p \|f\|_*$$

where C_p does not depend on f.

(2) *For every λ such that $0 < \lambda < \frac{1}{2^{d+1} e \|f\|_*}$, d is dimension*

$$\sup \frac{1}{|B|} \int\limits_B e^{\lambda |f(x) - f_B|} dx < \infty$$

1.2. The Hardy-Littlewood Maximal Operator and the Strong Maximal Operator

The Hardy-Littlewood maximal operator and its variants, along with so-called square functions and singular integrals, form the central objects of study in harmonic analysis [**12**]. It is defined as follows.

Definition. *Let f be a locally integrable function on $\mathbb{R}^d$. The (uncentered) Hardy-Littlewood maximal function is given by*

$$Mf(x) \coloneqq \sup_{x \in Q} \frac{1}{|Q|} \int\limits_Q \left|f(x)\right| \, dx,$$

where the supremum is taken over cubes containing x with sides parallel to the axes.

The Hardy-Littlewoord maximal operator has a wide variety of applications. For example, using the Vitali Covering Lemma that we will show in section 2.1, one can give a quick proof of the Lebesgue differentiation theorem that for almost every point, the value of an integrable function is the limit of infinitesimal averages taken about the point [**12**].

The properties of the *one-parameter* Hardy-Littlewood maximal operator M have been well studied, but there is still ongoing research to understand how multiparameter versions (that is, versions where the supremum is taken over multiparameter families of geometrical objects) of the operator behave. The one we are interested in is called the *strong maximal function* M_s, where the operator takes in f as a function of $\vec{x} \in \mathbb{R}^2$ and the supremum is taken over rectangles with sides parallel to axes. The function space analogous to BMO is taken to be *rectangular* BMO (denoted $bmo(\mathbb{R}^2)$).

1.3. Overview of the Thesis

The purpose of this thesis is to understand the behavior of the one-parameter Hardy-Littlewood maximal function and its two-parameter generalization, the strong maximal function, on the space BMO of functions of bounded mean oscillation. The Hardy-Littlewood maximal function and its generalizations, because of their tight relation with so-called singular integrals (operators that can be realized as a convolution with a singular kernel), are some of the most central and studied constructions in harmonic analysis. In this thesis, chapters 2, 3, and 4 will present the one-parameter results and chapter 5 will focus on the multiparameter case.

In chapter 2, we will begin with the Vitali Covering Lemma, a result of the engulfing property of balls (or cubes). The lemma has two important consequences for the Hardy-Littlewood maximal operator: the weak-type $(1,1)$ estimate and the strong-type (p,p) estimate via interpolation of the weak-type one.

In chapter 3, we then will discuss the boundedness of M on $L_p(wdx)$ if we insert a weight w (non-negative locally integrable function) into our measure. This question turns out to be closely related to the study of Muckenhoupt weights A_p. We will show the weights that make M bounded on $L_p(wdx)$ are exactly the A_p weights. Then we will utilize two important properties of A_p weights to prove the boundedness of M on BMO, but we will build up some machinery to get us there. The first property is the Coifman-Rochberg proposition that the fractional power of the maximal function applied to any locally integrable function is an A_1 weight, which in turn implies M maps any A_p weight into an A_1 weight (a special limiting class of weights). The second useful fact is that the logarithm of any A_p weight is in BMO.

If f is a BMO function, suppose we can control the value of Mf by the logarithm of some A_p weight, maybe in conjunction with some other obvious BMO function. This is exactly what Ou's commutation lemma says; and as a consequence, we are able to show that M is bounded on BMO. In chapter 4, we will present the proof of this main result, where we will integrate the results developed in chapter 3.

In chapter 5, we will explore the multiparameter maximal function following a similar mindset to that shown in the one-parameter result. In this paper, we mainly focus on the two-dimensional rectangles with sides parallel to the axes as the multiparameter object. We will first provide the definitions of the multiparameter analogues of the maximal function, bounded mean oscillation, and Muckenhoupt weights, namely the strong maximal function M_s, rectangular bmo, strong A_p weights A_p^*, respectively. We will also show that rectangles have an alternative covering lemma, which only gives the weak-type estimate of M_s from the Orlicz space $L(\log L)^{d-1}$ to $L^{1,\infty}$.

We then will continue our discussion of M_s by working on some open questions. Even though less is known for the boundedness of M_s on bmo, this problem is equivalent to the behavior that M_s maps (A_∞^*) into a A_1^*. However, still little is known about this conjecture. For our exploration, we will first explain a counterexample constructed by Soria showing that the Coifman-Rochberg proposition does not hold for the strong maximal function. In order to disprove the conjecture, we attempted to construct an A_∞^* weight counterexample by modifying Soria's example, but in this respect we were unsuccessful. At the end of this chapter, we will discuss the reason why this problem is inherently difficult.

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CHAPTER 2

The Weak-type Estimate of Hardy-Littlewood Maximal Function

To give some sense of the distinction between the one-parameter and the two-parameter strong maximal functions, we begin with the Vitali Covering Lemma of the geometric objects defined in M, namely balls or cubes. We then show the weak type $(1,1)$ estimate of M using the covering lemma and provide the proof of the lemma to highlight the engulfing property and the doubling property. Lastly, we prove that M is bounded on L^p.

Proposition 2. *(Weak Type (1,1) Estimate for Hardy-Littlewood Maximal Function) For a given constant α, let $E_\alpha := \{x \mid Mf(x) > \alpha\}$. There exists a constant c satisfying,*

$$|E_\alpha| \leq \frac{c}{\alpha}\|f\|_{L^1} \text{ (weak (1,1) condition).}$$

2.1. Vitali Covering Lemma

Lemma 2.1.1. *(The Vitali Covering Lemma) Let $E \subset \mathbb{R}^d$. Given any finite collection of balls, $E = \bigcup_{i=1}^{n} B_i$, then exists a disjoint subcollection $B_{n_1}, B_{n_2}, \cdots, B_{n_k}$ of these balls which satisfy*

$$c|E| \leq \sum_{i=1}^{k} |B_{n_i}|$$

where c is a universal constant.

The lemma implies the weak type $(1,1)$ estimate as follows. If $x \in E_\alpha$, namely $Mf(x) > \alpha$, given E_α is open, there exists a ball B containing x such that $\frac{1}{|B|}\int_B |f|\,dx > \alpha$. Every point in this ball is in E_α, since for any point x_o in the ball $Mf(x_o) \geq \frac{1}{|B|}\int |f|dx \geq \alpha$. One can thus obtain a collection of balls that satisfy $\bigcup B_i = E_\alpha$. Applying the covering lemma, there exists a subcollection of balls such that

$$|E_\alpha| \leq c\sum_k |B_{n_i}|.$$

In addition, for each ball B_{n_i}, $\frac{1}{|B_{n_i}|}\int_{B_{n_i}} |f|\,dx > \alpha$; so $\frac{1}{\alpha}\int_{B_{n_i}} |f|\,dx > |B_{n_i}|$. Thus,

$$|E_\alpha| < \frac{c}{\alpha}\sum_i \int_{B_{n_i}} |f|dx \leq \frac{c}{\alpha}\int_{\mathbb{R}^d} |f|\,dx,$$

and then we get the proposition.

PROOF OF LEMMA. One can select the biggest ball in the collection, denote it as B_{n_1}, and then remove all other balls that intersect with B_{n_1}. Repeat this procedure until one runs out of balls. After this greedy algorithm, one ends up with a subcollection of disjoint balls, namely $B_{n_1}, B_{n_2}, \cdots, B_{n_k}$, and each ball in B_i is either selected or removed. If one expands the radius of each ball in B_{n_i} by a factor of 3, each ball in B_i that has been removed is covered by some ball in $3B_{n_i}$, consequently

$$E = \bigcup B_i \subseteq \bigcup 3B_{n_i}.$$

Since all balls in B_{n_i} are disjoint, $\left|\bigcup B_{n_i}\right| = \sum|B_{n_i}|$, so

$$|E| \leq \left|\bigcup_{i=1}^{k} 3B_{n_i}\right| \leq \sum_{i=1}^{k}|3B_{n_i}| \leq 3^d \sum_{i=1}^{k}|B_{n_i}|,$$

where d is the dimension of the set. Thus the claim follows with an universal constant of $\frac{1}{3^d}$. □

The Vitali Covering Lemma can be generalized by using other objects like cubes. The main properties of those objects are the engulfing property and the doubling property. Engulfing property states that if two balls (or cubes) intersect, one is contained in some dilated form of the other one. The doubling property simply allows us to exploit the first property. In the proof of lemma, to cover all the removed balls that intersect with the selected one, we can choose the constant 3 that works for any size of balls. However, this does not apply to the rectangles used to define M_s. If two rectangles intersect, i.e. $R_1 \cap R_2 \neq \emptyset$, but one cannot obtain a fixed constant c like 3 such that $R_1 \subset cR_2$ for all possible rectangles. For example, one can consider R_1 as an arbitrarily narrow and long rectangle and R_2 as a cube, so that one cannot find a constant factor to expand the sides of R_1 to cover R_2. However, rectangles do satisfy another covering lemma, and we will discuss it further in chapter 5.

2.2. The Boundedness of Hardy-Littlewood Maximal Function on L_p

The weak-type estimate has powerful consequence as it implies the strong-type estimate of M. Next theorem says that M is bounded on any L_p space where $p > 1$.

Theorem 2.2.1. *(Boundedness of M on L_p) Suppose $f \in L^p$ with $1 < p \leq \infty$, then $\|Mf\|_{L^p} \leq C_p\|f\|_{L^p}$, where C_p depends on p but not on f.*

To start off the proof, it is easy to see that M is bounded on L^∞ (type (∞, ∞)). If $|f|$ is bounded by some constant, certainly all averages of $|f|$ are still bounded by the same constant. When $1 < p < \infty$, the proof combines the weak-type (1,1) estimate, L^∞ boundedness, and Marcinkiewicz interpolation [**13**].

PROOF. To prove $Mf \in L^p$, we make a stronger version of the weak-type inequality:

$$|\{x : Mf(x) > \alpha\}| \leq \frac{2c}{\alpha} \int\limits_{|f|>\frac{\alpha}{2}} |f|dx \qquad (2.2.1)$$

for all $\alpha > 0$. Decompose $f = f_1 + f_\infty$, where

$$f_1(x) = \begin{cases} f(x), & \text{if } \left|f(x)\right| > \frac{\alpha}{2} \\ 0, & \text{otherwise} \end{cases}$$

and similarly,

$$f_\infty(x) = \begin{cases} f(x), & \text{if } \left|f(x)\right| \leq \frac{\alpha}{2} \\ 0, & \text{otherwise} \end{cases}$$

So we have $f_1 \in L_1$ and $f_\infty \in L_\infty$. By the sub-additivity of M, we have $Mf < Mf_1 + Mf_\infty$. Given that f_∞ is essentially bounded by $\frac{\alpha}{2}$, so $Mf < Mf_1 + \frac{\alpha}{2}$. Since $\{x : Mf(x) > \alpha\} \subset \{x : Mf_1(x) > \frac{\alpha}{2}\}$, apply weak-type estimate to f_1 to yield inequality (2.2.1). Now rewrite $\|Mf\|_{L^p}$ with respect to α,

$$\int |Mf|^p dx = p \int_0^\infty |Mf > \alpha| \alpha^{p-1} d\alpha$$

Thus, apply inequality (2.2.1) to complete the proof,

$$\int |Mf|^p \leq 2cp \int (\int_0^{2|f|} \alpha^{p-2} d\alpha)|f| dx = \frac{cp}{p-1} 2^p \int |f|^p dx.$$

The theorem holds with the constant $C_p = \frac{p\, 2^p}{3^d (p-1)}$, which only depends on p and dimension d. □

CHAPTER 3

Weighted Inequalities

3.1. Muckenhoupt Weights

In Linear Algebra, we introduced the norm of a vector and then the weighted norm of a vector that is

$$\|\vec{v}\|_{\vec{w}} =< \vec{v}, \vec{v} >_{\vec{w}}= (\sum v_i^2 w_i)^{\frac{1}{2}} \text{ where } \vec{w} \geq 0.$$

The weighted norm of a function f is defined similarly as

$$\|f\|_w =< f, f >_w= (\int |f|^2 w)^{\frac{1}{2}},$$

which can be considered as the weighted L_2 norm of f. Having shown M is bounded on L_p, in other words, the L_p norm of Mf is bounded by the L_p norm of f, it is a natural question to ask: for what weights $w(x)$ (locally integrable positive function) that M is bounded on $L_p(wdx)$, i.e. there exists a constant C that satisfies the following inequality,

$$\int |Mf(x)|^p w(x)dx \leq C \int |f(x)|^p w(x)dx. \tag{3.1.1}$$

Surprisingly, the following weights turn out to provide the answer.

Definition. *(A_p weight, $p > 1$) For $p > 1$, a weight $w \in A_p$ class if*

$$A_p(w) := \sup_B (\frac{1}{|B|} \int_B wdx)(\frac{1}{|B|} \int_B w^{-\frac{1}{p-1}} dx)^{p-1} \leq \infty. \tag{3.1.2}$$

$A_p(w)$ is called the A_p characteristic where A is constant.

We remark that for $p_1 < p_2$, through simple calculation, $A_{p_1} \subset A_{p_2}$. Therefore, we denote by A_∞ the union of all A_p classes where $p > 1$. To build more intuition about the A_∞ weight, we give the following theorem of crucial characterizations of the class A_∞.

Theorem 3.1.1. *(R. Feferrman Notes) The following statements are equivalent:*

(0) $w \in A_\infty$.

(1) *There exists $\epsilon > 0$ such that for any subset $E \subseteq B$, if $\frac{|E|}{|B|} \geq \frac{1}{2}$, then $\frac{w(|E|)}{w(|B|)} \geq \epsilon$.*

(2) *If $\frac{|E|}{|B|} \leq \frac{1}{2}$, then $\frac{w(|E|)}{w(|B|)} \leq 1 - \epsilon$.*

(3) *For all E, there exists a constant δ such that*

$$(\int_E w^{1+\delta} dx)^{\frac{1}{1+\delta}} \leq C \int_E wdx.$$

(4) *For all* $E \subseteq B$*, there exists a* C_η *such that* $\frac{w(|E|)}{w(|B|)} \leq C_\eta(\frac{|E|}{|B|})^\eta$*, where* $\eta = \frac{1}{(1+\delta)'} = \frac{\delta}{1+\delta}$.
(5) $w \in A_p(\mathbb{R}^d)$ *for some* $1 < p < \infty$.
(6) w *satisfies a Reverse Jensen's inequality, namely*

$$\frac{1}{|B|}\int\limits_B w \leq Ce^{\frac{1}{|B|}\int\limits_B \log w}$$

where we denote the smallest such C *by* $A_\infty(w)$*. In fact,* $\lim\limits_{p\to\infty} A_p(w) \to A_\infty(w)$.

One can use any of above statements to check whether a weight is in A_∞ class. The statement (1) gives us an important observation: a A_∞ weight assigns to the subset E of a ball B a fair share of the weight of B; "fair" in terms of the ratio of the Lebesgue measure of E and B. This observation combined with statement (3) also entails that a A_∞ weight does not grow or degenerate too quickly.

The set A_∞ of all A_p weights is also called Muckenhoupt weights, since Muckenhoupt proved that (3.1.1) is equivalent to (3.1.2), i.e. the weights that make M bounded on $L_p(wdx)(1 < p < \infty)$ are exactly the A_p weights [**8**]. The proof comes from the following important proposition.

Proposition 3. *(Reverse Hölder inequality) If* $w \in A_\infty$*, there exists two constants* $s > 1$ *and* $c > 0$ *depending on* w *such that*

$$(\frac{1}{|B|}\int\limits_B w^s dx)^{1/s} \leq \frac{c}{|B|}\int\limits_B wdx \tag{3.1.3}$$

for all balls B*, i.e.* w *lies in the* reverse Hölder class RH_s*, where the Reverse Hölder characteristic, denoted as* $RH_s(w)$*, is defined to be the infimum of* c.

Proposition 3 is a powerful tool that will be used in many proofs in this thesis. It also yields an important corollary that the A_p class of Muckenhoupt weight can always be improved.

Corollary 3.1.1. *If* $w \in A_p$ *where* $1 < p < \infty$*, then there exists a* $p_1 < p$ *such that* $w \in A_{p_1}$

Proof. For $w \in A_p$, $\sigma = w^{-\frac{p'}{p}}$. $A_p(w)^{\frac{1}{p}} = A_{p'}(\sigma)^{\frac{1}{p'}}$, so σ is in $A_{p'} \subset A_\infty$. The Reverse Hölder inequality tells us that for some $s > 1$,

$$(\frac{1}{|B|}\int\limits_B \sigma^s dx)^{1/s} \leq \frac{c}{|B|}\int\limits_B \sigma dx.$$

Let $1 < p_1 < p$ satisfy $s = \frac{p_1'}{p_1}\frac{p}{p'}$, and the inequality translates into

$$(\frac{1}{|B|}\int\limits_B w^{\frac{-p_1'}{p_1}} dx)^{\frac{p_1}{p_1'}} \leq (\frac{C}{|B|}\int\limits_B w^{\frac{-p'}{p}} dx)^{\frac{p}{p'}}$$

Then multiply both sides by $\frac{1}{|B|}\int\limits_B w(x)$ to see that we have $w \in A_{p_1}$. □

Theorem 3.1.2. *Suppose $1 < p < \infty$ and w is a nonnegative integrable function. Then, for all $f \in L^p(w(x)dx)$, there exists a constant C such that*

$$\int (Mf(x))^p w(x)dx \le C \int |f(x)|^p w(x)dx,$$

if and only if $w \in A_p$.

Proof of Theorem. Suppose (3.1.1) holds for a locally integrable function $f = w^{-\frac{1}{p-1}}\chi_B$. By definition of M, for all $B \ni x$, $Mf(x) \ge \frac{1}{|B|}\int\limits_B w^{-\frac{1}{p-1}}dx$. Then, rewriting the inequality (3.1.1) in w,

$$
\begin{aligned}
(\frac{1}{|B|}\int\limits_B w^{\frac{-1}{p-1}}dx)^p \int\limits_B wdx &\le \int (Mf)^p wdx \\
&\le C\int\limits_B w^{\frac{-p+(p-1)}{p-1}}dx \qquad (3.1.4)\\
&= C\int\limits_B w^{-\frac{1}{p-1}}dx.
\end{aligned}
$$

Dividing both sides by $|B|$ and rearranging the terms, we obtain

$$(\frac{1}{|B|}\int\limits_B w^{\frac{-1}{p-1}}dx)^{p-1}\frac{1}{|B|}\int\limits_B wdx \le C$$

Thus f satisfies the A_p condition.

For the reverse direction, if $w \in A_p$, we claim that M satifies a *weak-type* $(L^p(wdx), L^p(wdx))$ condition, i.e.

$$|\{x : M(f)w > \alpha\}| \le \frac{C}{\alpha^p}\int |f(x)|^p wdx \text{ for all } \alpha > 0$$

We define M_w by

$$M_w f(x) := \sup_B \frac{1}{\int\limits_B w(x)dx}\int\limits_B |f(y)|^p w(y)dy.$$

Recall the weak-type (1,1) estimate of M, for any $a > 0$,

$$|\{x : M_w(f) > \alpha\}| \le \frac{C}{\alpha}\int |f(x)|w(x)dx$$

Replace f with $|f|^p$ and α with α^p,

$$|\{x : M_w(|f|^p) > \alpha^p\}| \le \frac{C}{\alpha^p}\int |f(x)^p|w(x)dx$$

If $w \in A_p$, The condition A_p is exactly when the p^{th} power of the mean value of f_B is bounded by the mean value of f^p with respect to $w(x)dx$, namely,

$$(f_B)^p \le \frac{c}{\int\limits_B w(x)dx}\int\limits_B f^p wdx, \text{ where } c \text{ is constant.}$$

Take the supremum of all balls B on both sides, the inequality translates into

$$(Mf(x))^p \le cM_w(|f|^p).$$

$|\{x : M(f)w > \alpha\}| \subset |\{x : M_w(|f|^p) > \alpha^p\}|$, thus the weak type estimate holds. Then from above corollary, $w \in A_{p_1}$ for some $p_1 < p$ and satisify the weak type estimate on $L^{p_1}(wdx)$. Simialr to the proof of the boundedness of M onL^p, using Marcinkiewicz interpolation together with the boundedness of $L^\infty(wdx)$, we are able to conlcude that M is bounded on $L^p(wdx)$. □

3.2. Coifman-Rochberg Proposition

Now let us consider another limiting case of A_p classes. We define the A_1 class as all weights w, for which there exists a constant C such that $Mw(X) \leq Cw(X)$, where the smallest C is the A_1 charateristic, denote it as $A_1(w)$. $A_1 \subset A_p$ for all $p > 1$, so A_1 class is a subset of the intersection of all A_p classes [**5**]. To build some intuition, we can imagine the class of Muckenhoupt weights to be a shooting target. Given that p does not have a upper limit, the target techinically does not have an outer rim. The size of A_p class shrinks as p decreases. Thus, as the set inside all A_p weights, A_1 is the bullseye of the target.

Coifman and Rochberg [**3**] provided an important proposition regarding the relation between the Hardy-Littlewood maximal function and the A_1 weights, when they tried to express a BMO function using the logarithm of the M of some function. We state the main theorem of their paper as follows, but in this thesis, we mainly use the proposition.

Theorem 3.2.1. *There is a constant c (which depends only on dimension d) such that if α and β are positive constants, g and h are weights with finite Mg and Mh a.e., and b is any bounded function, then the function*

$$f(x) = \alpha \log Mg - \beta \log Mh + b(x)$$

is in BMO and $\|f\|_* \leq c(\alpha + \beta + \|b\|_\infty)$.

Conversely, if f is any function in BMO then f can be written in above equation with

$$\alpha + \beta + \|b\|_\infty \leq c\|f\|_*$$

.

Proposition 4. *(The Coifman-Rochberg Proposition) Let $\mu(x)$ be a locally integrable function such that $M\mu(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. For any $0 < \delta < 1$, $w(x) = (M\mu(x))^\delta$ is an A_1 weight, i.e. there exists a constant C s.t.*

$$M(M\mu(x))^\delta \leq C(M\mu(x))^\delta$$

C only depends on δ and the dimension d.

Using the Hölder inequality and the reverse Hölder inequality, it is easy to get the following corollary regarding the behavior of M on A_∞.

Corollary 3.2.1. *If $w(x) \in A_\infty$, $Mw(x) \in A_1$.*

Proof. By the reverse Hölder inequality and Hölder inequality, if $w \in A_\infty$, then for $s > 1$,

$$c(\frac{1}{|B|}\int\limits_B w^s dx)^{1/s} \leq \frac{1}{|B|}\int\limits_B wdx \leq (\frac{1}{|B|}\int\limits_B w^s dx)^{1/s}$$

Rewrite the inequality in terms of M,

$$c(Mw^s)^{\frac{1}{s}} \leq Mw \leq (Mw^s)^{\frac{1}{s}}$$

By reverse Hölder inequality, Mw^s is bounded by Mw, so $Mw^s \in A_\infty$. By the Coifman-Rochberg proposition, $(Mw^s)^{\frac{1}{s}}$ is an A_1 weight with $0 < \frac{1}{s} < 1$. Since Mw is comparable to $(Mw^s)^{\frac{1}{s}}$, thus $Mw \in A_1$. □

3.3. The Logarithm of Muckenhoupt weights

We have seen L_∞ and $\log|x|$ are in BMO. In this section, we first show that BMO function can be produced from the logarithms of A_1 weights; we then show that in fact the logarithms of A_p for any $p \geq 1$ is in BMO.

Theorem 3.3.1. $\log A_1 \in BMO$

PROOF. Let $w(x) \in A_1$ and $\phi = \log(w)$, that is $w = e^\phi$. By definition of A_1,

$$\frac{1}{|B|}\int_B e^\phi dx \leq Ce^\phi$$

$$\implies (\frac{1}{|B|}\int_B e^\phi dx)e^{-\inf_B \phi(x)} \leq C$$

By Jensen's Inequality,

$$\frac{1}{|B|}\int_B e^\phi dx \geq e^{\frac{1}{|B|}\int_B \phi dx} = e^{\phi_B}$$

Then we have $e^{\phi_B - \inf \phi(x)} \leq C$, equivalently,

$$0 < \phi_B - \inf\ \phi(x) \leq C$$

For $\phi \in BMO$, $\frac{1}{|B|}\int_B \left|\phi(x) - \phi_B\right| dx$ needs to be finite for all balls B. By the triangle inequality, $\left|\phi(x) - \phi_B\right| \leq \left|\phi(x) - \inf\ \phi(x)\right| + \left|\inf\ \phi(x) - \phi_B\right|$. Taking the average over all balls $|B|$,

$$\frac{1}{|B|}\int_B \left|\phi(x) - \phi_B\right| dx \leq 2|\phi_B - \inf\ \phi(x)| = 2C < \infty.$$

Therefore $\phi = \log w(x) \in BMO$. □

If ϕ satisfies $\frac{1}{|B|}\int_B \phi - \inf_B \phi \leq C$ over all balls B, we say ϕ is in BLO (*functions of bounded lower oscillation*) and denote the smallest constant C by $\|\phi\|_{BLO}$. In the proof above, by the triangle inequality, we see that BLO is a subset of BMO. Indeed, if $w \in A_1$, $\log w$ satisfies a strong condition that $\log w \in BLO$.

In order to show $\log(A_\infty) \in BMO$, we state the next theorem that gives the characterization of A_p weights in terms of their logarithms.

Theorem 3.3.2. *If ϕ is locally integrable on $\mathbb{R}^d$ and $1 < p < \infty$.*

(1) $e^\phi \in A_p$ *if and only if:*

(a) $\frac{1}{B}\int_B e^{\phi(x)-\phi_B} dx \leq C$,

(b) $\frac{1}{B}\int\limits_B e^{-\frac{\phi(x)-\phi_B}{p-1}}dx \le C.$

(2) *Given that (1b) vanishes, $e^\phi \in A_\infty$ if and only if (1a) is true.*

Proof. The conditions (1a) and (1b) together implies the A_p condition,

$$(\frac{1}{B}\int\limits_B e^{\phi(x)-\phi_B}dx)(\frac{1}{B}\int\limits_B e^{-\frac{\phi(x)-\phi_B}{p-1}}dx)^{p-1}$$

$$= (\frac{e^{-\phi_B}}{B}\int\limits_B e^{\phi(x)}dx)(\frac{e^{\frac{\phi_B}{p-1}}}{B}\int\limits_B e^{-\frac{\phi(x)}{p-1}}dx)^{p-1}$$

$$= (\frac{1}{B}\int\limits_B e^{\phi(x)}dx)(\frac{1}{B}\int\limits_B (e^{\phi(x)})^{-\frac{1}{p-1}}dx)^{p-1} \le C \times C^{p-1}$$

Conversely, if $\phi \in A_p$, condition (1a) is satisfied,

$$\frac{1}{B}\int\limits_B e^{\phi(x)-\phi_B}dx = \frac{e^{-\phi_B}}{B}\int\limits_B e^{\phi(x)}dx$$

$$= (e^{-\frac{\phi_B}{p-1}}dx)^{p-1}\frac{1}{B}\int\limits_B e^{\phi(x)}dx \le (\frac{1}{B}\int\limits_B e^{-\frac{\phi}{p-1}}dx)^{p-1}\frac{1}{B}\int\limits_B e^{\phi(x)}dx \le C$$

Similarly, condition (1b) can be obtained,

$$\frac{1}{B}\int\limits_B e^{-\frac{\phi(x)-\phi_B}{p-1}}dx = (e^{\phi_B})^{\frac{1}{p-1}}(\frac{1}{B}\int\limits_B e^{-\frac{\phi(x)}{p-1}}dx)$$

$$\le (\frac{1}{B}\int\limits_B e^{\phi}dx)^{\frac{1}{p-1}}(\frac{1}{B}\int\limits_B e^{-\frac{\phi(x)}{p-1}}dx) \le C$$

For second part of the theorem, when $\phi \in A_\infty$, $\phi(x) - \phi_B$ is bounded by some constant, so condition (1a) is satisfied. □

The following corollary simplies Theorem 3.3.2, because the A_2 condition has a simpler and more symmetric appearance. We will use this corollary later in this thesis to determine whether a weight falls into the A_2 class.

Corollary 3.3.1. *If ϕ is locally integrable on $\mathbb{R}^d$. $e^\phi \in A_2$ if and only if there exists a constant C such that*

$$\frac{1}{B}\int\limits_Q e^{|\phi(x)-\phi_B|}dx < C.$$

Proof. When $p = 2$, by Theorem 3.3.2, the two conditions are $\frac{1}{B}\int\limits_Q e^{\phi(x)-\phi_B}dx \le C$ and $\frac{1}{B}\int\limits_Q e^{-(\phi(x)-\phi_B)}dx \le C$. Combining both terms, we get $\frac{1}{B}\int\limits_Q e^{|\phi(x)-\phi_B|}dx \le C$, which is equivalent to the A_2 condition. □

The next theorem shows that the averagae fluctuation of the magnitude of A_p weights w on every ball is uniformly controlled.

Theorem 3.3.3. $\log A_\infty \in BMO$

PROOF. Let $w(x) \in A_\infty$ and $w = e^\phi$. For the case $p < 2$, $w(x)$ lives in some $A_p \subset A_2$. Applying Corollary 3.3.1, $\frac{1}{B} \int\limits_Q e^{\left|\phi(x) - \phi_B\right|} dx$ is bounded by some constant. By definition of BMO, we can see $\phi \in BMO$ as follows,

$$\|\phi\|_* = \sup \frac{1}{B} \int\limits_Q \left|\phi(x) - \phi_B\right| \leq \sup \frac{1}{B} \int\limits_Q e^{\left|\phi(x) - \phi_B\right|} = C.$$

For the other case when $p > 2$, there exists $p' < 2$ such that function $\sigma = w^{-\frac{p'}{p}} \in A_{p'} \subset A_2$, so $\log(\sigma)$ lives in BMO. Also, $\log(\sigma) = \log(w^{-\frac{1}{1-p}}) = -\frac{1}{1-p} \log(w)$. It follows that $\log(w) \in BMO$ for both cases. $\square$

In fact, the converse of Theorem 3.3.3 is true: for every real-valued function $f \in BMO$, if we fix $p > 1$, then $f = c \log w$ for some $w \in A_p$ and some constant c. A short version of the proof of this statement is included in the proof of Theorem 4.0.1 when we fix $p = 2$.

CHAPTER 4

The Boundedness of the Hardy-Littlewood Maximal Operator on BMO

In previous chapters, we have shown M is bounded on L_p as well as $L_p(wdx)$ if w is an A_p weight. Now we are ready to show M is bounded on the space BMO. Bennett, DeVore and Sharpley first showed the following result in 1981 [**2**].

Theorem 4.0.1. *$M : BMO \to BMO$ is bounded.*

4.1. Commutation Lemma and Proof of the Main Result

The proof has several versions, and this thesis relies on the one given by Ou [**9**]. Ou used a generalization of M called *natural maximal operator* $M^\natural$, first introduced by Bennett in [**1**]. $M^\natural$ takes the average of the function itself, i.e.

$$M^\natural := \sup_{x\in Q} \frac{1}{|Q|} \int\limits_Q f(x)\, dx$$

Obviously, $M^\natural$ and M are the same for nonegative functions like weights. Notice that $Mf = M^\natural |f|$, so it suffices to show the boundedness of $M^\natural$ to prove Theorem 4.0.1. Ou's proof begins with a commutation lemma as follows.

Lemma 4.1.1. *For $w \in A_\infty$, $0 \le [\log M^\natural - M^\natural \log]w(x) \le \log A_\infty(w)$.*

Proof of Lemma. From Jensen's inequality, over all balls B,

$$e^{\frac{1}{|B|}\int\limits_B \log w} \le \frac{1}{|B|}\int\limits_B w.$$

By Reverse Jensen's inequality, we have

$$\frac{1}{|B|}\int\limits_B w \le C e^{\frac{1}{|B|}\int\limits_B \log w};$$

where, recall, $A_\infty(w)$ denotes the smallest such C. Now we have,

$$\frac{1}{|B|}\int\limits_B w \le A_\infty(w) e^{\frac{1}{|B|}\int\limits_B \log w} \le A_\infty(w)\frac{1}{|B|}\int\limits_B w.$$

Take the supremum over all $B \ni x$,

$$M^\natural w(x) \le A_\infty(w) e^{M^\natural \log w} \le A_\infty(w) M^\natural w.$$

Take the logarithm,

$$\log M^\natural w \le \log A_\infty(w) + M^\natural \log w \le \log A_\infty(w) + \log M^\natural w.$$

□

The commutation lemma tells us that $\log M^\natural$ is bounded by $M^\natural \log$ and logarithm of the A_∞ characteristic of w. To get the boundedness of $M^\natural$ from BMO into BLO, we combine the above lemma with what we have established: 1) the John-Nirenberg Inequality; 2) the corollary of the Coifman-Rochberg proposition; 3) the fact that the logarithm of any A_1 weight is in BLO.

Proof. First, we want to show that for every $\phi \in BMO$, if we fix $p = 2$, ϕ can be rewritten as a multiple of the logarithm of an A_2 weight, i.e. if $w \in A_2$, $\phi = c \log w(x)$ for some c.

By Corollary 1.1.1 of John-Nirenberg Inequality, if $\phi \in BMO$,

$$\frac{1}{B} \int_B e^{\lambda \left|\phi(x) - \phi_B\right|} \leq c$$

for all balls B. The inequality holds when $0 < \lambda < \frac{c_d}{\|\phi_*\|}$ and $c_d = \frac{1}{2^{d+1}e}$, d denotes dimension [**5**]. By Corollary 3.3.1, we observe $w(x) = e^{\lambda\phi} \in A_2$ and $A_2(w) \leq \sqrt{e}$. Thus,

$$\phi(x) = \lambda^{-1} \log w(x) = \frac{\|\phi\|_*}{c_d} \log w(x). \tag{4.1.1}$$

Apply the commutation lemma to (4.1.1),

$$M^\natural \phi(x) = \frac{\|\phi\|_*}{c_d} M^\natural \log w(x) \leq \frac{\|\phi\|_*}{c_d} [\log M^\natural w(x) + \log A_\infty(w)].$$

The term $\log A_\infty(w)$ is finite, as $A_\infty(w) \leq A_2(w)$. Consider $\log M^\natural w(x)$, $w(x) > 0$ allows us to switch $M^\natural$ to M. M maps A_∞(containing A_2) to A_1 and $\log A_1 \in BLO$, so $\log M^\natural$ eventually maps w to BLO. □

Thus, to complete the proof of Theorem 4.0.1, for $f \in BMO$, simply let $Mf = M^\natural(|f|) \in BLO \subset BMO$.

4.2. Question: The Boundedness of the Strong Maximal Function

In this chapter, we have finally connected our discussion on weights back to the original question on the boundedness of operators on BMO. We have seen that one could use the Coifman-Rochberg proposition that $(Mf)^\delta \in A$, for any $f \in L^1_{loc}(\mathbb{R})$ and any $0 < \delta < 1$, combined with the John-Nirenberg inequality, to show that the Hardy-Littlewood maximal operator M, the one-parameter case, is bounded on BMO. However, the analogous statement in the multi-parameter case is still not yet understood. In the next chapter, we would like to explore the boundedness of M_s on *bmo* by utilizing an equivalent relationship between Theorem 4.0.1 and Corollary 3.2.1.

CHAPTER 5

Behaviors of the Strong Maximal Function

The Hardy-Littlewood maximal function is called the one-parameter case, because we take the supremum over a family of the geometric objects like balls or cubes, which in essence are described by one piece of data (the radius or the side length). However, the strong maximal function considers a more general set of objects M_s is defined over a 2-parameter family. We define it as,

$$M_s f(x) := \sup_{x \in R} \frac{1}{|R|} \int_R |f(x)| \, dx,$$

where the supremum is taken over all rectangles on $\mathbb{R}^2$ containing x with sides parallel to the axes, denote the collection of these rectangles as $\mathscr{R}$.

If we denote M_j as the one-parameter maximal function, we observe

$$M_s f \leq M_1 M_2 f.$$

The multiparameter version of the space BMO is known as the *rectangular* BMO, denoted as $bmo(\mathbb{R})$, where the supremum of the mean oscillation is taken over all rectangles in $\mathscr{R}$.

5.1. The Boundedness of the Strong Maximal Function on L_p

The failure of Vitali covering lemma on rectangles demonstrates a crucial difference between M and M_s, and consequently M_s is not of weak-type $(1,1)$ as M is. However, Corboda and Fefferman gave a refined covering lemma for M_s [**4**].

Theorem 5.1.1. *Let $\{R_j\}_{j\in J}$ be a collection of all rectangles in $\mathscr{R}$. There exists a subcollection of those rectangles $\{R_{j_k}\} \subset \{R_j\}$ such that*

(1) $|\cup_{j\in J} R_j| \leq c_n |\cup_k R_{j_k}|$

(2) $\left\| \exp \frac{\sum_k \chi_{R_{j_k}}}{n-1} \right\|_{L^1} \leq C_n |\cup_{j\in J} R_j|$

The above lemma provides a geometric proof of the weak-type estimate of the strong maximal function, which was first studied by Jessen, Marcinkiewicz, and Zygmund [**6**]. However, the weak-type estimate for M_s is defined only from the Orlicz space $L(\log L)^{n-1}(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$:

Theorem 5.1.2. *For any $f \in L(\log L)^{n-1}(\mathbb{R}^d)$ and any $\lambda > 0$,*

$$\left|\{x \in \mathbb{R}^d |\, M_s f(x) > \lambda\}\right| \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(x)| \left(1 + \left(\log \frac{|f(x)|}{\lambda}\right)^{n-1}\right)$$

In our discussion on M, we utilize the interpolation to obtain the strong-type estimate from the weak-type one. However, one can obtain the boundedness of

M_s on L_p via an inductive argument. Analogous to A_p, we define the *strong A_p weight* (denoted as A_p^*) that preserves the (3.1.2) when the supremum is taken over rectangles in $\mathscr{R}$. Note that $w \in A_p^*$ implies that $w \in A_p(\mathbb{R})$ uniformly on any line parallel to the coordinate axis.

Similar to how we use the weights theory to arrive at the boundedness of M on BMO, we are naturally led to ask: is M_s bounded on *bmo*?

5.2. Equivalent Problems

In the one-parameter result, the behavior of M on A_∞ allows us to conclude the boundedness of M on BMO. Precisely, Ou [**9**] pointed that $M^\natural$ maps BMO function into BLO corresponds exactly to the behavior that M maps A_∞ into A_1.

Theorem 5.2.1. *$M^\natural : BMO \to BLO$ is equivalent to $M : A_\infty \to A_1$.*

The reverse direction was already shown in the proof of the boundedness of M on BMO. The forward direction can be shown using the commutation lemma together with a special characterization of BLO given in the following proposition. We leave the proof of this proposition for the readers [**1**]. Remark that the multiparameter version of this proposition is also true for *blo* (functions of bounded lower oscillaton with supremum taken over rectangles in $\mathscr{R}$).

Proposition 5. *$\phi(x) \in BLO$ if and only if $M\phi(x) \le \phi(x) + \|\phi\|_{BLO}$.*

Proof Of Theorem. When $w \in A_\infty$, apply the commutation lemma twice to w,

$$e^{M^\natural M^\natural \log w(x)} \le MMw(x) \le A_\infty(Mw)A_\infty(w)e^{M^\natural M^\natural \log w(x)}$$

Since $M^\natural w \in BLO$ by hypothesis, use the Proposition 5 to yield

$$\begin{aligned} MMw(x) \approx e^{M^\natural M^\natural \log w(x)} &\le e^{M^\natural \log w(x) + \left\|M^\natural \log w(x)\right\|_{BLO}} \\ &\approx Mw(x)e^{\left\|M^\natural \log w(x)\right\|_{BLO}} \end{aligned}$$

where $e^{\left\|M^\natural \log w(x)\right\|_{BLO}}$ is finite by hypothesis. So $Mw(x)$ is an A_1 weight. □

Notice that the above equivalence still works for the multiparameter maximal operator: the commutation lemma holds for the strong A_p weights, as the definition of A_p^* guarantees that A_p^* weights will satisfy the Jensen's inequality and the reverse Jensen's inequality. The theorem that $\log(A_\infty^*) \in bmo$ is also true, given the proof of the one parameter result does not utilize the engulfing property of the geometric objects defined in the maximal function. One can obtain the proof of this theorem by simply following the proof of Theorem 3.3.1 shown in the previous chapter and replacing balls B with rectangles R. Thus in order to prove M_s is bounded on *bmo*, due to above equivalence, it suffices to find whether the following statement is true:

Conjecture. $M_s(A_\infty^*) \in A_1^*$.

5.3. Soria's Counterexample

In previous chapters, the Corollary 3.2.1 that $M(A_\infty) \in A_1$ is shown as a simple corollary of the Coifman-Rochberg proposition (Proposition 4). Notice that Proposition 4 holds in $\mathbb{R}^d$, where the maximal function is taken over hyper-cubes or hyper-balls with the doubling properties. However, the strong maximal function considers rectangles of arbitrary eccentricity. The statement analogous to Proposition 4 might not be true in multiparameter case as the other theorems do shown in previous section. One distinguishment of Proposition 4 is that the proof relied on the weak-type $(1, 1)$ estimate of M, but M_s does not have this property.

Indeed, Soria [**10**] provides a function g which, when operated by M_s and raised to *any* fractional power, is not an $A_1(M_s)$ weight. To state it precisely, there exists a function $g \in \cap_{0<p<\infty} L^p(\mathbb{R}^d)$ such that, for every $0 < \delta$ and every constant $C > 0$, there is a set S of positive measure such that

$$M_s(M_s g)^\delta(x) \geq C(M_s g)^\delta(x), \forall x \in S. \tag{5.3.1}$$

Soria's counterexample is constructed using a sequence of functions $v(x) = \{v_N(x)\}$ with compact support. We define each function $v_N(x)$ as follows. For $k, j \in \mathbb{Z}$, let $Q_{k,j}$ denote the unit square (assuming $d = 2$ for simplicity) with the lower left corner at the point (k, j). For $k = 1, 2, \ldots, N$ and $d_k = \frac{2^k}{k}$, we let the height of the function V_N on the unit square Q_{k,d_k} be 2^k, i.e.

$$v_N(x) = \sum_{1\leq k\leq N} 2^k \chi_{Q_{k,d_k}}(x)$$

To give a picturesque description, let us call $v_N(x)$ *the N-th city* and define the area of city as $[1, N+1] \times [1, \frac{2^N}{N} + 1]$, thus denote $|city| = |[1, N+1] \times [1, \frac{2^N}{N} + 1]| = 2^N$ (that is the area of its support). Each city contains n number of *towers*, and the height of k-th tower is 2^k. The essential supremum of the city is the height of the tallest tower, so $\|v\|_\infty = 2^N$. Those cities satisfy the following lemma, which is essential in constructing the counterexample.

Lemma 5.3.1. *For all $N \in \mathbb{N}$, there exists $v = v_N$ with compact support so that*

$$\{M_s(M_s v)^\delta(x)\}^{1/\delta} \geq C_\delta N M_s v(x),\ \forall x \in [0, 1]^2$$

Observe that $\|v\|_{L_p}^p = \sum_1^N 2^{kp}$ for $0 < p < \infty$. One can use the formula for geometric series to show $\|v_N\|_{L_p} \leq c_p 2^N$:

$$\|v_N\|_{L_p} = \Big(\frac{2 * p(2^{Np} - 1)}{2^p - 1}\Big)^{1/p} \leq \frac{2(2^{N\dot{p}})^{1/p}}{(2^p - 1)^{1/p}} = c_p 2^N$$

with $c_p = \frac{2}{(2^p-1)^{1/p}}$. We scale the cities to make $v(x)$ a L_p function, $2^{-2N} v_N \in L_p$ for $0 < p \leq \infty$ and for all N. Place those scaled cities distant from another, for example at points $x_N = (2^N, 2^N)$, i.e.

$$g(x) = \sum 2^{-2N} v_N(x - x_N)$$

Notice that the values of $M_s g$ for any point in unit square $[x_N, x_N + 1]^2$ are the same as the values of $M_s(2^{-2N} v_N)$ in $[0, 1]^2$. g does not have compact support outside of the cities, and cities are sufficiently far from each other. To calculate $M_s(g(x))$ where x is in $[x_N, x_N + 1]^2$, one does not have to consider rectangles that stretch to other cities. In other words, if we look at each city individually, in the N-th city v_N, for x in $[0, 1]^2$, $M_s(v_N(x))$ is the supremum taken over some rectangle inside $[0, N+1] \times [0, \frac{2^N}{N} + 1]$. When N goes to infinity, for any x on $[x_N, x_N + 1]^2$, the value of $\{M_s(M_s v)^\delta(x)\}^{1/\delta} = O(N)$ becomes infinitely large. One cannot find a multiple of $M_s v(x) = O(1)$ that is comparable to $M_s(M_s v)^\delta(x)$ for *any* x on $[x_N, x_N+1]^2$ and for *every* $N \in \mathbb{N}$. Thus, the inequality (5.3.1) holds on a set of infinite measure.

5.4. Attempts to Make a Strong Muckenhoupt Weight Counterexample

Soria's counterexample is exciting but does not necessarily allow us to conclude that the conjecture 5.2 is false. Notice g is locally integrable but not a weight. Following the proof of corollary 3.2.1, we hope to construct an A_∞ weight, based on g, that preserves two of its properties: 1) lemma 5.3.1. 2) for any $x \in [0,1]^2$ in v_N, the value of $M_s(v_N)$ does not change after we place an infinite number of cities (in appropriate locations) on the same plane.

As an initial step towards A_∞, we would like to make a non-negative function by adding a constant term $\alpha > 0$, like a "floor" under the city. Now we have $\tilde{v}(x) = v(x) + \alpha$, where $v(x)$ is inherited from (the same as in) Soria's example. The essential supremum only changes by α, i.e. $\|\tilde{v_N}\|_\infty = 2^N + \alpha$, and the L_p norm is still bounded by $O(2^N)$,

$$\|\tilde{v}_N\|_{L_p} \leq \|\tilde{v}_N\|_{L_p} + N\alpha \leq \tilde{c_p} 2^N$$

In order to show $\tilde{v}$ satisfies the lemma, we rely on Soria's proof of the lemma but with some modification. Consider the rectangle $R_k = [1, k+1] \times [1, d_k + 1]$. By subadditivity of M_s, for any $x \in [0,1]^2$,

$$M_s\tilde{v}(x) \leq M_s v(x) + M_s(\alpha) \leq M_s v((1,1)) + \alpha$$

$$= \sup_{1 \leq k \leq N} |R_k|^{-1} \int_{R_k} v(y)dy + \alpha \leq 2 + \alpha$$

Note that the strong maximal function of any constant function is still constant. Then consider the rectangle $\tilde{R_k} = [k, k+1] \times [1, d_k + 1]$, which looks like a strip on column k. The strong maximal function of any $x \in \tilde{R_k}$ is larger than $k + \alpha$,

$$M_s\tilde{v}(x) \geq |R_k|^{-1} \int_{R_k} (v(y) + \alpha)dy = \frac{2^k + d_k\alpha}{d_k} \geq k + \alpha$$

If we then take the rectangle $R = [0, N+1]^2$, and denote $f = M_s\tilde{v}$, we see that for $x \in [0,1]^2$

$$
\begin{aligned}
\{M_s(M_s\tilde{v})^\delta(x)\}^{1/\delta} &\geq \{M_s(f)^\delta(x)\}^{1/\delta} \\
&\geq (|R|^{-1} \int_R f^\delta)^{1/\delta} \\
&\geq ((N+1)^{-2} \int_R f^\delta)^{1/\delta} \leq N \qquad (5.4.1) \\
&\geq [(N+1)^{-2} \sum_{j=1}^{N} \sum_{k \geq 2\log_2 N}^{N} (k+\alpha)]^{1/\delta} \approx O(N)
\end{aligned}
$$

Recalling that $M_s\tilde{v} < 2 + \alpha$, we confirm that the new city constructed on a constant floor preserves the lemma. We then place each city with its left bottom corner at $X_N = (2^2N, 2^2N)$ without scaling down its height, i.e.

$$g'(x) = \sum \tilde{v}_N(x - x_N) = \sum (v + \alpha)(x - x_N)$$

Given the floor under all cities are the same, the value of the strong maximal operator decreases as the rectangle exceeds the city area, so $\tilde{g}(x)$ satisfies the second criteria. Obviously, $\tilde{g} \notin L_\infty$, as the tallest tower 2^N determines the essential supremum, which is not bounded when N becomes arbitrarily large. Further, $\tilde{g}(x)$ is not an A_∞ weight as it fails the first characterization of A_∞ stated in theorem 3.1.1: take Q as the unit square of the tallest tower combined with an adjacent unit square of the floor, and let E be that unit square of floor, so $|E| \subset |Q|$ and $|E| = \frac{1}{2}|Q|$. Observe that $w(E) = \frac{\alpha}{2^N+\alpha} w(Q)$ and $\frac{\alpha}{2^N+\alpha}$ vanishes as N approaches infinity. A_∞ weights require a fixed ratio between the weights of two embedded sets.

To make the function bounded above, we mimic Soria's example by scaling each city by a multiplicative factor of 2^{-2N} and placing them at points $x_N = (2^N, 2^N)$, and then we add a universal constant α to all cities. However, this setting violates the lemma, as each city becomes $2^{-2N}v_N + \alpha = 2^{-2N}(v_N + 2^{2N}\alpha)$. To ensure that the lemma holds, we want each city to be of the form $2^{-2N}(v_N + \alpha)$, so that the floor needs to be short enough for every city. Let the height of the floor be $2^{-2N}\alpha$, but $2^{-2N}\alpha$ vanishes as N approaches ∞. Thus, we cannot make a weight using this strategy.

Another possibility would be to make the constant α dependent on the height of each tower. Take Soria's function g and filling in the gap among cities, we have

$$g^\natural = \sum \{2^{-2N}(v_N(x - x_N)) + 2^{-2N}\alpha\chi_{\tilde{x_N}}\}$$

$$\text{where } \tilde{x_N} = [2^N \le |x_1| < 2^{N+1}] \times [2^{N-1} \le |x_2| < 2^N]$$

Regrettably we find $g^\natural$ does not satify the second criteria. For x in $[2^N, 2^N + 1]^2$, $M_s(v_N(x))$ is the supremum taken over some rectangle that stretches to other cities. To be more precisely, consider the Nth city, based on Soria's reasoning, for x in $[2^N, 2^N + 1]^2$

$$M_s(v_N(x)) \le Ms(v_N(2^N + 1, 2^N + 1)) = \frac{1}{|city|} \int_{city} g$$

$$= \frac{2^{-2N}}{2^N} \int_{city} (v_N + \alpha) \approx \frac{2^{-2N}(2^N + 2^N)}{2^N} \approx \frac{1}{2^{2N}}$$

But consider the rectangle $R = [0, 2^N] \times [0, 2^N]$ containing all the cities and floors before Nth city, we obtain

$$\frac{1}{|R|} \int_R g'' \ge \frac{1}{2^{2N}} \int_R 2^{-2N}\alpha\chi_{\tilde{x_N}} \approx \frac{N}{2^{2N}} \ge \frac{1}{2^{2N}}$$

that is, the weight of the floors *overpowers* the weight of the towers.

From the above example, we notice when g becomes an A_∞^* weight, it gets harder to control $M_s v(x)$ where $x \in [0,1]^2$ of each city. The floor could certainly be scaled much shorter so that the total weight of floor does not grow at the rate of $O(N)$. But to construct an A_∞^* weight, we need to restrict the ratio between the height of tower and the height of floor, that is, the floor cannot be infinitely short compared to the tower. The results in this section could add to the discussion that maybe the characteristics of A_∞^* are so restrictive that they essentially force us to violate the lemma. The proof of this statement will make a huge step in studying the behavior of M_s.

It seems that it is not trivial to construct an A_∞^* weight example to meet Soria's proposition 5.3.1. Part of the trouble understanding the boundedness of M_s on *bmo* relates to the difficulty of constructing a non-trivial A_∞^* example. That is, if $u(x)$ and $v(x)$ are one-parameter A_p weights, then we know $w(x) = u(x)v(x)$ is a strong A_p weight. However, since $M_s w \leq M(w)M(w)$, obviously $M_s w \in A_1^*$, as the right-hand-side satisfies the Coifman-Rochberg proposition which holds true in higher dimension. In other words, we only know the well-behaving trivial examples. There do not seem to be any good examples in the literature of strong A_p weights that do not arise as the product of one-parameter A_p weights. Constructing a nontrivial strong A_p weight example can also be a breakthrough to understand the boundedness of the strong maximal function on BMO.

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