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Computationally Recovering Preferred Factorizations of Quantum Hilbert Space

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Computationally Recovering Preferred Factorizations of Quantum Hilbert Space

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by

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Contents

ABSTRACT: This thesis addresses the question of the preferred factorization of the quantum mechanical Hilbert Space into sub parts. Specifically, I computationally implement streamlined aspects of the recent "in principle" proposal of Carroll and Singh in their paper *Quantum Mereology: Factorizing Hilbert Space into Subsystems with Quasi-Classical Dynamics*. The goal is to link the selection of a preferred tensor product factorization to the appearance of quasi-classical behavior in this preferred factorization. Carroll and Singh quantify quasi-classical behavior through the criteria of "robustness" and "predictability" using the purity and pointer entropies. This work tests whether it is necessary to use both entropies or whether the minimization of purity entropy is sufficient in selecting quasi-classical behaviour. This coding platform sets the stage for the application of machine learning to the problem of preferred basis.

1 Introduction

Starting with two qubits with their respective self-Hamiltonians and Hilbert Spaces, constructing the quantum description of the joint system is simple. The total Hilbert Space is given by the tensor product of the two individual Hilbert Spaces. The joint Hamiltonian is the sum of both self-Hamiltonians, plus an interaction Hamiltonian coupling the two qubits.

The less examined question is the reverse direction: when given a composite Hamiltonian and a four dimensional Hilbert Space, is there a canonical way to recover the tensor product factorization that supports the system of two individual qubits? There are an infinite number of possible partitions or factorizations, induced by unitary change of bases on the total Hilbert Space. In our experience of the classical world, however, there is usually one clear preferred factorization of systems into subsystems (which is most likely correlated with locality in position, as seen in previous research)[1]. What then is the criteria to find the "correct", or perceived, factorization for a given physical system?

Max Tegmark refers to our perception of objects in the physical world as an object hierarchy [1]. To exemplify this, he describes drinking a beverage with ice. You perceive the ice cubes in the glass as separate objects from the drink

because they are "fairly integrated and fairly independent."[1]

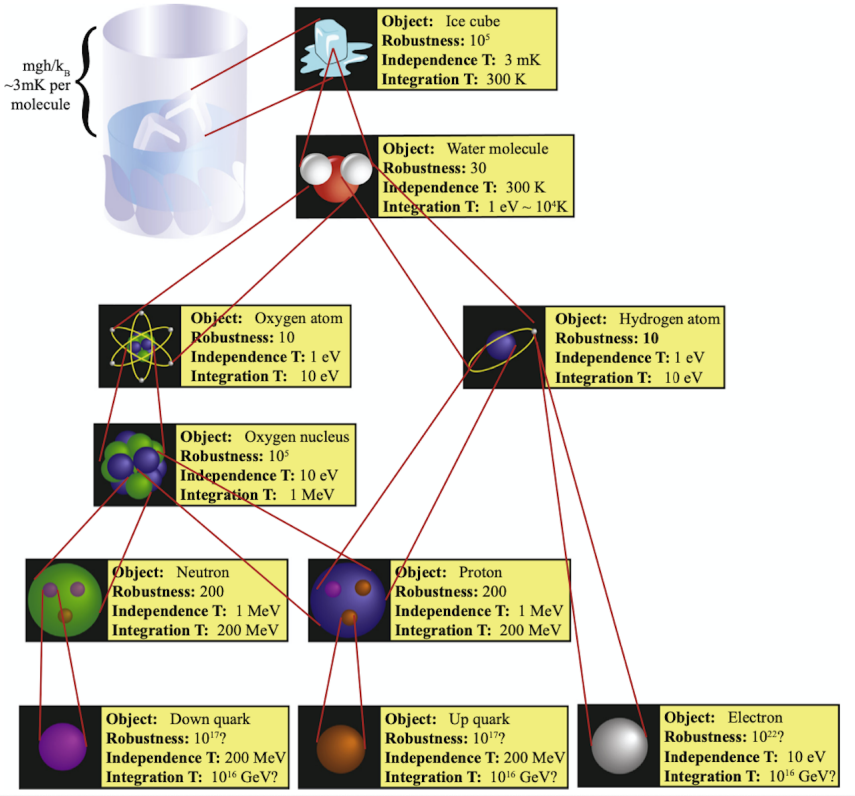


Figure 1: Conscious observers perceive the external world as a dynamic hierarchy of objects, whose parts are more strongly connected to each other than to the outside. This is illustrated by Tegmark using a glass of water with ice. Tegmark defines the robustness of an object as the ratio of the integration temperature (the energy per part needed to separate them) to the independence temperature (the energy per part needed to separate the parent object in the hierarchy). Figure from Tegmark [1].

This object hierarchy persists from the microscopic (electrons) to the macroscopic world (as large as planets and galaxies). The universe itself has no notion of a preferred separation, but the human brain perceives the universe as a tensor product of individual yet integrated subsystems. In the terminology of Tegmark, this thesis asks how beginning with a finite dimensional Hilbert Space, a Hamiltonian operator, and an initial state, we can determine the object hierarchy of the universe and recover subsystems that allow for quasi-classical description.

This is done by dividing the variables into groups with the greatest independence.

One algorithm to do so has recently been proposed by Carroll and Singh [2]; this thesis tests a streamlined modification of their approach which, in turn, builds on Zurek's idea of the "predictability sieve"[3] as a selection criterion. We begin with a decomposition and basis and then select an unentangled state with respect to this basis. Afterwards, the rate of growth of entanglement for the decomposition is measured by calculating the entropy. Treating one subsystem as the system and the other as the environment, the decomposition that minimizes entanglement growth between system and environment is the "correct" decomposition because subsystems retain their identity long enough to be registered as self-integrated objects. More precisely, it is the decomposition that identifies the object hierarchy, or cut between system and environment. This factorization is what conscious observers perceive.

The factorization into system and environment has potential implications on the specification of states in quantum error-correcting code [4, 5] and on Everett's interpretation of decoherence [6]. According to the Everett interpretation, there is no "collapse" of states during measurements and, therefore, no definition of a measuring device or observers. A challenge of this interpretation, then, is to explain where classical observers fit into the picture and why they perceive quasi-classical dynamics, characterized by relatively stable "systems" interacting with "environments". The quantum factorization problem is, therefore, important to complete this interpretive picture because it asks: how can one even divide the Hilbert Space into system and environment?

This thesis builds on various works in the field of the foundations of quantum mechanics and emergence of classicality. Tegmark [1] approaches this problem from the perspective of human consciousness and investigates the emergence of distinctive subsystem information processing abilities. Further, Zanardi develops an algebraic framework for the emergence of a multipartite tensor product factorization [7]. This framework is extended to structures in the Hilbert Space in the work of Kabernik, Pollack, and Singh [8]. Brun and Hartle [9] studied quantum harmonic oscillators and the emergence of preferred coarse-grained classical variables.

Ultimately, I successfully coded a Python platform to run the algorithm of Carroll and Singh. The framework involves evolving an arbitrary quantum system forward in time, taking the partial trace, and assigning a score to a given factorization based on rate of entropy growth. The code was run on the system

of two qubits and the model of the coupled harmonic oscillator. This sets the stage for applying machine learning to find the preferred factorization given a Hamiltonian for an arbitrary non-pointer basis tensor product space. The first steps of applying the machine-learning search have been taken and have produced initial results, but the search is made difficult by long computation times.

The paper is organized as follows: Subsection 2.1 of the Background describes the foundations of quantum mechanics necessary to have a general understanding of the problem of preferred factorization. Specifically, it describes the principle of superposition and notion of entanglement in relation to the emergence of quasi-classical behavior. Subsection 2.2 outlines the relevance of decoherence in helping to find the pointer basis, and the methods motivating the algorithm we use. Subsection 2.3 explains density matrices and the equations used throughout the Python platform to calculate the entropy and time-evolution. Section 2.4 grounds all of these concepts and equations in a concrete example of a book interacting with photons. In Section 3, the algorithm implemented in the Python coding platform is extensively explained. Section 4 explains methods of verifying the coding platform and Section 5 discusses the results found thus far, including the new result of a two qubit system.

2 Background

2.1 Foundations of Quantum Mechanics

2.1.1 The Superposition Principle

A fundamental axiom of quantum mechanics is the principle of superposition. Namely, given a Hilbert Space of states, any linear combination of states in the Hilbert Space is also a possible state. A quantum state vector can be expressed in Dirac notation as $|\Psi\rangle$. Therefore, we can express the set of states in a Hilbert Space as $|\Psi_n\rangle$ and superposition states as

$$|\Psi\rangle = \sum_n c_n |\Psi_n\rangle \ , \tag{1}$$

where c_n are arbitrary complex coefficients and $|\Psi\rangle$ is a new and valid quantum state vector.

The principle of superposition raises a puzzle as it seems at odds with the

familiar classical world of our perception. This discord is dramatically illustrated by the thought experiment of Schrödinger's cat. A cat has states $|Alive\rangle$ and $|Dead\rangle$ [10], yet, because a cat is a collection of particles, superposition implies that an equally valid state is $|Alive\rangle + |Dead\rangle$. In fact, the rules of time evolution predict that interactions between microscopic particles and the cat should cause it to evolve into such a superposition. Yet, we never observe cats in a superposition of alive and dead, we simply see them in one or the other. There is something special about the states $|Alive\rangle$ and $|Dead\rangle$ that is not true of the state $|Alive\rangle + |Dead\rangle$. We have no measuring devices, in this case, our eyes, to observe superposition states. Initially, it was thought that superposition was a strange phenomenon limited to microscopic systems because of how much it contradicts our everyday macroscopic experience. It was theorized that there existed a "cut" between the microscopic world where the laws of quantum mechanics applied and the macroscopic world where classical theory applied [10].

While superposition has been experimentally proven to be true for systems of microscopic particles, over time it has been validated on increasingly larger scales, challenging the notion of a quantum/classical "cut"[10]. It has become clear that a key role in resolving the above contradiction is the particular form of interaction of "system" (cat) with "environment" (photons, etc.). As I detail below in Section 2.2, the phenomena of decoherence offers at least a partial answer, identifying certain states of the cat as "pointer states" robust to interactions with the environment. However, this leaves an even more fundamental question unanswered: why do we even perceive a self-integrated system we label "cat" interacting with another set of states we label "environment"?

2.1.2 Entanglement

A nontrivial consequence of the principle of superposition is quantum entanglement, when a set of particles exist in such a way that the state of each particle cannot be described independently from the others. For example, given a system composed of two subsystems A and B, if system A is in state $|\Phi\rangle_A$ and system B is in state $|\Psi\rangle_B$ the state of the complete system is

$$|\Phi\rangle_A \otimes |\Psi\rangle_B \ . \tag{2}$$

A state vector of the composite system is entangled if it cannot be written as a tensor product of state vectors of A and B.

For two spin-$\frac{1}{2}$ qubit subsystems A and B, an unentangled state is

$$|0\rangle_A \otimes |1\rangle_B \ . \tag{3}$$

In this notation, the basis states are $|0\rangle$ and $|1\rangle$ which corresponds to spin-up and spin-down. Conceptually, an unentangled state can be visualized as two systems, existing in states individually.

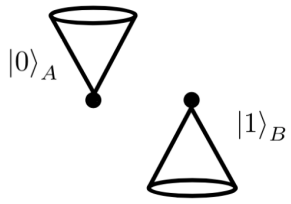


Figure 2: The complete system state is described by $|0\rangle_A \otimes |1\rangle_B$.

An example of a maximally entangled state is

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \ . \tag{4}$$

This state is a Bell state. The four Bell states comprise a maximally entangled basis of the four-dimensional Hilbert Space for spin-$\frac{1}{2}$ particles.

Once entangled, the individual state of the two systems has disappeared and what exists is a joint object. This interaction can be visualized as a line now linking the two subsystems and illustrating their loss of individuality.

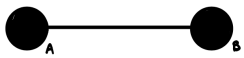


Figure 3: $|\Phi^+\rangle$.

There are degrees of entanglement, and as stated before, the Bell basis states are maximally entangled. As system A and system B become increasingly entangled, the total state of A is encoded in the joint state of subsystem A and B. The coherence that was in the subsystem A becomes a feature of the joint state, leading to decoherence. Although the joint system is in a "pure" state, if we are confined to making observations on subsystem A alone, then we are reduced to speaking of classical probabilities. To completely specify the state, we need to

be able to make measurements on the joint system. In a maximally entangled state, there is a 50% chance outcome of any measurement on subsystem A alone, making it completely random. The structure of the world, though, is such that measurements do not have a 50% chance outcome all of the time.

It is important to note that the notion of entanglement is factorization dependent. For example, given the state $|\Phi^+\rangle$, one can redefine the basis such that it is no longer an entangled state. In equations 3 and 4, the basis is defined in terms of subsystems A and B, where A and B are the qubits [7]:

$$\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B \ . \tag{5}$$

Or written in terms of product basis:

$$|A\rangle \otimes |B\rangle \equiv |A, B\rangle \, (A, B \in 0, 1) \ . \tag{6}$$

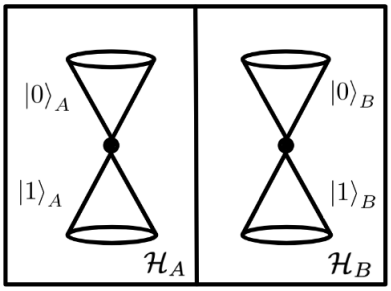


Figure 4: This factorization allows for the clear division of the joint Hilbert space into subsystems A and B.

It is just as valid to factor the Hamiltonian as [7]

$$\mathcal{H} \equiv \mathcal{H}_{\Phi,\Psi} \otimes \mathcal{H}_{+,-} \ . \tag{7}$$

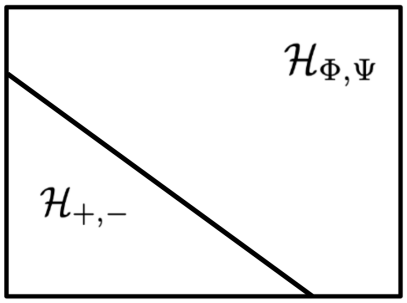


Figure 5: This factorization of the joint Hilbert Space is not adapted to our perception of A and B as classical observers. To tell apart A and B in this example, one would have to appeal to other characteristics aside from spin, such as the particles' locations.

In this new factorization, the maximally entangled Bell basis states become product states and the subsystems become the $\Phi, \Psi, +, -$ degrees of freedom. The state depicted in Figure 3 is no longer entangled with respect to this new basis because it can be written as a tensor product:

$$\left|\Phi^{+}\right\rangle = |\Phi\rangle \otimes |+\rangle \ . \tag{8}$$

In the other direction, some states that were not entangled in the original basis are now entangled with respect to the new basis. By changing the factorization, one is changing how the total 4-state system is composed of two 2-state systems. In terms of Schrödinger's cat, a new factorization would destroy the split between cat and universe that we classically perceive. This split is almost always implicit, but it is not obvious how this division comes from quantum mechanics itself, nor is it clear why classical observers even perceive one collection of states "cat" separately from another set of states "photons", or qubit A separately from qubit B.

To illustrate the growth of entanglement due to interactions, I selected a simple interaction that is easy enough to do by hand Later, this will be coded. We start with a Hamiltonian that will induce interactions between qubit A and qubit B [3]:

$$\hat{H} = \lambda \left|1\right\rangle \left\langle 1\right|_{A} \otimes \frac{1}{2}(\left|0\right\rangle \left\langle 0\right| + \left|1\right\rangle \left\langle 1\right| - \left|0\right\rangle \left\langle 1\right| - \left|1\right\rangle \left\langle 0\right|)_{B} \ , \tag{9}$$

where λ is the coupling coefficient. This is the physically motivated "controlled

NOT" (C-NOT) Hamiltonian that is an essential logic gate in quantum computing. After interaction time π, the C-NOT toggles the value of the target qubit when the value of the control qubit is 1, and does nothing if the value of the control qubit is 0. Setting λ to 1 and writing the Hamiltonian explicitly in matrix form yields

$$\hat{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} . \tag{10}$$

Starting in the state $|1\rangle \otimes |0\rangle$ at t = 0 and evolving it forward in time to t = π using the Schrödinger equation results in the state $|1\rangle \otimes |1\rangle$.

Using this same Hamiltonian, and beginning qubit A in the pointer basis states $|0\rangle_A$ or $|1\rangle_A$ and qubit B in a uniform superposition of all basis states, $\frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B)$, we can evolve the joint qubit starting state forward in time and show that the resulting qubit A state produces an entropy of 0. This resulting state is unentangled with qubit B.

If we were to apply a scrambling that did not respect the factorization of qubit A and qubit B, we could still get a final entropy of 0 for the evolved qubit A state, showing us that the form of the spin Hamiltonian is insufficient to select a preferred factorization. Concretely, scrambling the Hamiltonian with the unitary matrix

$$\hat{U}_{scr} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} , \tag{11}$$

we get the new Hamiltonian

$$\hat{H}' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{12}$$

The Hamiltonian in this new basis is diagonal, so the joint system-environment state will not evolve into an entangled state.

This result shows that there are at least two ways to factorize the Hamilto-

nian of two spin-$\frac{1}{2}$ qubits such that they exhibit quasi-classical behavior after interacting. This particular unitary matrix maps the unentangled product state to the entangled product state. The initial joint starting state $\frac{1}{\sqrt{2}}(|0\rangle_A + |0\rangle_B)$ is mapped to the Bell basis state $|\Phi^+\rangle$. Since our unscrambled result was unentangled and therefore produced an entropy of 0, this scrambled result should be entangled. Why, then, does it produce an entropy of 0? For two spin particles, there is no notion of position, so there are various factorizations that will produce an entropy of 0. The criteria of robustness alone is insufficient to tell us the quasi-classical tensor factorization. In other words, the 4-dimensional space of two qubits has no obvious decomposition into sub-parts that describes the quasi-classical world.

2.2 Decoherence Theory and Foundations of Quantum Mechanics

The principle of superposition, entanglement, and decoherence are fundamental to the measurement problem of quantum mechanics, which works to explain how the quantum domain gives way to classical systems and properties. The part of the measurement problem that is focused on in this thesis is the problem of preferred basis of the tensor product (or preferred factorization), which asks how our perceived split of objects emerges from quantum formulations. This is closely linked to justifying the emergence of classical observables from quantum states, or asking why we only observe a subset of states in the preferred basis even though all are equally valid in quantum mechanics. Decoherence straddles the border between the quantum and classical and is the field which studies part of this problem and explains how the system goes from a quantum superposition to classical probabilities of states. It seems to have the potential to single out certain states (such as $|Alive\rangle$ and $|Dead\rangle$ in the case of Schrödinger's cat) as the quasi-classical ones of our experience.

As alluded to, decoherence is the study of the interactions between systems and their environments. Specifically, it investigates the loss of system information to the environment. Given an initial quantum state, one can evolve it forward in time like a classical system using the Schrödinger equation. As time moves forward the system interacts with the environment and the reduced density matrix of the system goes toward diagonal form when it is in a specific basis. The diagonal terms describe the probability distribution for the outcomes of measurements on the system. This phenomenon is environment-induced de-

coherence and implies that in a specific basis, a subset of states pre-selected by the environment emerge. This subset of states constitutes what we see in the classical world. Therefore, decoherence is what leads to the environment-induced superselection, or einselection.

This specific basis is referred to as the pointer basis, as introduced by Zurek and Paz [11]. The term pointer here comes from the pointer display on many measuring devices. The pointer basis is comprised of pointer states which are robust to repeated measurement and stable to environmental interactions. In a basis that is not the pointer basis, the degree of entanglement will reach a maximum value instantaneously because decoherence occurs instantly. When this occurs, classical observers effectively see the system snap onto a certain state. Hence, the pointer basis is the preferred basis such that decoherence, or loss of information to the environment, is minimized.

In this work, we attempt to define the pointer basis of the tensor product based on the criteria of robustness. The tensor product describes the way that the Hilbert Space can be viewed as a state space of a multipartite quantum system [7].

Defining this in terms of linear algebra, given a Hilbert Space $\mathcal{H} \cong C^n$ with no tensor product structure (TPS), if n is not prime, the Hilbert Space has a variety of different tensor factorizations. These factorizations of n are of the form [7]:

$$n = \prod_{i=1}^{r} p_i^{n_i} (p_i < p_{i+1}) \ . \tag{13}$$

When the exponent n_i of the prime factor p_i of n is not equal to one, there are several possible regroupings. For example, when $r = 1$, $p_1 = 2$ and $n_1 = 3$, there are valid groupings $3 = 1 + 1 + 1$ and $3 = 1 + 2$. These groupings correspond to the state space factorizations $C^8 \cong C^2 \otimes C^2 \otimes C^2$ and $C^8 \cong C^2 \otimes C^4$. If n is decomposed into more than one prime number, the number of factorizations of n as a product of integers grows. We can define the set of factorizations for a given n as

$$\mathcal{P}_n = \{P \subset \boldsymbol{N} / \prod_{m \in P} m = n\} \ , \tag{14}$$

where $\boldsymbol{N}$ is the set of natural numbers.

Given a factorization of n in the set $\mathcal{P}_n$, one can construct the isomorphisms $\varphi : \mathcal{H} \mapsto \otimes_{j=1}^{|P|} C^{n_j}$. These isomorphisms form a TPS over $\mathcal{H}$. A TPS can be used to define a group of unitaries. For the TPS φ_0, the unitaries $\mathcal{U}(\mathcal{H})$ and $\mathcal{U}(\otimes_j C^{n_j})$

can be defined through the algebraic isomorphism $U \mapsto \varphi_0^{-1} \circ U \circ \varphi_0$. Two unitaries $U(\vec{\theta})$ and $U(\vec{\theta'})$ define the same factorization into subsystems when $U(\vec{\theta'}) = (\vec{U}_A \otimes \vec{I}_B) \cdot U(\vec{\theta}) \cdot (\vec{I}_A \otimes \vec{U}_B)$ for some $\vec{U}_A : \mathcal{H}_A \to \mathcal{H}_A$ and $\vec{U}_B : \mathcal{H}_B \to \mathcal{H}_B$. In the code produced for this thesis, I search over the parameter space of all unitary transformations, some of which change factorization and some of which are of the form $U = (\vec{U}_A \otimes \vec{I}_B) \cdot (\vec{I}_A \otimes \vec{U}_B)$ (merely changing basis within a factorization).

Our strategy to find the preferred factorization is to search the space of all unitary matrices for the one that exhibits quasi-classical behavior. For ease, we measure robustness, and therefore quasi-classical behavior, using the purity ($Tr\hat{\rho}^2$) entropy of a reduced state rather than the von Neumann entropy, although both are equally valid. There are many other potential criteria for selecting a pointer basis.

This notion of searching is the idea of the predictability sieve from Zurek [12]. The predictability sieve traverses the Hilbert Space of system interacting with environment and selects states that are most predictable as classical states. This concept is grounded in the fact that classical states evolve predictably. Finding states that are robust to entropy generation is equivalent to finding predictable states. Specifically, Zurek does this by fixing factorization and optimizing for pointer basis only. Taking this a step further, this thesis notes that the factorization into system and environment can itself be subjected to a kind of sieve, the factorization sieve, perhaps. The notion of a preferred factorization of system and environment is a human artifice that the universe has no notion of. By allowing the scrambling matrix to act on the entire tensor product space (the Hamiltonian of the system and environment) and not just on the system itself, we are mixing up the divide between system and environment. After scrambling, part of the system may be part of the environment and vice versa. The sieve we implement optimizes for basis and factorization.

One can conceptualize the landscape we traverse as being divided into a horizontal and vertical direction. Moving vertically in the landscape does not alter the factorization into system and environment, but does alter what we are considering the pointer basis of the system and ready state of the environment. Moving horizontally in this landscape moves to a new factorization. By setting up the code framework to apply machine learning and traverse the landscape in all directions, we are finding the lowest entropy across both criteria.

In sum, the focus of this paper is to explore the criteria to determine the emergence of the preferred pointer basis for the tensor product. Specifically, we

investigate how to find the preferred pointer basis given a scrambled system in a finite amount of time.

2.3 Density Matrices

We use density matrices to represent states of the system and joint system-environment. Density matrices are a key tool in the formal description of decoherence. When the system and environment become entangled, it is no longer possible to represent the system as an individual quantum state vector. Reduced density matrices allow us to represent the results of all possible measurements on the system alone (i.e., if we are ignorant of the exact environment state, as we most often are). This, in turn, lets us quantify the degree of entanglement with the environment.

A pure or mixed state of a physical system can be represented as a series of bra and ket quantum state vectors in Dirac notation. Using this combination of quantum state vectors, one can define the density operator. So, rather than thinking of a state of a system as a vector, the density operator represents it as a projection operator onto a state for a pure state, or sum of projectors for a mixed state, acting on the corresponding Hilbert Space. The density matrix is a matrix representation of the density operator in a chosen basis. The two terms are used interchangeably. Given a state $|\Psi\rangle$ in a Hilbert Space, one can construct the density operator representation of this state through the simple operation

$$\hat{\rho} = |\Psi\rangle \langle\Psi| \ . \tag{15}$$

Given a state expressed as a superposition of basis states (of the form of equation 1 in 2.1.1), the density matrix of this state can be written as

$$\hat{\rho} = |\Psi\rangle \langle\Psi| = \sum_{ij} c_i c_j^* |\Psi_i\rangle \langle\Psi_j| \ . \tag{16}$$

The density matrix provides an efficient means to calculating expectation values, entropy and reducing the entire system to a subsystem state. Given an observable A, the procedure for calculating the expectation value of A using state vector notation is

$$\langle A\rangle = \langle\Psi| \hat{A} |\Psi\rangle \ . \tag{17}$$

Using the corresponding density matrix, $\hat{\rho}$, the equation becomes:

$$\langle A \rangle = tr(\hat{A}\hat{\rho}) \ . \tag{18}$$

Therefore, it is clear that when two ensembles have the same density matrix, it is impossible to distinguish them. For example, given two boxes of identically prepared particles, the result of measuring the same physical quantity of both will likely result in different outcomes. In other words, quantum states and in turn density matrices do not specify the exact result of measuring the state of the system; they allow us to calculate the probability distributions of a set of possible measurement outcomes.

The real benefit of using density matrices comes when distinguishing the subsystem from the whole using the trace operation. The separation of the world into subsystem and environment is crucial for decoherence and entanglement. If the universe is not divided into individual subsystems, the measurement problem does not exist. This is because the state vector of the entire universe evolves deterministically according to the Schrödinger equation [13].

Illustrating with a concrete example, our simulation begins with a pure density matrix and Hamiltonian operator corresponding to the joint system-environment. A density matrix that is pure relative to the subsystem is in principle completely known. The distinction between mixed and pure corresponds to the classical superposition describing our state of ignorance of the system. In a mixed state, you have a set of pure states with unknown associated classical probabilities, but in a pure state, we have exact information about the system. Mathematically, this translates to $\hat{\rho}$ having eigenvalues 0 or 1. Obtaining this joint pure density matrix is straightforward. Given two density matrices $\hat{\rho}_{sys}$ and $\hat{\rho}_{env}$ corresponding to the the system and environment in their own respective Hilbert Spaces, we obtain the density matrix of the whole system by taking their tensor product:

$$\hat{\rho} = \hat{\rho}_{sys} \otimes \hat{\rho}_{env} \ . \tag{19}$$

Now, suppose we only have access to the system to perform measurements, a scenario that is very possible due to the fact that the environment may be inaccessible or too large to completely measure. By looking at the system only, one is essentially "snipping" the entanglement chord between the system and environment and ending in a mixed state. Obtaining the matrix that contains all the information that is held in the system is done by finding the reduced

density matrix of the composite state. In order to get the reduced density matrix of the particle, we take the partial trace over the environment of our full density matrix $\hat{\rho}$ which traces over the degrees of freedom of the environment. The result is an operator acting in the Hilbert Space of the system. We can write this as

$$\hat{\rho}_{sys} = tr_{env}(\hat{\rho}) \ . \tag{20}$$

When we trace over $\hat{\rho}$ our reduced density matrix $\hat{\rho}_{sys}$ will grow into a mixed state as it evolves with time, while $\hat{\rho}$ remains pure. Entanglement with the environment is making it look like the subsystem is mixed because some of the information is in joint correlation with the environment. It is no longer in a definite quantum state. This loss of information due to entanglement with the wider world is a mechanism for explaining how things look classical for the subsystem. The subsystem begins to behave like a classical mixture of states as opposed to quantum pure state, when in reality we are seeing the Schrödinger equation evolving on the whole system. This is the fundamental problem of decoherence, the loss of information between quantum and classical probabilities.

An important distinction to make is between classical and quantum uncertainties. Take, for example, the uncertainty of flipping a coin. When you flip a coin, there is a 50% chance that the coin will have value heads and a 50% chance that the coin will have value tails. The coin has a definite state following the flip, however, you are ignorant of what it is until you look at the result, so there is a $\frac{1}{2}$ chance it is in either state. This is the classical probability. A quantum probability could arise if the coin was entangled with another coin somewhere else. Both coins are in a definite entangled state, but you only have access to your original coin to measure. For all intents and purposes, there is $\frac{1}{2}$ chance your coin is heads or tails again because the subsystem of the entangled state looks classical. The quantum probabilities have been converted into effective classical probabilities because the reduced density matrices in both cases are the same. Therefore, given a reduced density matrix, you cannot distinguish whether the total density matrix of the larger system is mixed or pure (from whence the reduced density matrix came). Density matrices do not tell us the state of the system, but, rather the probability distribution of measurement outcomes. In an entangled state, it still holds that no definite state can be attributed to either subsystem.

The degree of entanglement can be quantified by calculating the von Neu-

mann entropy of the system. As in other fields of physics, the entropy denotes the degree of uncertainty of the particular state of the system. The entropy is calculated using the density matrix as follows:

$$S_{von\ Neumann}(\hat{\rho}) = -tr(\hat{\rho}\ ln\ \hat{\rho})\ , \tag{21}$$

where ln is the logarithm of the density operator. For a pure state, $S(\hat{\rho})$ will be 0 because we know the exact state of the system and, therefore, there is no entanglement. For a maximally mixed state where all microstates are equally probable, $S_{von\ Neumann}(\hat{\rho})$ will equal $ln(N)$, where N is the dimension of the Hilbert Space. This will look the same as a maximally entangled state where the eigenvalues of the density operator are $1/N$.

In this case, we will use an approximation of the von Neumann entropy called the purity. It is a first order approximation of the von Neumann entropy and is calculated as follows:

$$S_{purity}(\hat{\rho}) = 1 - tr(\hat{\rho}^2)\ . \tag{22}$$

Calculating the time evolution of a density operator, or rotating it within its Hilbert Space, is straightforward. This equation, the von Neumann equation, is analogous to the Schrödinger's equation for pure states. Given a time independent Hamiltonian operator, the von Neumann equation is

$$\hat{\rho}(t) = e^{-i\hat{H}t/\hbar}\hat{\rho}(0)e^{i\hat{H}t/\hbar}\ . \tag{23}$$

This operation is used to see how the reduced density matrix of the subsystem evolves in time, with the goal of finding the factorization that minimizes entropy growth of a coupled system as time progresses.

2.4 Example

To put this in terms of a concrete example of a system and environment and reiterate key points, consider a system of a book [10]. Every instant, a huge number of photons (the environment) are colliding with the book and then scattering at a certain angle with respect to the orientation of the book and their original path. If you were to change the orientation of the book, the resulting photon path would be different from the original scenario. The resulting scattered photons give us information about the spatial orientation of the book. Each orientation is an eigenstate of the position operator. The

millions of photons that are constantly colliding with the book are measuring the position of the system. Therefore, it is helpful to think of the environment as a measuring device that continuously measures the position of the system.

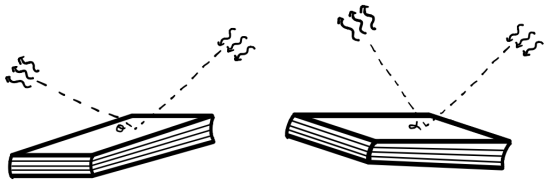


Figure 6: The orientation of the book produces different resulting states of initially identical photons.

In Dirac notation, the interaction between the book (the system, ψ) and the photons (the environment, E) can be represented as

$$|\psi_1\rangle \otimes |E_0\rangle \rightarrow |\psi_1\rangle \otimes |E_1\rangle \ . \tag{24}$$

The resulting E state will vary depending on ψ. In other words,

$$|\psi_2\rangle \otimes |E_0\rangle \rightarrow |\psi_2\rangle \otimes |E_2\rangle \ . \tag{25}$$

For simplicity, the $\otimes$ symbol will be dropped going forward. If the system begins in state $|\psi_1\rangle$ and the environment in ready state $|E_0\rangle$, the starting state is

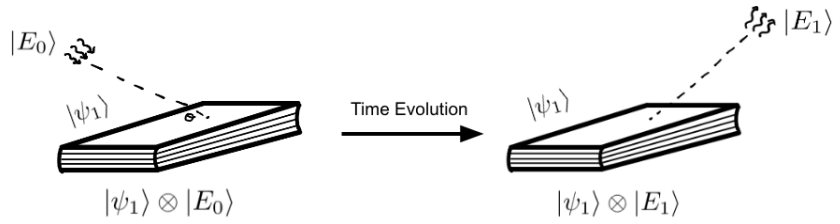
$$|\psi_1\rangle |E_0\rangle \ . \tag{26}$$

And the density matrix representation of this state is

$$\hat{\rho}_0 = |\psi_1\rangle |E_0\rangle \langle\psi_1| \langle E_0| \ . \tag{27}$$

Evolving the state forward in time according to the Schrödinger equation gives the final state of the joint-system environment as

$$|\Psi\rangle = |\psi_1\rangle |E_1\rangle \ . \tag{28}$$



As can be seen, there is a one-to-one correspondence between the state of the book and the state of the photons. The interaction process does not change the state of the system, and does not lead to entanglement. The result is a well defined quantum state where the system and environment are separable. The reduced density matrix of the the joint-system environment is

$$\hat{\rho} = |\Psi\rangle \langle\Psi| = |\psi_1\rangle |E_1\rangle \langle\psi_1| \langle E_1| \ . \tag{29}$$

Once the state is evolved, we will assume that sufficient time has passed for the environment states $|E_1\rangle$ and $|E_2\rangle$ to be orthogonal to each other. This means

$$\langle E_1|E_2\rangle \approx 0 \ . \tag{30}$$

To obtain the reduced density matrix for the system, we must trace over a complete, orthonormal basis of environment states. We can choose a basis where $|E_1\rangle$ and $|E_2\rangle$ are the first two members. The calculation is as follows:

$$\hat{\rho}_{sys} = Tr_{env} \ \hat{\rho} = \sum_i \langle E_i| \hat{\rho} |E_i\rangle \ . \tag{31}$$

All but two terms vanish due to the orthogonal relationship of the environment basis states. Therefore,

$$\hat{\rho}_{sys} = \langle E_1| \hat{\rho} |E_1\rangle + \langle E_2| \hat{\rho} |E_2\rangle \ . \tag{32}$$

From here, we calculate $\hat{\rho}_{sys}$ explicitly by plugging in $\hat{\rho}$ from equation 29 and obtain

$$\hat{\rho}_{sys} = |\psi_1\rangle \langle\psi_1| \ . \tag{33}$$

The result is a pure density matrix. Computing the entropy $1 - Tr\hat{\rho}_{sys}^2$:

$$\begin{aligned} S_{sys} &= 1 - Tr((|\psi_1\rangle \langle\psi_1|)(|\psi_1\rangle \langle\psi_1|) \\ &= 1 - Tr(|\psi_1\rangle \langle\psi_1|) \end{aligned} \tag{34}$$

Now we take the trace over the complete basis of system states to get the trace term:

$$\begin{aligned} Tr\ \hat{\rho}_{sys} &= \sum_i \langle\psi_i| \hat{\rho}_{sys} |\psi_i\rangle \\ &= \langle\psi_1| \hat{\rho}_{sys} |\psi_1\rangle + \langle\psi_2| \hat{\rho}_{sys} |\psi_2\rangle \\ &= 1 \end{aligned} \tag{35}$$

This results in an entropy of 0.

The calculations will be repeated for the scenario where the system is in a superposition of basis states before interaction. After time evolution, the superposition will spread to the environment following their interaction. For example, starting in a state of the form

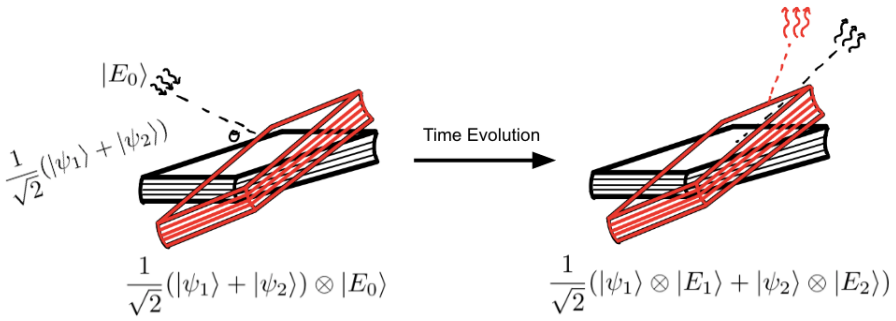
$$|\psi\rangle |E_0\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle) |E_0\rangle \ . \tag{36}$$

The density matrix representation of this state is therefore

$$\hat{\rho}_0 = |\psi\rangle |E_0\rangle \langle\psi| \langle E_0| \ . \tag{37}$$

Evolving this state forward according to the Schrödinger equation gives the final state of the joint-system environment as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle |E_1\rangle + |\psi_2\rangle |E_2\rangle) \ , \tag{38}$$



where the environment will evolve into E_1 or E_2 depending on the state of the system. As can be seen, the resulting state is one of the maximally entangled Bell basis states. The coherence that was initially within the system alone has now spread to the system-environment state through the process of decoherence. Since the final state of the book and photons are entangled, we can no longer depict the book and photons as separate entities (individual state vectors).

The reduced density matrix of the system in the final state is then:

$$\begin{aligned}\hat{\rho} &= |\Psi\rangle \langle\Psi| \\ &= \frac{1}{2}(|E_1\rangle |\psi_1\rangle \langle\psi_1| \langle E_1| + |E_2\rangle |\psi_2\rangle \langle\psi_1| \langle E_1| + \\ &\quad |E_1\rangle |\psi_1\rangle \langle\psi_2| \langle E_2| + |E_2\rangle |\psi_2\rangle \langle\psi_2| \langle E_2|\end{aligned} \tag{39}$$

Taking the trace over the basis of environment states to obtain the reduced density matrix for the system gives the result

$$\hat{\rho}_{sys} = \frac{1}{2}(|\psi_1\rangle \langle\psi_1| + |\psi_2\rangle \langle\psi_2|) \ . \tag{40}$$

This result is a mixed reduced density matrix. Using this to compute the purity entropy, $1 - Tr\hat{\rho}_{sys}^2$, :

$$\begin{aligned}S_{sys} &= 1 - Tr(\frac{1}{4}(|\psi_1\rangle \langle\psi_1| + |\psi_2\rangle \langle\psi_2|)(|\psi_1\rangle \langle\psi_1| + |\psi_2\rangle \langle\psi_2|)) \\ &= 1 - Tr(\frac{1}{4}(|\psi_1\rangle \langle\psi_1| + |\psi_2\rangle \langle\psi_2|))\end{aligned} \tag{41}$$

Now we take the trace over the complete basis of system states to get the trace

term:

$$\begin{aligned} Tr\hat{\rho}_{sys} &= \sum_i \langle\psi_i| \hat{\rho}_{sys} |\psi_i\rangle \\ &= \langle\psi_1| \hat{\rho}_{sys} |\psi_1\rangle + \langle\psi_2| \hat{\rho}_{sys} |\psi_2\rangle \\ &= \frac{1}{2} \end{aligned} \tag{42}$$

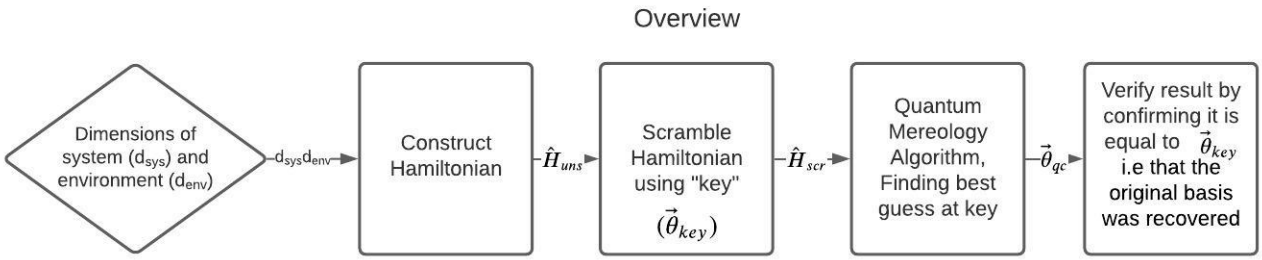
This results in an entropy of $\frac{1}{2}$. As can be seen, when the system begins in a superimposed state, the final entropy of the system after interaction with the environment will increase, indicating we are no longer in a quasi-classical state. This helps to explain why in the classical world, you will never see the book in a superposition of two different orientations even though that superposition is a perfectly valid state in quantum mechanics. In a particular basis, the pointer basis, classically observable orientations are robust to entanglement and entropy generation whereas superposition states are not.

In a special basis, the pointer basis, the classical universe is reflected in quantum formulations. Before measurement, the environment is in a ready state. As the photons collide with the book in any classically observable orientation, no entanglement is generated and entropy stays low. As was just shown, if the photons were to collide with a book in a superposition of orientations, unitary evolution predicts that the system and environment will branch into an entangled state and that there will be an increase in entropy. Therefore, we search through the field of possible system and environment orientations for one that is robust to entanglement and entropy growth as time evolves if the system does not start in a superimposed state.

3 The Algorithm

First, I will begin with a high level view of the algorithm and then I will explain the individual pieces.

3.1 Overview



The algorithm begins by specifying the dimensions of the Hilbert Space of the system (d_{sys}) and the Hilbert Space of the environment (d_{env}). The dimensions are used to construct the Hamiltonian operators corresponding to the system, the environment, and the interaction between the system and environment. Summing these three matrices, we get the Hamiltonian of our complete system in the pointer basis. From here, we scramble the Hamiltonian using a specified unitary matrix, changing the axes of the Hilbert Space to a random basis. Once we have the scrambled Hamiltonian, we can run the Quantum Mereology algorithm. The algorithm selects an initial state in the form of a pure density matrix and evolves it forward to a characteristic moment in time using the scrambled Hamiltonian. Next, it takes the partial trace to get the reduced density matrix and then calculates the purity entropy at the characteristic time, yielding the "score" for this basis. The algorithm loops over each element of the basis states for the system tensored with the ready state (a ready state as described in Section 2.4, by the photons before they hit the book) of the environment and averages all the scores (there are d_{sys}). The environment is in an equal superposition of the d_{env} basis states. This algorithm is then run over the field of possible scrambling unitary matrices using machine learning, and the scrambling that produces the lowest average entropy (whose average entropy does not instantly reach a maximum) is our result for a basis that reflects the classical world.

3.2 Constructing the Hamiltonian

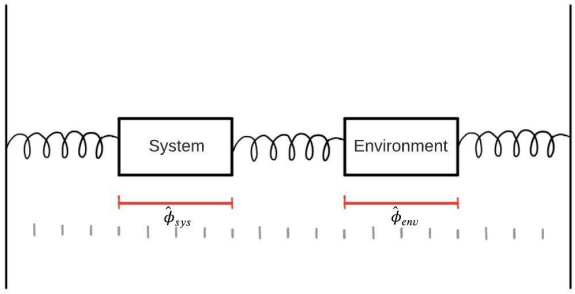
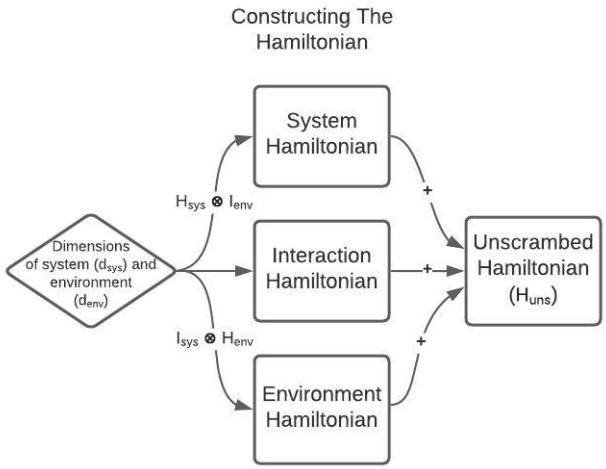


Figure 7: Two coupled harmonic oscillators.

The work done in this thesis is motivated by the classical system of two coupled harmonic oscillators and the system of two qubits (as described in Section 2.1.2), but the coding framework is generalizable to any quantum system. The coupled harmonic oscillator system is illustrated in the image above.

This system was chosen because both oscillators have a notion of position (at least in the large dimension limit) and momentum in the plane of reference. From other studies of decoherence, and from our observations in Section 2.1.2, position seems important for the existence of a preferred factorization. Space is constructed as a lattice in order to use a finite matrix to represent an infinite dimensional system. In other words, position is discretized. The position of the system is represented by $\hat{\phi}_{sys}$ and the position of the environment is represented by $\hat{\phi}_{env}$, as labeled in Figure 1. The momentums are represented by $\hat{\pi}_{sys}$ and $\hat{\pi}_{env}$ respectively. We wish to treat the system of two coupled harmonic oscillators on par with all quantum systems (continuous or discrete) and not elevate position to a special status a priori. Accordingly, we follow the strategy of Carroll and Singh [2] in discretizing space in a certain way, representing $\hat{\pi}$ and $\hat{\phi}$ as $d_{sys} \times d_{sys}$ matrices for the system and $d_{env} \times d_{env}$ matrices for the environment.



To construct the Hamiltonian of our complete system, $\hat{H}_{uns}$, we must begin by specifying the dimensions of the system and environment. The dimensionality of the complete system D will equal $d_{sys}d_{env}$.

The first step in constructing $\hat{H}_{uns}$ is constructing the explicit matrix representations for the momentum ($\hat{\pi}$) and position ($\hat{\phi}$) operators in the given dimensions. The momentum and position matrices can then be used to construct the self-Hamiltonians for the system and environment. For oscillators corresponding to the system and environment of the same mass m and frequency ω, their respective self-Hamiltonians are defined as [2]:

$$\hat{H}_{sys} = \frac{\hat{\pi}_{sys}^2}{2m} + \frac{1}{2}m\omega^2\hat{\phi}_{sys}^2 \tag{43}$$

$$\hat{H}_{env} = \frac{\hat{\pi}_{env}^2}{2m} + \frac{1}{2}m\omega^2\hat{\phi}_{env}^2 \tag{44}$$

The self-Hamiltonian for the system, $\hat{H}_{sys}$, is a $d_{sys} \times d_{sys}$ matrix, so it is tensored with the $d_{env} \times d_{env}$ identity matrix to bring it into the dimensionality of the complete tensor space. The $d_{sys} \times d_{sys}$ identity matrix is tensored with the $d_{env} \times d_{env}$ self-Hamiltonian of the environment, $\hat{H}_{env}$, for the same reason. The momentum operator can also be used to construct a Hamiltonian operator for the interaction between the two masses. In the diagram, this interaction is the spring between the system and environment oscillators. The interaction term is modeled as the position of the system coupled with the position of the

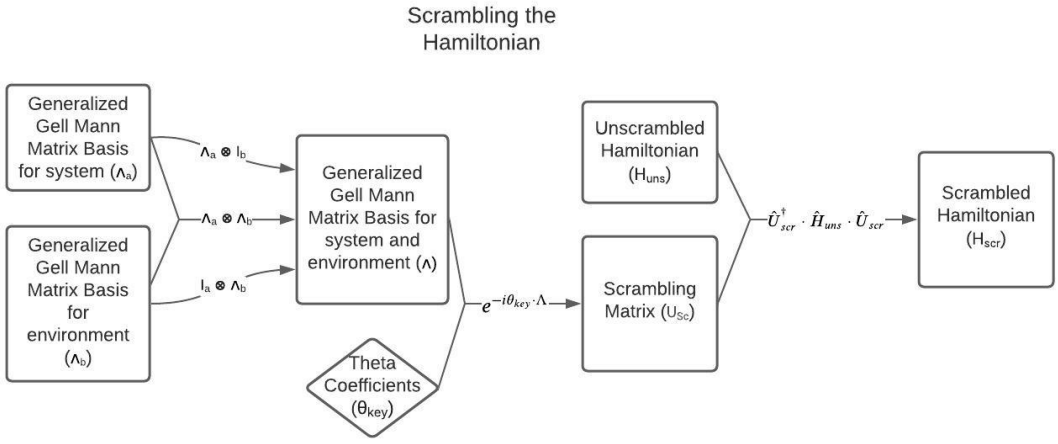
environment with a certain interaction strength λ between 0 and 1, [2]

$$\hat{H}_{int} = \lambda(\hat{\phi}_A \otimes \hat{\phi}_B) \ . \tag{45}$$

By summing these three matrices, we arrive at the unscrambled Hamiltonian of the complete system, written as

$$\hat{H}_{uns} = \hat{H}_{sys} + \hat{H}_{env} + \hat{H}_{int} \ . \tag{46}$$

3.3 Scrambling the Hamiltonian



The next step in the algorithm is scrambling the complete Hamiltonian with a unitary matrix. The change of basis of the complex Hilbert Space is implemented by a unitary operator, $\hat{U}$.

Every unitary matrix is the exponential of i times a traceless Hermitian matrix. The first step in constructing a unitary matrix is building a basis of traceless Hermitian generators that will be used to construct a traceless Hermitian matrix. The traceless Hermitian matrix will be a linear combination of the basis matrices. Following Carroll and Singh, the Hermitian generators we use are the Generalized-Gell Mann matrices (GGMM). The GGMMs come in groups of three forms: symmetric, anti-symmetric and diagonal matrices.

Constructing the complete set of GGMMs can be done by constructing the GGMMs for the system in the d_{sys} dimension and the GGMMs for the environ-

ment in the d_{env} dimension. As before, it is necessary to tensor both subsets of the GGMMs with the identity matrix of the correct size so that they are the dimensions of the complete tensor space. The result of these tensor products is part of the complete set. The last part of the set is found by tensoring all the GGMMs for the system (Λ_{sys}) with all the GGMMs of the environment (Λ_{env}). This results in $D^2 - 1$ generator matrices, Λ.

You then must pick $D^2 - 1$ real parameters (the theta coefficients) that will be used to scale all the Λ matrices. The theta coefficients are what modify the resulting scrambling matrix for each run of the algorithm while the GGMMs remain the same.

The theta coefficients and generator matrices (Λ) can then be used to construct the scrambling matrix, $\hat{U}_{scr}$. The equation to construct the scrambling matrix is

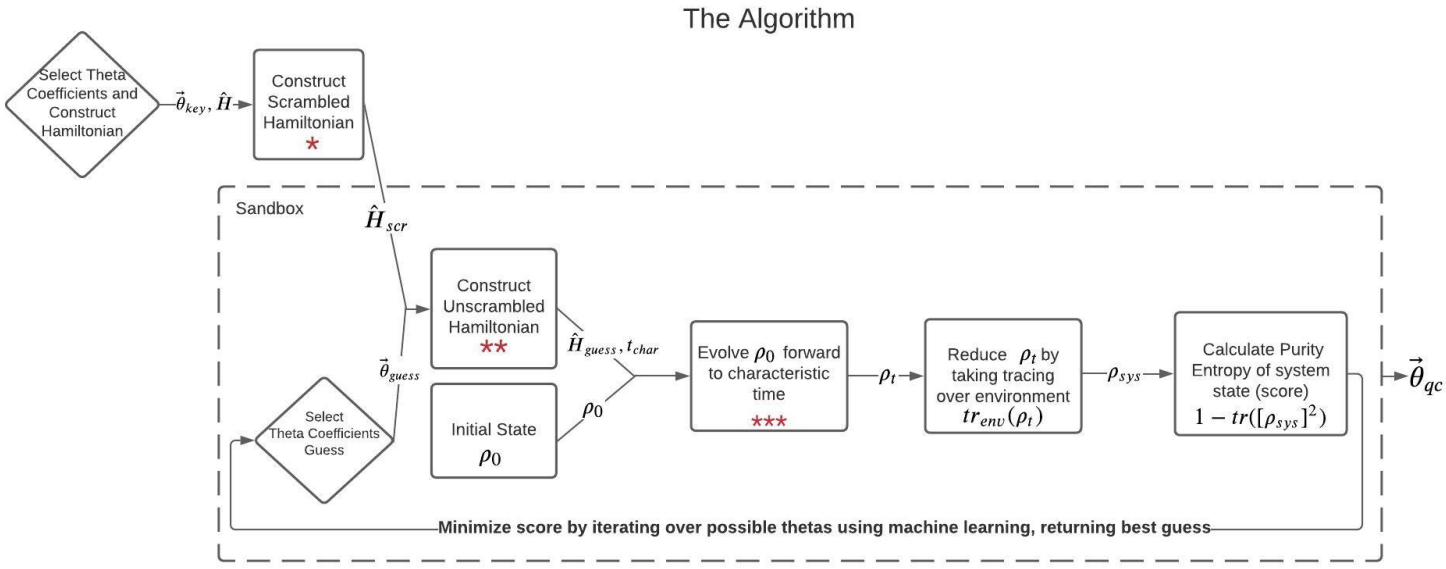
$$\hat{U}_{scr}(\vec{\theta}) = exp(\sum_{n=1}^{D^2-1} i\ \vec{\theta}_n \Lambda_n)\ . \tag{47}$$

Once the scrambling matrix is constructed, it can be used to find the complete scrambled Hamiltonian, H_{scr}. The equation to do so is

$$\hat{H}_{scr} = \hat{U}_{scr}(\vec{\theta})^\dagger \cdot \hat{H}_{uns} \cdot \hat{U}_{scr}(\vec{\theta})\ , \tag{48}$$

which is a similarity transformation in matrix algebra.

3.4 The Algorithm



$$* \ \hat{H}_{scr} = \hat{U}(\vec{\theta}_{key})^{\dagger} \cdot \hat{H} \cdot \hat{U}(\vec{\theta}_{key})$$

$$** \ \hat{H}_{guess} = \hat{U}(-\vec{\theta}_{guess})^{\dagger} \cdot \hat{H}_{scr} \cdot \hat{U}(-\vec{\theta}_{guess})$$

$$*** \rho_t = exp(-it_{char}\hat{H}_{guess}) \cdot \rho_0 \cdot exp(it_{char}\hat{H}_{guess})$$

The complete algorithm uses machine learning to try different bases ranging over the complete tensor space. The algorithm selects a guess, or list of theta coefficients that specify the scrambling matrix, or orientation of space. It then unscrambles the scrambled Hamiltonian using the guess, and evolves an initial state forward in time using the resulting Hamiltonian. Next, it takes the partial trace over the environment to get the evolved system matrix, and calculates the purity entropy of this reduced state. The result is the guess theta list that produces the lowest purity entropy, indicating that the state has evolved predictably and is quasi-classical, as the predictability sieve explains.

The system starts in the $d_{sys} \times d_{sys}$ initial states:

$$\hat{\rho}_{sys,1} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{\rho}_{sys,2} = \begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, .., \hat{\rho}_{sys,d_{sys}} = \begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{49}$$

The environment always begins in the $d_{env} \times d_{env}$ ready state of an equal superposition of basis states:

$$\hat{\rho}_{ready} = \frac{1}{d_{env}} \begin{bmatrix} 1 & 1 & 1 & ... \\ 1 & 1 & 1 & ... \\ 1 & 1 & 1 & ... \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{50}$$

For every guess, the algorithm loops over the system in every possible pointer basis starting state and keeps the same environment state. In the book and photon example, the system states are the angle orientations of the book and the environment is the ready state of the photons. There are d_{sys} joint system-environment states per theta guess. It then averages all of the scores to get an entropy score for the whole basis corresponding to the theta guess, as suggested by Carroll and Singh [2]. Writing this out,

$$score = \frac{\sum_{n=1}^{d_{sys}} score(\hat{\rho}_{sys,n} \otimes \hat{\rho}_{ready})}{d_{sys}} \ . \tag{51}$$

The original Hamiltonian is scrambled using a key, or preselected list of theta coefficients that are used to construct a unitary scrambling matrix by the process described in Section 2.3. The algorithm then attempts to unscramble the scrambled Hamiltonian through the same process as the original scrambling. It guesses a list of theta coefficients to construct the unscrambling matrix and from there can construct the guess Hamiltonian. Following Carroll and Singh, the Hamiltonian operator is used to define the characteristic time. When the Hamiltonian is applied to a wave function, the result has units of energy. Therefore, the Hamiltonian can be thought of as having units of energy. Energy and frequency have a direct relationship, when one increases so does the other and vice versa. Therefore, one over frequency or one over the magnitude of the Hamiltonian has units of time. By units, it is reasonable, then, to guess some-

thing of this form as a time where the entropy of a quasi-classical basis will remain low and the entropy of quantum states will be large. The characteristic time is defined as

$$t_{char} = \frac{1}{||\hat{H}||_2} \ , \tag{52}$$

where the operation being done on the Hamiltonian is the 2-norm of matrices. The 2-norm of a matrix A is defined as

$$||A||_2 = max_{i=1:n}\sqrt{\lambda(A^T A)} \ , \tag{53}$$

where $\lambda_i(A^T A)$ is the i^{th} eigenvalue of $A^T A$. The characteristic time is a consistent benchmark because it will be the same for a scrambled or unscrambled Hamiltonian since they have the same eigenvalues.

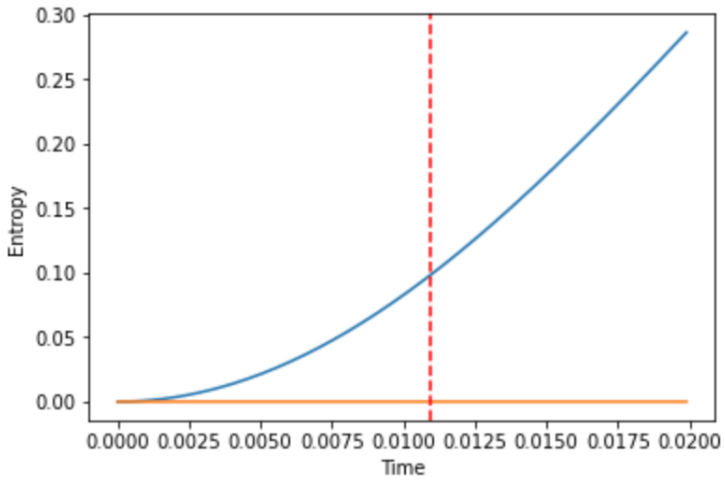


Figure 8: This figure depicts one run of the algorithm with $\lambda = 0.8$ and the characteristic time marked by the dotted red vertical line. The blue line is the scrambled system and the orange line is the unscrambled system, illustrating how entropy stays low for the unscrambled system but increases rapidly for the scrambled.

The guess Hamiltonian is used to evolve the initial state forward to the characteristic time as described by equation 23. Lastly, the algorithm takes the partial trace of the evolved state and finds the purity entropy of this state. Machine learning is implemented to guess theta coefficients spanning theta space and then compare the resulting purity entropy values. It will return the theta

coefficients that result in the smallest purity entropy. This result should match the pre-selected key.

In the work of Carroll and Singh [2], the initial joint system-environment state factorization is optimized by calculating a state referred to as the candidate pointer observable. Afterwards, the pointer basis of the system is optimized for. We do not calculate the candidate pointer observable because, by virtue of ranging over every possible unitary matrix (every theta list) using machine learning, we optimize for factorization into system and environment, system pointer basis, and ready state of the environment. This is because the system-environment initial state will be adapted to every particular theta. Therefore, as you range over all possible scrambling matrices, one matrix will produce a factorization where the system is in the candidate pointer observable state. This method takes care of all optimizations at the same time.

Throughout we have seen a number of mathematical objects. A brief inventory of key ones is provided below.

Glossary

Characteristic time Time that the initial state is evolved to at which entropy of the preferred factorization should remain low.

D Integer dimension of the joint system-environment Hilbert Space, equal to $d_{sys}d_{env}$.

Generalized Gell Mann Matrices Basis of size $D^2 - 1$ for all traceless Hermitian matrices of a given size.

Guess Theta list of real numbers of length $D^2 - 1$ chosen to unscramble the scrambled Hamiltonian ($\vec{\theta}_{guess}$).

Key Theta list of real numbers of length $D^2 - 1$ chosen to scramble the original Hamiltonian ($\vec{\theta}_{key}$).

Scrambling Matrix D by D unitary matrix constructed using the GGMM's and theta.

4 Methods

To test the code for producing the Hamiltonian, the eigenvalues of the generated Hamiltonian were checked against the exact, infinite-dimensional results. Throughout the code I work in units where $\hbar, \omega, m = 1$. Using these units, the lowest four eigenvalues for the exact results are $1, 2, 3$, and 4. When the interaction strength is 0 and d_{sys} and d_{env} is 61, the produced results closely resemble that of the infinite-dimensional case where eigenvalues incrementally increase by 1. The lowest four eigenvalues were

$$0.99999781 - 1.08302455e - 14j$$

$$2.0000275 - 2.86437540e - 14j$$

$$2.99957675 + 5.95242131e - 15j$$

$$3.99960644 - 2.70913204e - 14j$$

where the imaginary terms are so small they are negligible.

Afterwards, the time evolution of the system of two coupled harmonic oscillators with varying interaction magnitudes was confirmed. The interaction term, λ, is a value between zero and one. As the term gets larger, the entanglement between the oscillators occurs at a faster rate. When the interaction term is zero, there is no interaction between the oscillators and they will remain unentangled. Entropy will remain at zero.

The unscrambled system with varying lambda values was compared to the same system now scrambled with a randomly generated unitary matrix prior to time evolution. The entropy of the scrambled system immediately reaches a maximum entropy value because it is no longer in a preferred basis or factorization.

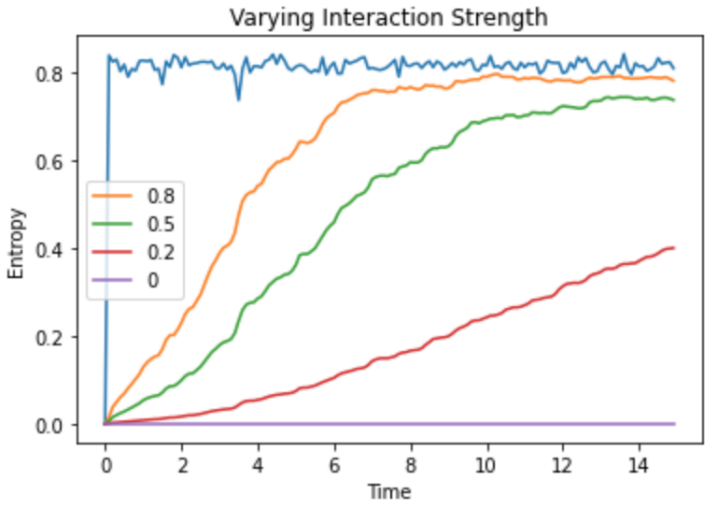


Figure 9: As lambda increases, entropy grows at a faster rate. For the scrambled system (the blue line), entropy reaches a maximum immediately.

5 Results and Discussion

One illustrative result was achieved by perturbing the theta coefficients away from the origin of theta space and taking the entropy of the evolved scrambled state at the characteristic time for the system of two qubits and two harmonic oscillators. This was done by producing the theta vector by specifying its variance. With each step, the variance increased by 0.0001.

For the system of two qubits, there are many entropy minima as the theta vector, and, consequently, unitary scrambling matrix, are moved away from the origin. These minima correspond to multiple notions of a pointer basis. At various points on the graph, the entropy returns back to zero rather than consistently increasing the farther away from the origin it gets.

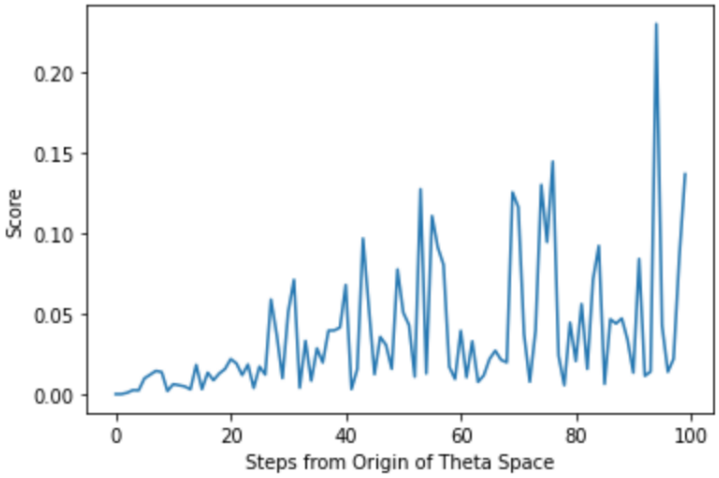


Figure 10: The score of a two qubit system goes to 0 at multiple points.

Therefore, to distinguish factorizations that produce an entropy of zero (as mentioned in 2.1.2) it is necessary to appeal to different fundamental properties of the classical world, such as position and locality, to see the classical world emerging. The system of two harmonic oscillators differs from two qubits because it does have a notion of position built into the Hamiltonian. Therefore, the entropy score will not return to zero once the scrambling matrix has moved away from the origin. This result reflects the fact that in the classical world, we perceive the harmonic oscillators only in one preferred factorization and basis. Different valid factorizations would alter the split between the oscillators, or system and environment, that we perceive when observing.

This expected result was verified for the system of two harmonic oscillators. Only the unscrambled state produces the minimum entropy. As the scrambling matrix moves away from the origin of theta space (the basis moves away from the pointer basis or the factorization from the preferred factorization), the entropy of the evolved state increases.

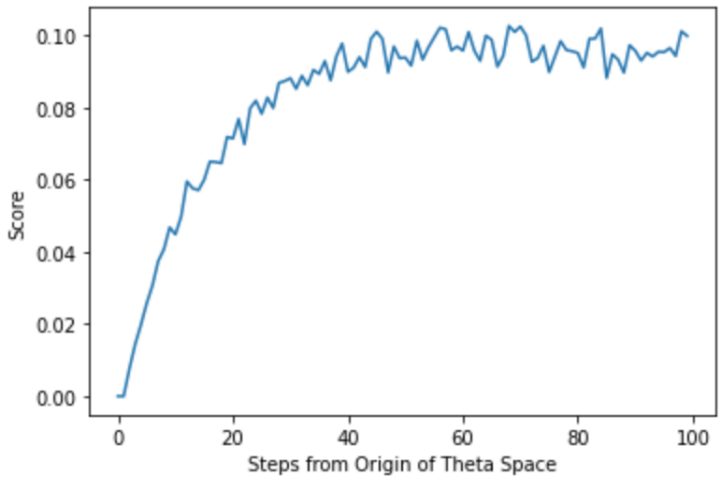


Figure 11: The score of a coupled harmonic oscillator system does not return to zero.

All iterations of the machine learning algorithm were run with the dimension of the system and environment both set to three to minimize computation time.

Initially, the machine learning code was run on the unscrambled system of two harmonic oscillators with the coupling coefficient set to 0.8. This outputted a theta list that produced a score lower than the score produced by the original pointer basis, indicating that the machine learning traversal had entered the uncoupled regime outside of the quantum measurement limit. The form of the interaction between the two harmonic oscillators is such that the algorithm produced a factorization that dialed down the lambda coefficient. Following this, the machine learning was run on the same system with lambda set to 0. This outputted a theta list that was all zeroes, a promising result that the lowest entropy was produced by the original basis.

Afterwards, the system was changed to that of two coupled anharmonic oscillators where dialing down lambda is not possible because of the form of the interaction.

For the the unscrambled system of two anharmonic oscillators with lambda set to 0, the machine learning outputted a theta list that was all zeroes, once again reflecting that the lowest entropy was produced by the original basis.

Next, the machine learning was run starting with a scrambled basis on two anharmonic oscillators. The algorithm produced a theta list corresponding to an entropy minimum. The first steps to verify if the outputted theta result is

equal to the scrambling key were taken by comparing the norm of the difference of the scrambling unitary matrix (produced with the scrambling key) and the unscrambling matrix (produced with the list outputted by the algorithm). This can be written as

$$abs(||\hat{U}(\vec{\theta}_{key}) - \hat{U}(\vec{\theta}_{qc})||_1) \ . \tag{54}$$

This distance was then compared to the typical differences of the norm of unitary matrices in theta space to see if the unscrambling and scrambling matrices were significantly closer together than other unitary matrices. The results did not show this to be the case, but deciding equivalence of two theta lists requires further investigation.

More specifically, one factor that could be making the theta lists differ is that the machine learning algorithm is detecting a different ready state of the environment that produces a low entropy with a given factorization and pointer basis. In the example of the book, this would mean the photons are coming in to collide with the book at a different initial angle. This angle would be just as effective in distinguishing the pointer basis of the book. Mathematically, if this were the case, this would correspond to the theta key and outputted theta differing in some entries, specifically the entries that refer to GGMMs of the form $I_{sys} \otimes \Lambda_{env}$. The manner in which the GGMMs were constructed makes it straightforward to identify these entries and to generally separate out directions in theta space that correspond to solely a change in system or change in environment.

Another thing that could be occurring is that the outputted theta list reorders the pointer basis without changing the basis. This can be thought of as relabeling the axes of space and would result in a different theta list corresponding to the same basis. In fact, it is expected that the minima are degenerate. For d_{sys} basis elements, there are at least $d_{sys}!$ minima. Shuffling the ordering of the system basis elements would allow us to determine if this is what was occurring in the machine learning result.

Examining these two scenarios would allow us to more conclusively determine if the outputted theta list was equivalent to the theta key.

6 Conclusion and Future Work

In this thesis project, I have coded a Python test platform for any quantum mechanics system. Every valid quantum system has a Hermitian Hamiltonian

operator that can, therefore, be generated using the Generalized Gell Mann Matrices. This fact allows a user to fit the platform to any quantum system and search for bases where the entropy and entanglement stays low. In illustrating that the measurement of purity entropy corresponding to the entanglement between the system and environment produced the expected results, it draws into question whether the second measurement of entropy Carroll and Singh used is necessary. This second entropy corresponds to the spread of the system's position. In relation to the predictability sieve, Zurek found that predictability and being robust to purity entropy growth were correlated. Similarly, perhaps, our measure of slow growth of entanglement is correlated with the measure of spread, making purity entropy a sufficient criteria.

The next critical step in this project is more extensively applying the machine learning to the sandbox created. So far, we have been able to verify the growth of entropy when starting at a pointer basis and perturbing away from it. The machine learning algorithm works in the opposite direction, starting with the Hamiltonian in a scrambled basis and attempting to unscramble it to recover the original pointer basis. Recovering the original pointer basis is what justifies the emergence of the classical world, and by Tegmark [1], human consciousness. I have begun implementing the machine learning and obtained results, but the main roadblocks are that the sandbox mechanism is computationally extensive and there are many subtleties in verifying the results.

This project can be expanded in a variety of other ways. One future direction is exploring how low the dimensionality of the system can be such that the algorithm still produces a preferred basis. Another direction may entail attempting to unscramble systems with more than two subsystems, or trying to separate bound states into subsystems. One last expansion would be further optimizing the sandbox to run the machine learning at a faster rate and on higher dimensions.

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Appendices

A Constructing and Verifying the Hamiltonian Eigenvalues in Different Dimensions

Utilizing the framework of Carroll and Singh [2], space is discretized in the construction of the Hamiltonian (which requires operators $\hat{\phi}$ and $\hat{\pi}$). In the case of the odd dimension $d = 2l + 1$ one can find explicit representations for $\hat{\phi}$ and $\hat{\pi}$. The matrix representation for $\hat{\pi}$ in the $|\phi_j\rangle$ basis as given by Carroll and Singh is

$$\langle\phi_j|\,\hat{\pi}\,|\phi'_j\rangle = \left(\frac{2\pi}{(2l+1)^2\alpha}\right)\sum_{n=-l}^{l} n\, exp\left(\frac{2\pi i(j-j')n}{2l+1}\right) = \begin{cases} 0, & \text{if } j = j' \\ \left(\frac{i\pi}{(2l+1)\alpha}\right)\text{cosec}\left(\frac{2\pi l(j-j')}{2l+1}\right), & \text{if } j \neq j' \ . \end{cases} \tag{55}$$

The eigenvalue equations for $\hat{\phi}$ and $\hat{\pi}$ are

$$\hat{\phi}\,|\phi_j\rangle = j\left(\frac{2\pi}{(2l+1)\beta}\right)|\phi_j\rangle \ \ , \ \ \ j = -l, ..., 0, ..., l \ , \tag{56}$$

$$\hat{\pi}\,|\pi_j\rangle = j\left(\frac{2\pi}{(2l+1)\alpha}\right)|\pi_j\rangle \ \ , \ \ \ j = -l, ..., 0, ..., l \ , \tag{57}$$

These can then be used to define the Hamiltonian operator as described in Section 3.2. In the large dimension limit $d \to \infty$, one recovers the Heisenberg form of the canonical commutation relation if the parameters α and β are constrained to obey $\alpha\beta = 2\pi/d$. In my code, I specifically define $\beta = 2\pi/\sqrt{d}$ and $\alpha = 2\pi/\beta d$.

As mentioned in Section 4, I verified the accuracy of the discretizing framework and code I wrote by comparing the eigenvalues of the generated Hamiltonian to the exact eigenvalues of the infinite dimensional system. From these tests, I observed that high dimension seems to be needed for good agreement with the expected results. Using the system of two harmonic oscillators with the interaction coefficient set to 0, the exact eigenvalues in ascending order begin at 1 and and increase by increments of 1. Starting with d_{sys} and d_{env} both set to 3, the lowest unique eigenvalues of the generated Hamiltonian in descending

order are:

$$\begin{bmatrix} 13.49280587 + 0.j \\ 13.38264077 + 0.j \\ 13.27247567 + 0.j \\ 6.85656803 + 0.j \\ 6.74640293 + 0.j \\ 0.2203302 + 0.j \end{bmatrix}$$

These eigenvalues clearly do not resemble the expected results.

Dialing d_{sys} and d_{env} both to 21, the smallest six unique eigenvalues generated in descending order are:

$$\begin{bmatrix} 4.3622925 + 0.j \\ 3.78841125 + 0.j \\ 3.21453 + 0.j \\ 2.66581302 + 0.j \\ 2.09193177 + 0.j \\ 0.96933354 + 0.j \end{bmatrix}$$

While these eigenvalues perhaps more closely resemble what was expected, they are still far from the expected consistent increment size of 1. Only when dialing d_{sys} and d_{env} to a high dimension, for example setting both to 101, do the results start to resemble the expected values. The smallest nine unique eigenvalues generated in descending order are:

$$\begin{bmatrix} 5.99998879 + 1.38696582e - 14j \\ 4.99998879 + 5.41635312e - 14j \\ 3.99999993 + 5.52484257e - 14j \\ 2.99999993 - 1.78823613e - 18j \\ 2. + 1.64317320e - 15j \\ 1. - 6.89464431e - 28j \end{bmatrix}$$

Increasing the dimensionality only improved the agreement with the expected results, illustrating that high dimensionality of the discretized system was needed to resemble the infinite dimensional system.

B Generalized Gell Mann Matrix Construction

Every scrambling matrix is constructed using D^2-1 real parameters $\{\vec{\theta}_a | a = 1, 2, ..., (D^2-1)\}$ and D^2-1 traceless Hermitian generators $\{\Lambda_a | a = 1, 2, ..., (D^2-1)\}$, which can be identified by the Generalized Gell-Mann matrices (GGMM) [2]. The Generalized Gell-Mann matrices come in three groups: symmetric, anti-symmetric and diagonal matrices. Using the notion where E^{jk} is the $d \times d$ matrix with all zeros except a 1 in the (j, k) location, one can construct the GGMMs of dimensions $d \times d$ as follows [2]:

$$\Lambda_{sym}^{jk} = E^{kj} + E^{jk} \ ; \ 1 \leq j < k \leq d \tag{58}$$

$$\Lambda_{antisym}^{jk} = -i(E^{jk} - E^{kj}) \ ; \ 1 \leq j < k \leq d \tag{59}$$

$$\Lambda_{diag}^{l} = \sqrt{\frac{2}{l(l+1)}}(-l \ E^{l+1,l+1} + \sum_{l}^{j=1} E^{jj}) \ ; \ 1 \leq l \leq d-1 \tag{60}$$

As outlined, I constructed the GGMMs in the system dimension where $d = d_{sys}$, the list of these is $\Lambda_{d_{sys}}$. Then I constructed the GGMMs in the environment dimension where $d = d_{env}$, the list of these is $\Lambda_{d_{env}}$. The complete $D^2 - 1$ generator GGMMs (Λ_A) for the joint system-environment is then composed of every matrix in $\Lambda_{d_{sys}}$ tensored with the $d_{env} \times d_{env}$ identity matrix, the $d_{sys} \times d_{sys}$ identity matrix tensored with every matrix in $\Lambda_{d_{env}}$, and every matrix in $\Lambda_{d_{sys}}$ tensored with every matrix in $\Lambda_{d_{env}}$. The factorization change unitary $\hat{U}(\theta)$ can then be expressed as [2]

$$\hat{U}_{scr}(\vec{\theta}) = exp(\sum_{a=1}^{D^2-1} i \ \vec{\theta}_a \Lambda_a) \ . \tag{61}$$

C Code and Implementation

The entire platform was coded from scratch in Python, and the machine learning aspects were implemented in SciPy. The code was run in Jupyter notebooks and is available in the following GitHub repository: `https://github.com/louisa-cornelis/Thesis-Code.git`.