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Eigenvalues and Approximation Numbers

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Abstract

While the spectral theory of compact operators is known to many, knowledge regarding the relationship between eigenvalues and approximation numbers might be less known. By examining these numbers in tandem, one may develop a link between eigenvalues and ℓ^p spaces. In this paper, we develop the background of this connection with in-depth examples.

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1 Preliminaries

Spectral theory is a famously studied field in mathematics. Its applications are so broad that it is used in various applied fields. However, spectral theory is far from solved. While we know much about the spectral theory of compact operators, there is still much to explore regarding the relationship between eigenvalues and approximation numbers.

The aim of this paper is to explore the theory relating approximation numbers to eigenvalues. The paper proceeds as follows. We begin by establishing preliminaries, before discussing compact operators and their various representations. Next, we introduce approximation numbers and explore their properties, before discussing H-operators. Once we understand H-operators, we can prove our desired result connecting approximation numbers to eigenvalues. We finish by discussing a few open problems.

We begin our discussion by introducing Hilbert spaces. These spaces are of interest for their inner products: an analogue for orthogonality. For instance, given a basis of a Hilbert space, we can orthogonalize and normalize it. Many of the theorems defined on Hilbert spaces will rely on the use of this "orthonormal basis."

Definition 1. *A Hilbert space is a complete inner product space.*

Note that when we say completeness, we mean the following:

Definition 2. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of numbers. This sequence is Cauchy if and only if $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ that for $n, m > N$,*

$$|x_n - x_m| < \varepsilon.$$

A space X is complete if and only if every Cauchy sequence in it converges within it.

We will assume that the reader is familiar with the basic properties of inner products and vector spaces. While we will not focus on these properties, they serve as building blocks to important definitions. For example, to know what a "right angle" is in a non-Euclidean setting, an understanding of inner products is necessary.

In this paper, when we refer to a vector space, we will assume that it is infinite-dimensional. Two common examples of these spaces follow:

Example 1. *The space $\ell^2(X)$ consists of all complex-valued sequences $\{x_i\}_{i=1}^{\infty}$ such that*

$$\ell^2(X) = \left\{ x \in X : ||x|| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} < \infty \right\}$$

with inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} |x_i y_i|.$$

Similarly, the space $L^2(X)$ consists of all complex-valued functions f such that

$$L^2(X) = \left\{ f \in X : ||f|| = \left(\int_X |f(x)|^2 dx \right)^{1/2} < \infty \right\}$$

with inner product

$$\langle f, g\rangle = \int_X |f(x)g(x)|dx.$$

However, inner products are the exception, not the norm. It is then useful to study these spaces which lack orthogonality. We define them as follows.

Definition 3. *A Banach space is a complete normed space.*

It is clear that every Hilbert space is a Banach space, as an inner product induces a norm. In other words, if H is a Hilbert space, and $x \in H$, then

$$||x|| = \sqrt{\langle x, x\rangle}.$$

One can easily verify that the above satisfies the properties of a norm. One consequence of the above result is that a Banach space is a generalized Hilbert space. In the following, we provide a common examples of Banach spaces. Notice how both $\ell^2(X)$ and $L^2(X)$ are included.

Example 2. *If $1 \leq p < \infty$, the space $\ell^p(X)$ consists of all complex-valued sequences $\{x_i\}_{i=1}^{\infty}$ such that*

$$\ell^p(X) = \left\{ x \in X : ||x|| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.$$

Similarly, the space $L^p(X)$ consists of all complex-valued functions f such that

$$L^p(X) = \left\{ f \in X : ||f|| = \left(\int_X |f|^p dx \right)^{1/p} < \infty \right\}.$$

There are many different types of Hilbert and Banach spaces. Consider what we call "separability." This property is very powerful; many mathematicians exclusively work with separable spaces.

Definition 4. *[16] A Banach space X is separable if there exists a countable collection $\{f_i\}_{i=1}^{\infty}$ of elements in X such that their linear combinations are dense in X.*

In effect, separability allows us to assume a countable basis. ℓ^p and L^p for $1 \leq p < \infty$ are examples of separable spaces [16]. We will omit an example of a non-separable space as we will not work with them in this paper and they are challenging to construct. Another example of this type of space follows.

Example 3. *[5] c_0 is the separable Banach space of all real sequences $\{u_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} u_n = 0$. This space is equipped with the essential supremum norm,*

$$||\{u_n\}_{n=1}^{\infty}||_\infty = \sup\{|u_n| : n \geq 1\}.$$

It is clear from the above definition that c_0 is the space of bounded null sequences. We find c_0 particularly interesting due to its relationship with the following space.

Example 4. *The separable space* $\ell^\infty(X)$ *consists of all real sequences* $\{u_n\}_{n=1}^\infty$ *such that*

$$||\{u_n\}_{n=1}^\infty||_\infty = \sup\{|u_n| : n \geq 1\} < \infty.$$

In other words, ℓ^∞ is the space of bounded sequences. Note that $c_0 \subset \ell^\infty$. After we introduce a few more definitions, we will return to these spaces to show that they have even more in common.

We will now discuss operators on these spaces.

Definition 5. *Suppose we have a Banach space* X *and* Y*. The operator* $T : X \to Y$ *is linear if for all* $x, y \in X$ *and* $\alpha \in \mathbb{F}$,

$$T(x+y) = Tx + Ty \text{ and } T(\alpha x) = \alpha Tx,$$

where $\mathbb{F}$ *is a scalar field.*

Let $\mathcal{L}(X,Y)$ denote the space of such operators. One can show that it is then a normed linear space under the following norm.

Definition 6. *The operator norm of* $T : X \to Y$ *is defined as*

$$||T|| = \sup\{||Tx||_Y : x \in X, ||x|| \leq 1\}.$$

Or equivalently,

$$||T|| = \sup\left\{\frac{|T(x)|}{||x||} : x \neq 0\right\}.$$

This norm has a variety of interesting properties. We will focus on one we will use throughout this paper. Suppose X, Y, and Z are Banach spaces. Let $T : X \to Y$ and $G : Y \to Z$. Let GT denote the operation of composition, such that $GT : X \to Z$. Their relationship is shown by the following diagram:

$$\underbrace{X \xrightarrow{T} Y \xrightarrow{G} Z}_{GT}.$$

Then we have that $||GT|| \leq ||G||||T||$. This result is achieved by iteratively applying the definition of the operator norm.

We have used the operator norm to construct the normed linear space $\mathcal{L}(X,Y)$. A natural question that arises is if there are other spaces we can define with the operator norm. Consider the following:

Definition 7. *Let* $T : X \to Y$ *such that for all* $x \in X$

$$||Tx||_Y \leq M||x||_X,$$

where M *is a real number. These functions are called bounded and their space is denoted as* $\mathcal{B}(X,Y)$ *under the operator norm.*

$\mathcal{B}(X, Y)$ is also a normed linear space. In the next section, we will continue to use the operator norm to define spaces for useful operators. Not all operators act between two Banach spaces. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. We call all linear operators $T : X \to \mathbb{F}$ *linear functionals*. It is not hard to show that a linear functional is bounded if and only if its continuous. Due to this equivalence, we are interested in looking at their Banach space.

Definition 8. *The dual space of X is the space of all linear functionals on X. It is commonly denoted as X^*.*

In the remainder of this section, we will need the Hahn-Banach theorem to extend linear functionals.

Theorem 1. *Let V be a vector space. Suppose V_0 is a linear subspace of V, and that we are given a linear functional ℓ_0 on V_0 that satisfies*

$$\ell_0(v) \leq p(v)$$

for all $v \in V_0$, where p is a real-valued sublinear function on V. Then ℓ_0 can be extended to a linear functional ℓ on V that satisfies

$$\ell(v) \leq p(v)$$

for all $v \in V$.

A proof of the Hahn-Banach theorem may be found in [16]. We can use the above theorem to show that the dual space of a non-trivial space X is also non-trivial.

Theorem 2. *If $X \neq \{0\}$, then $X^* \neq \{0\}$.*

Proof. By assumption, we know that there exists a non-zero element $x \in X$. We then have some subspace $Y \subset X$ such that $Y = \text{span}(x)$. Define $\varphi(x) = 1$. Therefore, $\varphi : Y \to \mathbb{R}$ and $\varphi \in Y^*$. As this is a bounded linear functional, by the Hahn-Banach theorem there exists a linear functional $\tilde{\varphi} \in X^*$ that extends φ. This linear functional is non-zero, so $X^* \neq \{0\}$. $\square$

There is a large amount of literature dedicated to identifying these spaces. For instance, let q be a positive number such that $1/p + 1/q = 1$, i.e. they are conjugate exponents. Then the dual space of L^p is L^q and vice versa [16]. One interesting consequence of this theorem is that L_2 is its own dual space. Recall from the earlier examples that only when $p = 2$, L^p is a Hilbert space. It can be shown in general that every Hilbert space is its own dual space.

One might be curious as to whether the double dual of a space is the same as the original space. A common counter-example is as follows: $(c_0)^* = \ell^1$, but $(\ell^1)^* = \ell^\infty$. In the following, we provide an example of how to construct a dual basis in the vector space of polynomials of degree at most 2:

Example 5. *Suppose we want to find the dual space for $\mathbb{P}_2$, that is the space of second degree or less polynomials spanned by $v_1 = 1$, $v_2 = x$, and $v_3 = x^2$. By definition, this basis is ϕ_j for $1 \leq j \leq 3$ such that $\phi_j(v_k) = 1$ if $j = k$ and 0 otherwise. Define ϕ_j to be the linear* *functional which selects the $(j-1)^{th}$ coefficient of a polynomial in $\mathbb{P}_2(\mathbb{R})$. In other words, if $p(x) = a_0 + a_1 x + a_2 x^2$, then $\phi(p(x)) = a_{j-1}$. It follows immediately that*

$$\phi_1(1) = 1, \phi_2(1) = 0, \phi_3(1) = 0,$$
$$\phi_1(x) = 0, \phi_2(x) = 1, \phi_3(x) = 0,$$
$$\phi_1(x^2) = 0, \phi_2(x^2) = 0, \phi_3(x^2) = 1,$$

so ϕ_1, ϕ_2, ϕ_3 is the dual basis to $\mathbb{P}_2$.

Now that we have seen how to construct X^* from X, we will shift our attention to operators defined on dual spaces. A question we may ask is if we have an operator $T : X \to Y$, is it always possible to find an operator $T^* : Y^* \to X^*$? The answer is yes, and we call this operator the *adjoint* of T.

Definition 9. *Let X and Y be Banach spaces and let $T \in L(X,Y)$. Its adjoint T^* is defined as*

$$T^*(f) = f \circ T.$$

For a Hilbert space, it follows that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

We proceed by proving elementary properties of T^*.

Theorem 3. *Let X and Y be Banach spaces and $T : X \to Y$ such that $T \in \mathcal{B}(X,Y)$. Then the following is true:*

1. *T^* is a map with domain Y^* and codomain X^*.*
2. *T^* is a linear operator.*
3. *$||T^*|| = ||T||$ under the operator norm.*

Proof. 1. Note that for all $f^* \in Y^*$, the map

$$\underbrace{X \xrightarrow{T} Y \xrightarrow{f} \mathbb{R}}_{f \circ T}.$$

We see then that $f \circ T$ is a linear functional on X. Since for all $x \in X$, we have

$$|(f \circ T)(x)| = |f(Tx)| \leq ||f||||T(x)|| \leq ||f||||T||||x||,$$

we also have that $(f \circ T) \in X^*$ and $(f \circ T)$ is bounded. Therefore, $T^* : Y^* \to X^*$ is a bounded operator with the desired domain and codomain.

2. Let $f, g \in Y^*$ and $\alpha, \beta \in \mathbb{C}$. Since f and g are linear operators, we have that

$$T^*(\alpha f + \beta g) = (\alpha f + \beta g) \circ T = \alpha(f \circ T) + \beta(g \circ T) = \alpha T^*(f) + \beta T^*(g).$$

Therefore, T^* is a linear operator.

3. By the Hahn-Banach theorem, there exists a linear functional f with $||f|| \leq 1$ such that
$$||T|| = \sup\{||f(Tx)|| : ||x|| \leq 1, ||f|| \leq 1\}$$
$$= \sup\{||(T^*f)x|| : ||f|| \leq 1, ||x|| \leq 1\} = \sup\{||T^*f|| : ||f|| \leq 1\} = ||T^*||.$$

□

Adjoints typically are not too hard to compute. In the case of matrix operators, this computation is trivial.

Example 6. *Suppose we have the matrix operator*

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Its adjoint is then

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

In general, the adjoint of a matrix operator is its conjugate transpose.

There is one more class of operators we must familiarize ourselves with: self-adjoint operators. These operators are exclusively defined on Hilbert spaces.

Definition 10. *An operator T is self-adjoint if $T = T^*$.*

We note that T is self-adjoint if and only if its matrix representation is real-valued and symmetric. Self-adjoint operators are incredibly powerful for proving specific theorems. For instance, we have the following on Hilbert spaces:

Theorem 4. *Suppose that H is a real Hilbert space and let $T \in \mathcal{L}(H, H)$. Let T be a self-adjoint operator. Then T has real eigenvalues.*

Proof. Let λ be an eigenvalue of T with corresponding eigenvector x. Then

$$\lambda||x||^2 = \lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\langle x, x\rangle = \bar{\lambda}||x||^2.$$

Dividing both sides by $||x||^2$, we get that $\lambda = \bar{\lambda}$. This is true if and only if λ is real. Therefore, every eigenvalue of T is real.

□

This spectral theorem for self-adjoint operators is incredibly powerful as it classifies what kind of operators have real eigenvalues and eigenvectors. In the case of matrix operators, these are real symmetric matrices. Later in this paper, we will investigate operators on Banach spaces which share similar properties to self-adjoint operators.

2 Compact Operators

In this section, we will focus on an important class of operators we call "compact." These operators will allow us to connect important results which hold on Hilbert spaces to the more general Banach spaces. For instance, they provide a unifying force in spectral theory. We begin by defining what we mean by *compactness.*

Definition 11. *A set $A \subset X$ if compact if for every open cover of A, there exists a finite subcover.*

For example, fix $A \subset X$. Let $\{\mathcal{O}_i\}_{i=1}^{\infty}$ be a sequence of open sets such that

$$A \subset \bigcup_{i=1}^{\infty} \mathcal{O}_i.$$

A is compact if and only if there exists some subsequence $\{\mathcal{O}_{i_j}\}_{j=1}^{N}$ with $N \in \mathbb{N}$ such that

$$A \subset \bigcup_{j=1}^{N} \mathcal{O}_{i_j}.$$

Note that there could be an infinite number of elements in A, but A would still be compact. Intuitively speaking, compactness makes the infinite finite. For any compact set, we can find a finite number of sets to cover it. For $\mathbb{R}^n$ or $\mathbb{C}^n$ a set is compact if and only if it is closed and bounded.

While not all sets are compact, some sets are more compact than others. We find it useful to define a *measure of non-compactness* in the following way.

Definition 12. *Let $A \subset X$. Then*

$$\gamma(A) = \inf\left\{ r > 0 : A \subset \bigcup_{i=1}^{n} B(x_i, r) \right\}$$

is a measure of non-compactness.

The following theorem explains how to use the above definition to characterize the compactness of sets, their subsets, their closures, and their unions.

Theorem 5. *Let A, B be bounded subsets of X. Then*

1. $\gamma(A) = 0 \iff A$ *is compact.*
2. $A \subset B \implies \gamma(A) \leq \gamma(B)$.
3. $\gamma(\bar{A}) = \gamma(A)$.
4. $\gamma(A \cup B) = \max(\gamma(A), \gamma(B))$.

Proof. 1. Suppose $\gamma(A) = 0$. Then for all $r > 0$, there exists a finite set of open balls $\{B(x_i, r)\}_{i=i}^n$ such that

$$A \subset \bigcup_{i=1}^{n} B(x_i, r).$$

Then for any open covering of balls of any size, we can find a finite subcover of A. However, from analysis we know that for any open set, we can write it as a union of open balls. Let r denote the maximum radius of these balls. Then our open cover is a subset of this union of open balls with radius r. This set has a finite subcover, so the open cover has a finite subcover. Therefore, A is compact.

Suppose A is compact. Then for all $r > 0$, we have that if

$$A \subset \bigcup_{i=1}^{\infty} B(x_i, r)$$

then

$$A \subset \bigcup_{j=1}^{n} B(x_{i_j}, r).$$

As this is true for all balls with $r > 0$, the infinum of r is 0. Therefore, $\gamma(A) = 0$.

2. Let $A \subset B$. Note that then

$$B \subset \bigcup_{i=1}^{n} B(x_i, r) \implies A \subset \bigcup_{i=1}^{n} B(x_i, r).$$

It follows immediately that $\gamma(A) \leq \gamma(B)$ because any radius that works for B works for A.

3. Consider the sequence $\{x_i\}_{i=1}^n$ in A such that

$$\gamma(A) = \inf \left\{ r > 0 : A \subset \bigcup_{i=1}^{n} B(x_i, r) \right\}.$$

Note that we have $x \in \bar{A}$ and $x \notin A$ only if x is a limit point of some sequence of numbers in A. Therefore, there is a sequence of numbers $\{y_j\}_{j=1}^{\infty}$ in $\bigcup_{i=1}^{n} B(x_i, r)$ such that for all $\varepsilon > 0$, there exists a natural number N such that when $j > N$, we have $|y_j - x| < \varepsilon$. Therefore, there exists an ε such that

$$\bar{A} \subset \bigcup_{i=1}^{n} B(x_i, r + \varepsilon).$$

By the properties of the infinum, we have that

$$\gamma(A) = \inf \left\{ r > 0 : A \subset \bigcup_{i=1}^{n} B(x_i, r) \right\} = \inf \left\{ r > 0 : A \subset \bar{A} \subset \bigcup_{i=1}^{n} B(x_i, r + \varepsilon) \right\}$$

$$= \inf \left\{ r > 0 : \bar{A} \subset \bigcup_{i=1}^{n} B(x_i, r+\varepsilon) \right\} = \inf \left\{ r > 0 : \bar{A} \subset \bigcup_{i=1}^{n} B(x_i, r) \right\} = \gamma(\bar{A}).$$

4. Let $\gamma(A) = r_1$ and $\gamma(B) = r_2$. Without loss of generality, assume that $r_2 > r_1$. Then

$$A \subset \bigcup_{i=1}^{n} B(x_i, r_1) \subset \bigcup_{i=1}^{n} B(x_i, r) \text{ and } B \subset \bigcup_{i=1}^{m} B(x_i, r_2)$$

for $n, m < \infty$. Then

$$A \cup B \subset \bigcup_{i=1}^{n} B(x_i, r_2) \cup \bigcup_{i=1}^{m} B(x_i, r_2) = \bigcup_{i=1}^{m'} B(x_i, r_2).$$

This implies that $\gamma(A \cup B) \leq r_2$.

Suppose $\gamma(A \cup B) < r_2$. Then there exists an $r_2 > r > 0$ such that

$$B \subset A \cup B \subset \bigcup_{i=1}^{p} B(x_i, r) \implies \gamma(B) = r < r_2.$$

This is a contradiction. Therefore, $\gamma(A \cup B) \geq r_2$.

These two inequalities can only be true if $\gamma(A \cup B) = r_2$. Note that if $r_1 > r_2$, repeating the same proof we end up with $\gamma(A \cup B) = r_1$. Therefore, $\gamma(A \cup B) = \max(r_1, r_2) = \max(\gamma(A), \gamma(B))$.

□

In a later section, we will return to γ to explore more of its properties. However, for now, we shift our attention to *compact* operators.

Definition 13. *An operator $T : X \to Y$ is compact if and only if the closure of $T(B)$ is compact, where B is the closed unit ball in X. Alternatively, it is compact if for every bounded sequence $\{f_n\}_{n=1}^{\infty}$ in X, there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{Tf_{n_k}\}_{k=1}^{\infty}$ converges.*

Much like we observed for linear and bounded operators, compact operators under the operator norm form a normed linear space. We will denote this space throughout this thesis as $\mathcal{K}(X, Y)$.

Development of the theory of compact operators was initiated by research into what are known as *integral operators*. Before we delve into this further, we provide a theorem useful to this field of study.

Theorem 6. *[6] Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded and uniformly continuous sequence of functions on the closed interval $[a, b]$. There exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on $[a, b]$.*

The above is known at the *Arzela-Ascoli* theorem. Note that a uniformly bounded sequence is a sequence which shares a common bound and a uniformly continuous sequence is a sequence of functions whose ε in the $\varepsilon - \delta$ definition of continuity does not depend on x and y. With this theorem, we can prove the following theorem about integral operators.

Theorem 7. *Let $a, b \in \mathbb{R}$ with $b > a$ and let $I = [a, b]$. Suppose $k : I \times I \to \mathbb{C}$ is a continuous function. Define*

$$(Kx)(s) = \int_a^b k(s,t)x(t)dt$$

for all $s \in I$ and all $x \in C(I)$, where $C(I)$ denotes the continous functions $x : I \to I$. Then K is compact.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence of functions in $L^2[a,b]$ such that $||f_k|| \leq M$ for some $M \in \mathbb{R}$. Since k is continuous, for every $\epsilon > 0$, we can find a $\delta > 0$ such that when $|x_1 - x_2| < \delta$, then $|k(x_1,t) - k(x_2,t)| < \epsilon$. It follows that

$$|Kf_k(x_1) - Kf_k(x_2)| = |\int_a^b k(x_1,t)f_k(t)dt - \int_a^b k(x_2,t)f_k(t)dt|$$

$$= |\int_a^b (k(x_1,t) - k(x_2,t))f_k(t)dt| \leq \int_a^b |k(x_1,t) - k(x_2,t)||f_k(t)|dt$$

$$\leq \epsilon \int_a^b |f_k(t)|dt \leq M(a-b)\epsilon.$$

Therefore, we have that $\{Kf_k\}_{k=1}^{\infty}$ is equicontinuous. Moreover, this sequence is also uniformly bounded. Recall that the Arzela-Ascoli theorem states that given an uniformly bounded and equicontinuous sequence of real-valued functions $\{f_k\}_{k=1}^{\infty}$ defined on the interval $[a,b]$, then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ which converges uniformly. Therefore, there exists a convergent subsequence of $\{Kf_k\}_{k=1}^{\infty}$. This is the definition of compactness of K.

□

Due to the usefulness of integral operators, compact operators surged in popularity. The nicest part about compact operators are the simple forms with which we may represent them. For instance, consider the following class of operators:

Definition 14. *An operator T is called finite-rank if its bounded and the dimension of its range is finite-dimensional.*

From the above definition, we can arrive at the following theorem:

Theorem 8. *Let $\{T_n\}$ be a sequence of finite rank operators and let $||T - T_n|| \to 0$ as $\to \infty$ in the operator norm. Then $T \in \mathcal{K}(X,Y)$.*

Proof. We begin by showing that if $\{T_k\}_{i=1}^{\infty}$ is a sequence of compact operators which converges to T in the operator norm, then T is compact. Let $\{x_i\}_{n=1}^{\infty}$ be a sequence of numbers which converges to x. Since T_k is compact for all $k \in \mathbb{N}$, then we have that $||T_k x_{i_j} - T_k x|| < \frac{\epsilon}{2}$

when j is large enough. We also know by assumption that $||Tx - T_k x|| < \frac{\epsilon}{2}$ when k is large enough. Putting these together with the triangle inequality, we get that

$$||Tx - Tx_{i_j}|| \leq ||Tx - T_k x|| + ||T_k x - Tx_{i_j}|| < \epsilon.$$

Therefore, there exists a subsequence of $\{x_i\}_{j=1}^{\infty}$ such that Tx_{i_j} converges to Tx. Therefore, T is compact.

Now that we have proven that the limit of compact operators is compact, all that is left to show is that finite-rank operators are compact. Let T denote a finite-rank operator. Since T is bounded, given a sequence $\{x_i\}_{i=1}^{\infty}$, $\{Tx_i\}_{i=1}^{\infty}$ is also bounded. Moreover, since T is finite-dimensional, its range is finite-dimensional, and therefore its range is isomorphic to $\mathbb{R}^d$. We may then apply the Bolzano-Weierstrass theorem; there exists a convergent subsequences of $\{Tx_i\}_{i=1}^{\infty}$. So T is compact, completing the proof.

□

Note that the converse of this theorem is not always true. We say that spaces where the converse hold have the *approximation property*. The above theorem allows us to construct an example of a compact operators.

Theorem 9. *Let $x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in \ell^2$ with $T : \ell^2 \to \ell^2$ defined as $T(x) = y$ where $y_i = \frac{x_i}{i}$. Then T is a compact operator.*

Proof. Define $T_n : \ell^2 \to \ell^2$ such that

$$T_n(x) = (x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, 0, \dots).$$

We then have that

$$||(T - T_n)x||^2 = \langle (T - T_n)x, (T - T_n)x \rangle = (0, \dots, 0, \frac{x_{n+1}}{n+1}, \dots) \cdot (0, \dots, 0, \frac{x_{n+1}}{n+1}, \dots)$$

$$= \frac{x_{n+1}^2}{(n+1)^2} + \frac{x_{n+2}^2}{(n+2)^2} + \dots \leq \frac{x_{n+1}^2}{(n+1)^2} + \frac{x_{n+2}^2}{(n+1)^2} + \dots =$$

$$\leq \frac{x_1^2}{(n+1)^2} + \dots + \frac{x_n^2}{(n+1)^2} + \dots = \frac{1}{(n+1)^2}(x_1^2 + \dots + x_n^2 + \dots) = \frac{||x||^2}{(n+1)^2}.$$

Taking the square root of both sides, we arrive at

$$||(T - T_n)x|| \leq \frac{||x||}{n+1} \implies ||T - T_n|| < \frac{1}{n+1}.$$

As $n \to \infty$, we see that $T_n \to T$. As T_n is always rank n, $\{T_n\}_{n=1}^{\infty}$ is a sequence of finite-rank operators, so T is compact.

□

3 Schmidt Representation and Singular Values

Now that we have defined compact operators, we can start to look into some of their properties. It is often useful to find alternative representations and decompositions of a function if possible. This kind of result is achievable on a Hilbert space for compact operators.

Theorem 10. *[13] Let H and G be Hilbert spaces. For each $T \in K(H, G)$, there exists a decreasing null sequence $\{s_n\}_{n=1}^{\infty}$ in $[0, \infty)$ and orthonormal systems $\{e_n\}_{n=1}^{\infty}$ in H and $\{f_n\}_{n=1}^{\infty}$ in G, such that*

$$T = \sum_{n=0}^{\infty} s_n \langle \cdot, e_n \rangle f_n,$$

where the series converges in the operator norm.

Proof. T^*T is compact and self-adjoint. Take λ to be a non-zero element in the spectrum of this operator and x to be a corresponding eigenvector normalized such that $||x|| = 1$. We have that

$$\lambda = \langle \lambda x, x \rangle = \langle T^*Tx, x \rangle = \langle Tx, Tx \rangle \geq 0.$$

The above condition is equivalent to the spectrum of T^*T being contained in the interval $[0, ||A||^2]$. For compact self-adjoint operators, we have that there exists an orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$ corresponding to a null sequence of real eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $A = \sum_{n=0}^{\infty} \lambda \langle \cdot, e_n \rangle e_n$, where the series converges in the operator norm. Therefore, we have that

$$T^*T = \sum_{n=1}^{\infty} s_n^2 \langle \cdot, e_n \rangle e_n.$$

For $n \in \mathbb{N}$ with $s_n > 0$, define $f_n = s_n^{-1} T e_n$. Then we have that

$$\langle f_n, f_m \rangle = s_n^{-1} s_m {-1} \langle Te_n, Te_m \rangle = \frac{1}{s_n s_m} \langle T^*Te_n, e_m \rangle = \frac{s_n^2}{s_n s_m} \langle e_n, e_m \rangle = \begin{cases} \frac{s_n^2}{s_n s_m} & \text{if } m = n, \\ 0 & \text{else.} \end{cases}$$

If $N = \{n \in \mathbb{N} : s_n > 0\}$ is a finite set, then we extend the orthonormal system $\{f_n\}_{n\in N}$ to $\{f_n\}_{n\in\mathbb{N}}$. For $y \in H$ with $y \perp e_n$ for all $n \in \mathbb{N}$, we have that

$$||Ty||^2 = \langle Ty, Ty \rangle = \langle T^*Ty, y \rangle = 0.$$

Thus from the definition of $\{f_n\}_{n\in\mathbb{N}}$ we have

$$Tx = T\left(x - \sum_{n=0}^{\infty} \langle x, e_n \rangle e_n\right) + T\left(\sum_{n=0}^{\infty} \langle x, e_n \rangle e_n\right) = \sum_{n=0}^{\infty} \langle x, e_n \rangle Te_n = \sum_{n=0}^{\infty} s_n \langle x, e_n \rangle f_n.$$

$\square$

The above form of a linear operator is known as its *Schmidt representation*. Since each $s_n\langle\cdot, e_n\rangle f_n$ is finite-dimensional, we thereby have that every compact operator is the limit of finite-dimensional operators. To see this directly, consider the partial sum

$$T_k = \sum_{n=0}^{k} s_n\langle\cdot, e_n\rangle f_n.$$

Clearly, $T_k \to T$. Therefore, every Hilbert space has the approximation property. Note that while the sequences $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are not unique, as there are many orthonormal systems in a Hilbert space, the sequence $\{s_n\}_{n=1}^{\infty}$ is. These numbers are known as singular values.

Definition 15. *The singular values $\{s_n\}_{n=1}^{\infty}$ of an operator T are given by the eigenvalues of $\sqrt{T^*T}$. We will denote these as $s_n(T)$.*

We provide an example on how to compute the singular values of a matrix operator. This process is not difficult, but sometimes tedious, so we employ the assistance of a calculator.

Example 7. *Let the matrix operator A be given by*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies A^*A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}.$$

The eigenvalues of this matrix are $15 \pm \sqrt{221}$, so its singular values are $\sqrt{15 \pm \sqrt{221}}$.

Singular values see a wide array of applications in a variety of fields, from functional analysis to image processing. These tools, such as other decomposition similar to those above, are commonly used in both pure and applied contexts. This usefulness has led to a considerable development of theory around these values and some of their properties are explored below.

Definition 16. *[15] A map $s : X \to \Big(s_n(X)\Big)$ from a Banach space X into the set of sequences of non-negative numbers is called an s-number function if the following conditions are satisfied ($n = 1, 2, \dots$):*

1. $||S|| = s_1(S) \geq s_2(S) \geq \cdots \geq 0$ *for* $S \in \mathcal{L}(E, F)$.
2. $s_n(S + T) \leq s_n(S) + ||T||$ *for* $S, T \in \mathcal{L}(E, F)$
3. $s_n(RST) \leq ||R||s_n(S)||T||$ *for* $T \in \mathcal{L}(E_0, E), S \in \mathcal{L}(E, F), R \in \mathcal{L}(F, F_0)$
4. *If* $\dim S > n$ *and* $S \in \mathcal{L}(E, F)$, *then* $s_n(S) = 0$.
5. *If* $\dim E \geq n$, *then* $s_n(I_E) = 1$,

   where E_0, E, F, F_0 *are all Banach spaces.*

4 Approximation Numbers

On Hilbert spaces, the singular values are the only s-number [3]. However, this is not necessarily true on an arbitrary Banach space. In attempt to generalize s-numbers to Banach spaces, we start by looking at an equivalence of singular values on Hilbert spaces. Consider the following definition:

Definition 17. *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Then*

$$\alpha_n(T) = \inf\{||T - A|| : A \in \mathcal{L}(X, Y) \text{ with } rank(A) < n\}$$

be the n^{th} approximation number of T.

Theorem 11. *[14] For every $A \in K(H, G)$ and all $n \in \mathbb{N}$,*

$$s_n(A) = \alpha_n(A).$$

Proof. Since A is a compact operator, it has some Schmidt representation

$$A = \sum_{n=0}^{\infty} s_n \langle \cdot, e_n \rangle f_n.$$

By considering the partial sum of this series, we have an approximation for A that includes singular values, allowing us to connect $\alpha_n(A)$ and s_n. We see that

$$\alpha_n^2(A) \leq ||Ax - \sum_{j=0}^{n-1} s_j \langle x, e_j \rangle f_j||^2 = ||\sum_{j=n}^{\infty} s_j \langle x, e_j \rangle f_j||^2.$$

By the definition of the operator norm and the fact that $||f_j|| = 1$, we have that the above is less than

$$\sum_{j=n}^{\infty} ||s_j||^2 ||\langle x, e_j \rangle|| ||f_j||^2 = \sum_{j=n}^{\infty} s_j^2 |\langle x, e_j \rangle|^2.$$

By Bessel's inequality and the fact that $||x|| \leq 1$, the above is less than

$$s_n^2 ||x||^2 \leq s_n^2 \implies \alpha_n(A) \leq s_n.$$

To show the other direction, take $A \in \mathcal{L}(E, F)$ with $\text{rank}(A) < n$. As this operator is finite-dimensional, if we restrict it to the span of $\{e_j\}_{j=0}^{n}$ must have a non-trivial kernel. Let $y = \sum a_j e_j$ be in this kernel and normalize it so $||y|| = 1$. It follows by the definition of the operator norm that

$$||A - B||^2 \geq ||(A - B)y||^2.$$

Since y is in the kernel of B and due to the Schmidt representation of A, we have that the above is equal to

$$||Ay||^2 = ||\sum_{j=0}^{n} s_j a_j f_j|| \geq ||\sum_{j=0}^{n} s_j^2 |a_j|^2|| \geq s_n^2 \implies \alpha_n(A) \geq s_n.$$

Thus, we must have that $\alpha_n(A) = s_n(A)$.

$\square$

At the beginning of this section, we discussed a measure of non-compactness γ. Through the use of approximation numbers, we can compute an upperbound on its value.

Theorem 12. *Suppose X and Y are normed linear spaces and $T \in \mathcal{B}(X, Y)$. Define $\gamma(T(B_X)) = \gamma(T)$, where B_x denotes the unit ball in X. Then*

$$\gamma(T) \leq ||T||_{\mathcal{K}} \leq \alpha_n(T),$$

where $||T||_{\mathcal{K}} = d(T, \mathcal{K}(X, Y))$.

Proof. First, note that

$$\gamma(T) = \inf \left\{ r > 0 : B_X \subset \bigcup_{i=1}^{n} B(x_i, r) \right\}.$$

Recall that γ denotes the smallest positive radius such that we can find a finite subcover of a covering of open balls of radius γ. Then there exists an open cover $\mathcal{O}$ that cannot be written as a union of open balls of radius $\gamma(T)$ or greater. Let K denote the closest compact operator to T in the sense of $||T||_{\mathcal{K}}$. We know that for any open cover of K's unit ball, there exists a finite subcover as its unit ball is compact. As such, there must exists $x \in X$ with $||x|| \leq 1$ such that

$$Kx \in \bigcup_{i=1}^{n} B(x_i, r) \subset \mathcal{O},$$

but

$$Tx \notin \bigcup_{i=1}^{n} B(x_i, r),$$

where $x_i = x$ for some x. This implies that

$$||Tx - Kx|| \geq r \implies ||T - K|| \geq \gamma(T)$$

under the operator norm as Tx must exist outside of $B(x, r)$. However, this is equivalent to

$$\gamma(T) \leq ||T - K|| = \inf \{||T - K|| : K \in \mathcal{K}(X, Y)\} = ||T||_{\mathcal{K}}$$

.

Next, recall that

$$||T||_{\mathcal{K}} = \inf \{||T - K|| : K \in \mathcal{K}(X, Y)\}.$$

Therefore, it is the best approximation of T by a compact operator. Since all finite-rank operators are compact, we know that $||T||_{\mathcal{K}}$ is always at least as good as $\alpha_n(T)$. Therefore, we have the bound

$$||T||_{\mathcal{K}} \leq \alpha_n(T).$$

Putting these two inequalities together, we arrive at the desired result.

$\square$

We will take some time to dive into the properties of approximation numbers in the next section. We will find that they will be invaluable tools for developing a spectral theory for Banach spaces .

We observed in the preceding section that on a Hilbert space, singular values are equivalent to approximation numbers. We will spend some time now exploring some applications of approximation numbers. We are most interested in how they differ between Hilbert spaces and Banach spaces. We begin by showing the approximation numbers are indeed still s-numbers on a Banach space.

Theorem 13. *The approximation numbers are s-numbers.*

Proof. We proceed by proving that α_n satisfies all 5 necessary conditions of an s-number function.

1. Let $S \in \mathcal{L}(E, F)$. We first want to show that $\alpha_1(S) = ||S||$. There is exactly one operator with rank less than 1: the zero operator $0(x) = 0$. It follows immediately that $\alpha_1(S) = ||S - 0|| = ||S||$.

   We will now show that $\alpha_i(S) \geq \alpha_{i+1}(S)$. Let $A \in \mathcal{L}(E, F)$ with $\text{rank}(A) < i$ such that $\alpha_i(S) = ||S - A||$. Then we also know that $\text{rank}(A) < i + 1$. It is clear then that

   $$\alpha_{i+1}(S) = \inf_{\text{rank}(B)<i+1} ||S - B|| \leq ||S - A|| = \alpha_i(S).$$

   Of course, for any n, $\alpha_n(S) \geq 0$ as the operator norm is non-negative.

2. Let $S, T \in L(E, F)$. Let $A \in \mathcal{L}(E, F)$ with $\text{rank}(A) \leq n$ such that $\alpha_n(S) = ||S - A||$. By the first property of s-numbers and the triangle inequality,

   $$\alpha_n(S + T) \leq ||S + T - A|| \leq ||S - A|| + ||T|| = \alpha_n(S) + ||T||.$$

3. Let $T \in \mathcal{L}(E_0, E), S \in \mathcal{L}(E, F)$, and $R \in \mathcal{L}(F, F_0)$. Let $A \in \mathcal{L}(E, F)$ with $\text{rank}(A) \leq n$ such that $\alpha_n(S) = ||S - A||$. Note that $\text{rank}(RAT) \leq n$ due to A's own rank. By applying the definition of the operator norm twice, we get

   $$\alpha_n(RST) \leq ||RST - RAT|| = ||R(S - A)T|| \leq ||R||||S - A||||T|| = ||R||\alpha_n(S)||T||.$$

4. Let $\text{rank}(S) < n$. Then the best approximation of S with rank less than n is S itself, i.e. $\alpha_n(S) = ||S - S|| = 0$.

5. The proof for this 5^{th} property was taken from [15]. Let $\dim(E) \geq n$. Assume that $\alpha_n(I_E) \leq 1$. Then there exists an $A \in L(E, E)$ and $\text{rank}(A) \leq n$ such that $||I_E - A|| \leq 1$. Since we can write $A = I_E - (I_E - A)$, A is invertible by the Neumann series and thusly $\text{rank}(A) \geq n$. This is a contradiction.

□

We are now confident that there are some equivalences between approximation numbers and singular values between the two spaces. One might be interested precisely in the relationship between s-numbers and approximation numbers. This may be easily computed.

Theorem 14. *[15] For any s-number s_n and $S \in \mathcal{L}(E,F)$, $s_n(S) \leq \alpha_n(S)$.*

Proof. Let $S \in \mathcal{L}(E,F)$. Then for each s-number function and $A \in \mathcal{L}(E,F)$ with $\dim(A) < n$, we have

$$s_n(S) = s_n(A + (S - A)) \leq s_n(A) + ||S - A|| = ||S - A||$$

by the (2) and (4) axioms of s-numbers. Since the above inequality is true for an arbitrary $A \in \mathcal{L}(E,F)$ with $\dim(A) < n$, it follows that

$$s_n(S) \leq \inf_{A \in \mathcal{L}(E,F)} ||S - A|| = \alpha_n(S)$$

for all $S \in \mathcal{L}(E,F)$. $\square$

Approximation numbers serve as a tight upper-bound to s-numbers. Having such a special relationship to generalized singular values, we should expect theorems on Banach spaces and Hilbert spaces to be at least somewhat similar. In fact, this trend will continue for some time, where only minor changes to the hypotheses of theorems will be required to achieve similar results. The proofs, however, are typically more involved.

5 Approximations of Adjoint Operators

The previous discussion leads to the first property we will investigate:

Theorem 15. *Let E and F be Banach spaces and $T \in \mathcal{L}(E,F)$. Then*

$$\alpha_n(T) = \alpha_n(T^*).$$

A proof of the above result is easy when E and F are Hilbert spaces. The approximation numbers of T are found by computing the eigenvalues of $\sqrt{T^*T}$. Likewise, the approximation numbers of T^* are found by finding the eigenvalues of $\sqrt{TT^*}$. Note that the eigenvalues of T^*T and TT^* differ only by the multiplicity of 0 eigenvalues. As matrix operators are square in Hilbert spaces, the eigenvalues of T^*T and TT^* are the same.

On a Banach space, we can achieve a similar result if we restrict $T \in \mathcal{K}(X,Y)$. However, we must first build tools. We introduce the following:

Theorem 16. *[12] Let $E \subset X^{**}$ and $F \subset X^*$ be finite dimensional subspaces. Given $\varepsilon > 0$, there exists an ε-isometry $T : E \to X$ such that $T|_{E \cap X} = id|_{E \cap X}$ and $f(Te) = e(f)$ for all $f \in E$ and all $e \in E$.*

While the statement itself is esoteric, the result has an intuitive explanation. Every finite-dimensional subspace of X^{**} has to also be in X. In other words, X and its double dual only disagree in infinite dimensions. Even then, they only disagree by a distance of ε, due to the existence of an ε-isometry. We may now prove the desired result on Banach spaces.

Theorem 17. *[7] If T is compact, $\alpha_n(T) = \alpha_n(T^*)$.*

Proof. Let $T \in \mathcal{K}(X,Y)$, where X and Y are Banach spaces. First, note that it is easy to observe that $a_n(T^*) \leq \alpha_n(T)$. Let A be the best rank n approximation of T. Then

$$a_n(T) = ||T - A|| = ||(T-A)^*|| = ||T^* - A^*|| \geq \alpha_n(T^*).$$

All that is left is to show that $a_n(T) \leq \alpha_n(T^*)$. Let $E \subset Y^{**}$ and $F \subset Y^*$ be finite-dimensional subspaces. Since T is compact, both T^* and T^{**} are also compact. Let $B_{X^{**}}$ denote the unit ball in X^{**}. Then by the compactness of T^{**}, $T^{**}(B_{X^{**}})$ is a totally bounded set. Since $T^{**}(B_{X^{**}})$ is totally bounded, for any $\varepsilon > 0$ we can find

$$T^{**}(B_{X^{**}}) \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon),$$

where $x_i \in T^{**}(B_{X^{**}})$ for all i.

Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a sequence of positive numbers converging to 0. We then have a family of finite coverings of $T^{**}(B_{X^{**}})$ which depend on j, that is

$$T^{**}(B_{X^{**}}) \subset \bigcup_{i=1}^{N(j)} B(x_{i,j}, \varepsilon_j).$$

Let G_j denote the space spanned by the sequence $\{T^{**}x_{i,j}\}_{i=1}^{N(j)}$. By the principle of local reflextivity, there exists an ϵ_j-isometry $\phi_j : G_j \to Y$ such that $\phi_j|_{G_j \cap Y} = id|_{G_j \cap Y}$, where id is the canonical injection from G_j to Y. We will define

$$\varphi : \bigcup_{j=1}^{\infty} G_j \to Y \text{ such that } \varphi(x) = \phi_j(x) \text{ when } x \in G_j.$$

This function is well defined as if $x \in G_i$ and $x \in G_j$, then

$$\phi_i(x) = id|_{G_i \cap Y}(x) = x = id|_{G_j \cap Y}(x) = \phi_j(x)$$

as G_i is a subset of Y by the compactness of T^{**}. Naturally, we may extend φ to the closure of its domain and call this G. Therefore, $\varphi : G \to Y$.

Let $A : X^{**} \to Y^{**}$ be of rank at most n such that $||T - A|| \leq \alpha_n(T) + \varepsilon$. As A is of finite rank, $A(X^{**})$ is of finite-dimensional. We can then apply the principle of local reflextivity to find an ϵ-isometry $\psi : A(X^{**}) \to Y$ such that $\psi|_{A(X^{**}) \cap Y} = id|_{A(X^{**}) \cap Y}$. Define $\Psi : G \cup A(X^{**})$ such that if $x \in G$, $\Psi(x) = \varphi(x)$, and if $x \in A(X^{**})$, $\Psi(x) = \psi(x)$. This function is well-defined as

$$G \cap A(X^{**}) \subseteq G \subset Y,$$

so if $x \in G \cap A(X^{**})$, $\Psi(x) = id(x) = x$.

Let $I : X \to X^{**}$ be the canonical map and consider $\Psi A I : X \to Y$. This function can have at most rank n due to the presence of A. Let $J : Y \to Y^{**}$ denote the canonical injection on these spaces. We see that when $||x|| \leq 1$,

$$\alpha_n(T) \leq ||T - \Psi A I|| \leq ||Tx - \Psi A I x||_Y = ||JTx - J\Psi A I x||_{Y^{**}} = ||T^{**}Ix - J\Psi A I x||_{Y^{**}}.$$

Since $T^{**}Ix \in G$ as x is in the unit ball, we have that $J\Psi T^{**}Ix = T^{**}Ix$ as each operates like the identity in this domain. Therefore, the above is equal to

$$||J\Psi T^{**}Ix - J\Psi AIx||_{Y^{**}} \leq ||J||||\Psi||||T^{**}Ix - AIx|| \leq ||T^{**}Ix - AIx|| \leq \alpha_n(T^{**}).$$

This implies that $a_n(T) \leq \alpha_n(T^*)$ for all n, so the proof is complete.

□

Having now shown that the statement is true for compact operators, one might be curious if we could extend it to a more generalized group of operators. However, it turns out that this fact is not true. The counter-example follows:

Example 8. *[7] Let $I : \ell^1 \to c_0$ be the natural injection. Then $\alpha_k(I) = 1$ for each $k \in \mathbb{N}$. Let $I^* : \ell^1 \to \ell^\infty$ be its adjoint. Then $\alpha_k(I^*) = 1/2$ for each $k \in \mathbb{N}$. Therefore, $\alpha_k(I) \neq \alpha_k(I^*)$.*

6 Eigenvalues of Compact and Adjoint Operators

There are other properties of approximation numbers which have slightly different requirements between Hilbert and Banach spaces. As approximation numbers represent the error of the best approximation of an operator, it makes sense to consider whether or not a sequence of functions converges to the same error. If this is true, then we can possibly find the best approximation of a computationally easier class of operators and get accurate estimates for what they converge to.

Theorem 18. *[2] Let H be a separable complex Hilbert space, $T \in B(H, H)$ and P_n be the orthogonal projection onto the span of $\{e_j\}_{j=1}^n$, where this sequence denotes an orthonormal basis of H.Let $T_n = P_n T P_n$. Then for each $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

In other words, the truncations of T converge to T.

In [2], work is done to find an equivalent theorem on Banach spaces. While the theorem is quite cumbersome, a brief description of it will be included here. The author finds that for bounded operators, under specific kinds of convergence for $T_n \to T$ and if T_n is compact for every $n \in \mathbb{N}$, we have

$$s_k(T) = s_k(T^*) \iff \lim_{n\to\infty} s_k(T_n) = s_k(T).$$

Moreover, if T is compact,

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

Suppose now we require T to be self-adjoint. We can then approximate the spectrum of an operator from its singular values. This may be of particular interest if T^*T is an easier function to compute the eigenvalues for than T.

Theorem 19. *[4] Let H be a Hilbert space. If $T \in B(H, H)$ is self-adjoint and the essential spectrum of T is connected, then*

$$\liminf s_n(T) = \limsup s_n(T) = s(T).$$

Through the use of approximation numbers, we can find a similar theorem on Banach spaces for approximating only the spectral radius. While this is weaker than the above theorem, it is still useful as the spectral radius is the largest element of the spectrum of an operator.

Theorem 20. *[3] Let A be a complex $n \times n$ matrix and $||\cdot||$ be a norm in $\mathbb{C}^d$. Then*

$$\rho(A) = \lim_{m\to\infty} ||A^n||^{1/n} = \limsup_{n\to\infty} ||A^n||^{1/n} = \liminf_{n\to\infty} ||A^n||^{1/n},$$

where $\rho(A)$ denotes the spectral radius of A.

Now that we have developed a theory of approximation numbers, we may proceed to dive into more generalized results on Banach spaces. We mentioned earlier in this thesis that self-adjoint operators are extremely powerful for proving theorems. What would happen if we loosen the conditions on theses operators?

7 H-Operators

Let H be a separable Hilbert space. Hilbert and Schmidt proved a powerful decomposition theorem for all self-adjoint compact operators $T : H \to H$. This is known as the spectral decomposition. This theory amplified to a set of results which we call nowadays Riesz-Theory; this field is dedicated to the study of operators $S : X \to X$, where X is a complex Banach space, that can be expressed as $S = \lambda I_X - T$ with $\lambda \neq 0$ and $T \in \mathcal{K}(X, X)$. In such study, one may find that the spectral properties of the operator T is essential and these properties are related to the approximation quantities. However, on Banach spaces, the definition of self-adjoint operators is quite limited. As this property is still useful, this motivates the development of the concept of H-operators.

Definition 18. *[8] Let X denote a Banach space. The linear operator T acting on X is an H-operator if and only if its spectrum is real and its resolvent satisfies*

$$||(T - \lambda I)^{-1}|| \leq C|Im\ \lambda|^{-1},$$

where $Im\ \lambda \neq 0$.

H-operators are the needed generalization of self-adjoint operators to Banach spaces. In fact, $C = 1$ if and only if it is self-adjoint [11]. Below, we provide an example of this fact.

Example 9. *Consider the matrix operator*

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Note that the above operator is self-adjoint, because it is diagonal. Then its resolvent is given by

$$T - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \implies (T-\lambda I)^{-1} = \begin{bmatrix} \frac{1}{3-\lambda} & 0 \\ 0 & \frac{1}{3-\lambda} \end{bmatrix}.$$

The operator norm of this matrix is given by its largest singular value, which means that

$$||(T-\lambda I)^{-1}|| = \left| \frac{1}{3-\lambda} \right|.$$

Since $\lambda \in \mathbb{C}$, we have that $\lambda = a + bi$ for $a, b \in \mathbb{R}$. Then

$$\frac{1}{3-\lambda} = \frac{1}{(3-a)-bi} = \frac{3-a+bi}{(3-a)^2+b^2}$$

$$\implies \left| \frac{1}{3-\lambda} \right| = \frac{\sqrt{(3-a)^2+b^2}}{(3-a)^2+b^2} = \frac{1}{\sqrt{(3-a)^2+b^2}} \leq \frac{1}{b^2} = |Im\lambda|^{-2} \leq |Im\lambda|^{-1}$$

because $b^2 < \sqrt{(3-a)^2+b^2}$. Therefore, T is an H-operator with $C = 1$. Note that the above bound does not work if we take $C < 1$.

It is useful when defining a new class of operators to determine how to construct them specific classes of them. The following example provides a criteria for which closed linear operators are H-operators.

Theorem 21. *[11] Let T be a closed linear operator acting on X which has a monotonic sequence $\{\lambda_i\}$ of real eigenvalues, and let the corresponding sequence of eigenvectors $\{\phi\}$ form a basis of X. Then T is an H-operator.*

Proof. Define

$$\sigma = \sum_{i=1}^{\infty} \left| \frac{1}{\lambda_i - \lambda} - \frac{1}{\lambda_{i+1} - \lambda} \right|,$$

where $\text{Im}\lambda \neq 0$. By looking at the triangle defined by λ, λ_i, and λ_{i+1}, we can see that

$$\left| \frac{1}{\lambda_i - \lambda} - 1\lambda_{i+1} - \lambda \right| = \frac{|\lambda_{i+1} - \lambda_i|}{|\lambda_i - \lambda||\lambda_{i+1} - \lambda|} = \frac{|\sin \alpha_i|}{\text{Im}\lambda} \leq \frac{|\alpha_i|}{\text{Im}\alpha_i}$$

where $\alpha_i = \arg(\lambda - \lambda_{i+1}) - \arg(\lambda - \lambda_i)$. Since we assumed that $\{\lambda_i\}$ is a monotonic sequence, for all i, α_i has the same sign. As a result,

$$\sigma \leq |\text{Im}\lambda|^{-1} \sum_{i=1}^{\infty} |\alpha_i| = |\text{Im}\lambda|^{-1} |\arg(\lambda - \lambda_1) - \lim_{i \to \infty} \arg(\lambda - \lambda_i)| < \pi |\text{Im}\lambda|^{-1}$$

because the series is telescoping. Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of functionals biorthogonal to $\{\phi_i\}_{i=1}^{\infty}$ and define

$$P_n = \sum_{i=1}^{\infty} F_i(\cdot)\phi_i.$$

Since $\{\phi_i\}_{i=1}^{\infty}$ is a basis of X, the sum converges and the operator norm of P_n is bounded, say by some real number a. This implies that

$$\sum_{i=1}^{\infty}\left|\frac{1}{\lambda_i-\lambda}-\frac{1}{\lambda_{i+1}-\lambda}\right| ||P_i|| \leq a\sigma \leq a\pi|\text{Im}\lambda|^{-1}.$$

Because of this bound, the series

$$\frac{I}{\lambda_0-\lambda}+\sum_{i=1}^{\infty}\left(\frac{1}{\lambda_i-\lambda}-\frac{1}{\lambda_{i+1}-\lambda}\right)P_i$$

converges in the operator norm where $\lambda_0 = \lim_{i\to\infty}\lambda_i$. Note that this is the resolvant of T. We showed earlier its operator norm was bounded by $a\pi|Im\lambda|^{-1}$, so it is a H-operator and we are done.

□

Recall the spectral theorem for self-adjoint operators. We know that their eigenvalues are always real. The above example serves as a close reverse to the spectral theorem; closed operators with real eigenvalues are H-operators.

8 Eigenvalues of Compact H-Operators

H-operators have their own unique spectral properties. For instance, we can make strong statements regarding the multiplicity of their eigenvalues.

Theorem 22. *[11] Every generalized eigenvector of an H-operator is an eigenvector.*

While these properties are interesting in and of themselves, they are more valuable for how they connect to our main topic: approximation numbers. When we have compact H-operators, we can upper-bound the approximation numbers of an operator with their eigenvalues.

Theorem 23. *[11] If T is a compact H-operator, then*

$$\alpha_n(T) \leq 2\sqrt{2}C|\lambda_n(T)|.$$

Proof. Let $\rho > |\lambda_n(A)|$. Then no point in the spectrum of T lies on the path $\gamma = \rho e^{it}$ for $0 \leq t < 2\pi$. Define $R_\lambda = (T-\lambda I)^{-1}$, i.e. the resolvent of T. Consider the following integral,

$$P = -\frac{1}{2\pi i}\int_\gamma R_\lambda d\lambda.$$

P is a projection, so $P = P^2$. Therefore,

$$-\frac{1}{2\pi i}\int_\gamma R_\lambda d\lambda = -\frac{P}{2\pi i}\int_\gamma R_\lambda d\lambda.$$

If $z \in \gamma^*$, i.e. the image of γ, we know that $I/(\lambda - z)$ is a holomorphic function over γ, so its integral is zero. Therefore,

$$-\frac{P}{2\pi i}\int_\gamma R_\lambda d\lambda = -\frac{P}{2\pi i}\int_\gamma R_\lambda + \frac{I}{\lambda - z} d\lambda = 0$$

because R_λ is also holomorphic over γ. Combining the fractions, we find that

$$0 = \frac{P}{2\pi i}\int_\gamma R_\lambda + \frac{I}{\lambda - z} d\lambda = -(T - zI)\frac{P}{2\pi i}\int_\gamma \frac{R_\lambda}{\lambda - z} d\lambda.$$

Since $z \in \gamma^*$, and γ^* contains no eigenvalues of T, $(A - zI)^{-1}$ exists. This gives us

$$\frac{\rho^2 - z^2}{2\pi i} P \int_\gamma \frac{R_\lambda}{\lambda - z} d\lambda = 0.$$

It is a fact in matrix computation that

$$(T + zI)P = -\frac{P}{2\pi i}\int_\gamma (\lambda + z) R_\lambda d\lambda.$$

Combining this with the earlier equality, we have that

$$(T + zI)P = -\frac{P}{2\pi i}\int_\gamma \frac{\lambda^2 - \rho^2}{\lambda - z} R_\lambda d\lambda.$$

Define $Q = 1 - P$. Plugging this into the previous equality, we find that

$$T - QB_z = -zI - \frac{1}{2\pi i}\int_\gamma \frac{\lambda^2 - \rho^2}{\lambda - z} R_\lambda d\lambda$$

where B_z is a linear operator. Evaluating the right integral with the fact that $|\lambda - z| \geq p - |z|$, we arrive at

$$||T - QB_z|| \leq |z| + 2\sqrt{2} C \rho^2 (\rho - |z|)^{-1}.$$

Since $\dim Q \leq n - 1$, we get that

$$s_n(A) \leq |z| + 2\sqrt{2} C (\rho - z)^{-1}.$$

Letting $\rho \to |\lambda_n(T)|$ and $|z| \to 0$, we arrive at the desired result.

$\square$

Recall that in Hilbert spaces, we can strictly equate the approximation numbers of an operator with its singular values. The above theorem then is a generalization of this result to Banach spaces, where we can only provide a bound. Even though its not a strict equality, we will find it useful later.

Note that without compactness, we cannot find any relationship between $\alpha_n(T)$ and $|\lambda_n(T)|$. An example of this is below.

Example 10. *Take $T \in B(\mathbb{C}^2)$ such that its matrix representation is*

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

As this is a lower triangular matrix, we see that its eigenvalues are $\lambda_1(T) = 2$ and $\lambda_2(T) = 1$. We can further find that the matrix representation of T^*T is

$$\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

with eigenvalues $3 \pm \sqrt{5}$. Since we are on a Hilbert space, we know that $\alpha_n(T) = \lambda_n \left[(T^*T)^{1/2}\right]$. Therefore, $a_1(T) = \sqrt{3+\sqrt{5}} \approx 2.29 > 2 = \lambda_1(T)$, but $a_2(T) = \sqrt{3-\sqrt{5}} \approx 0.87 < 1 = \lambda_2(T)$. Therefore, there is not regular relationship between $\alpha_n(T)$ and $\lambda_n(T)$.

H-operators are also nice because we have some information regarding the conditions for which they can be perfectly approximated by operators of finite rank.

Theorem 24. *If T is an H-operator whose spectrum consists of zero and a sequence of eigenvalues of finite multiplicity which converges to 0, then T is compact and $\alpha_n(T) \to 0$ as $n \to 0$.*

The above theorem restates a fact we proved earlier about compact operators; if an operator can be approximated by a finite rank operator, then it is also compact. The difference now is that its extended to particular types of H-operators.

We will define a new kind of distance similar to the approximation numbers in order to strengthen the bounds found earlier.

Definition 19. *Let L be a Banach space and L_n be a subspace of L for all $n \in \mathbb{N}$. The Kolmogorov distance of an operator $d_n(T)$ is given by*

$$d_n(T) = \inf_{L_n} \sup_{||x|| \leq 1} \rho(Tx, L_n),$$

where $\rho(y, L) = \inf\limits_{z \in L} ||y - z||$.

Theorem 25. *[11] If T is a compact H-operator,*

$$|\lambda_n(T)| \leq 2\sqrt{2}(C+1) d_{n-1}(T)$$

Proof. The statement is clearly true when $|\lambda_n(T)| = 0$, as $d_{n-1}(T) \geq 0$. Therefore, suppose $|\lambda_n(T)| \neq 0$. Let e_n denote the eigenvector corresponding to $\lambda_n(T)$ and E be the space spanned by these eigenvectors. Let $\tilde{T}$ be a contraction of T onto E and let $G = (\tilde{T})^{-1}$. In [11], it is shown that under these conditions, G is also an H-operator with a constant less than or equal to $C + 1$. By Theorem 23, we have that

$$||G|| = \alpha_1(G) \leq 2\sqrt{2}(C+1)|\lambda_1(G)| = 2\sqrt{2}(C+1)|\lambda_n(T)|^{-1}.$$

If $T : X \to Y$, let $A \subset X$ such that $\dim(A) = n - 1$. Since $\dim(E) = n$, by the extension theorem in [10], there exists a non-zero vector $y \in E$ such that $\rho(y, A) = ||y||$. We define $x = Gy$. Therefore, $Tx = y$ and

$$\rho(y, A) = ||y|| = ||G^{-1}x|| \geq ||G||^{-1}||x|| \geq |2\sqrt{2}(C+1)|^{-1}|_n(T)|.$$

Supposing $||x|| = 1$ and the fact that A is arbitrary gives us

$$d_{n-1}(A) \geq (2\sqrt{2}(C+1))^{-1}|\lambda_n(A)|.$$

Multiplying both sides by $2\sqrt{2}(C+1)$ completes the proof.

□

With this new distance metric, we can find even more bounds for our approximation numbers.

Theorem 26. *T is a compact H-operator implies that*

$$d_{n-1}(T) \leq \alpha_n(T) \leq 2\sqrt{2}C|\lambda_n(T)| \leq 8C(C+1)d_{n-1}(T)$$

Proof. In [15], Pietsch shows that $d_n(T) \leq \alpha_{n+1}(T)$ for an operator T. Combining this with Theorems 23 and 25, we get

$$d_{n-1}(T) \leq \alpha_n(T) \leq 2\sqrt{2}C|\lambda_n(T)| \leq 8C(C+1)d_{n-1}(T).$$

□

One important consequence of the above theorem follows:

Corollary 1. *If T is a compact H-operator and $p > 0$, then the convergence of any of the series*

$$\sum_{n=1}^{\infty} |\lambda_n(T)|^p,\ \sum_{n=1}^{\infty} \alpha_n^p(T),\ \sum_{n=1}^{\infty} d_n^p(T)$$

implies the convergence of the others.

Proof. Suppose converges. Then by Theorem 26,

$$\sum_{n=1}^{\infty}(d_{n-1}(T))^p \leq \sum_{n=1}^{\infty} \alpha_n(T)^p \leq 2\sqrt{2}C \sum_{n=1}^{\infty} |\lambda_n(T)|^p < \infty.$$

So the other two converge.

Suppose $\sum_{n=1}^{\infty} |\lambda_n(T)|^p$ converges. Then by Theorem 26, we have that

$$\sum_{n=1}^{\infty} \alpha_n(T)^p \leq 2\sqrt{2}C \sum_{n=1}^{\infty} |\lambda_n(T)|^p \leq 8C(C+1) \sum_{n=1}^{\infty} d_{n-1}(T)^p < \infty.$$

So the other two converge.

Suppose $\sum_{n=1}^{\infty} \alpha_n(T)^p$ converges. Then by Theorem 26,

$$\sum_{n=1}^{\infty} d_{n-1}(T)^p \leq \sum_{n=1}^{\infty} \alpha_n(T)^p < \infty,$$

so $\sum_{n=1}^{\infty} |\lambda_n(T)|^p$ converges. However, we have already showed that if this converges, then $\sum_{n=1}^{\infty} |\lambda_n(T)|^p$ converges. We are done.

$\square$

In other words, the above corollary states that if any of the sequences $\{\alpha_n(T)\}_{n=1}^{\infty}$, $\{d_n(T)\}_{n=1}^{\infty}$, or $\{|\lambda_n(T)|\}_{n=1}^{\infty}$ are in ℓ^p, then they are all in ℓ^p. This result is especially powerful because each of the sums of these sequences as defined in the above corollary define a norm. Therefore, each space must also contain the same elements.

9 Q-Compactness

Over the course of this paper, we have seen how compactness serves as an invaluable tool in our analysis. The purpose of this section is to loosen the conditions on compactness and see which results generalize. However, to figure out how to loosen these conditions, we first have to view a different characterization of compactness. We begin with a definition.

Definition 20. *The convex hull of a set A is defined as*

$$co(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_1, \ldots, x_n \in A, \lambda_1, \ldots, \lambda_n \geq 0, \sum_{i=1}^{\infty} \lambda_i = 1 \right\}.$$

The convex hull of a set typically creates a polyhedra. In the case of our current problem, we will use it to define a new flavor of compactness. Before we get to this definition however, we want to explore a few more properties of the convex hull.

Theorem 27. *The closure of the convex hull, that is $\bar{co}(A)$, is the smallest closed convex set containing A. Moreover, $\bar{co}(A) = co(\bar{A})$.*

Proof. Let B be a closed convex set containing A. Then every convex combination of points in B is still in B. Since $A \subset B$, every convex combination of points of A is in B. However, this is precisely the definition of the convex hull. The limit points of this set must also be in B, as B is closed, so

$$\bar{co}(A) \subseteq B.$$

Thus, $\bar{co}(A)$ is a subset of every closed convex set containing A, so it is the smallest closed convex set containing A.

Consider $x \in \bar{co}(A)$. Note that x is a limit point of elements in $co(A)$ if and only if there exists the following sequence:

$$\lim_{j\to\infty} \left\{ \sum_{i=1}^{n} \lambda_{ij} x_{ij} \right\} \to \sum_{i=1}^{n} \lambda_i x_i.$$

However, then for every i we have a family of sequences $\{x_{ij}\}_{j=1}^{\infty}$ which converges to some $x_i \in \bar{A}$. x is the convex combination of points in $\bar{A}$. Therefore, a point is a limit point in $\bar{co}(A)$ if and only if it is in $co(\bar{A})$.

Now let $x \in \bar{co}(A)$, but not be a limit point. This is true if and only if $x \in co(A)$. Since $A \subset \bar{A}$, it follows that the previous statement is equivalent to saying that $co(A) \subset co(\bar{A})$ and $x \in co(\bar{A})$. Therefore, a point is not a limit point in $\bar{co}(A)$ if and only if it is in $co(\bar{A})$. As all points must either be limit points or not limit points, we have shown $\bar{co}(A) = co(\bar{A})$. $\square$

As one may see above, the convex hull has a multitude of useful properties. One may naturally try to find more. Investigations of the convex hull have led to an equivalent definition of compact sets, as follows:

Theorem 28. *[1] A subset of a Banach space is compact if and only if it is included in the convex hull of a sequence that converges in norm to zero.*

Compactness of sets is now linked to c_0 spaces and convexity. We can extend this definition to work for compact operators as well.

Theorem 29. *[1] An operator $T : X \to Y$ between two Banach spaces is compact if and only if there exists a sequence $\{u_n\}$ of linear functionals in X^* with $||u_n|| \to 0$ such that inequality*

$$||Tx|| \leq \sup_n |\langle u_n, x\rangle|$$

holds for all $x \in X$.

In this paper, we have shown that compact operators and approximation numbers are intrinsically linked. As we have just seen an equivalent definition to compact operators above, one may ask if there are approximation theorems which use it. These approximation theorems will deal with Q-compactness. When we say Q, we mean an *approximation scheme.*

Definition 21. *[1] Let X be a Banach space and $\{Q_n\}$ be a sequence of subsets of X satisfying*

1. $Q_1 \subseteq \cdots \subseteq Q_n \subseteq \cdots \subseteq X$
2. $\lambda Q_n \subset Q_n$ *for all scalars* λ *and* $n = 1, 2, \ldots$
3. $Q_m + Q_n \subset Q_{m+n}$ *for* $m, n = 1, 2, \ldots$

Then $\{Q_n\}$ is an approximation scheme of X.

With this definition of approximation schemes, we continue on to the definition of Q-compactness.

Definition 22. *[1] We say D is Q-compact if*

$$\lim_{n\to\infty} \delta_n(D,Q) = 0,$$

and similarly $T \in \mathcal{L}(Y,X)$ is a Q-compact map if

$$\lim_{n\to\infty} (T,Q) = 0$$

where

$$\delta_n(D,Q) = \inf\{r > 0 : D \subset rU_X + A \text{ for some } A \in Q_n(X)\}.$$

Earlier in this section, we presented an alternative definition for compact sets and compact operators. Using these definitions, we arrive at similar equivalencies for Q-compact sets and Q-compact operators.

Theorem 30. *[1] Let X be a Banach space with an approximation scheme with sets $A_n \in Q_n$ satisfying the condition $|\lambda|A_n \subset A_n$ for $|\lambda| \leq 1$. A bounded set D of X is Q-compact if and only if there is a c_0 sequence $\{x_{n,k}\}_k \subset A_n$ such that*

$$D \subset \left\{ \sum_{n=1}^{\infty} \lambda_n x_{n,k(n)} : x_{n,k(n)} \in (x_{n,k}), \sum_{n=1}^{\infty} |\lambda_n| \leq 1 \right\}.$$

Theorem 31. *[1] Let X and Y be Banach spaces, $T \in \mathcal{L}(X,Y)$, and assume that both T and T^* are both Q-compact maps. Then there exists a sequence $\{u_{n,k}\} \in Q_n$ with $||u_{n,k}|| \to 0$ as $n \to \infty$ uniformly in k, such that the inequality*

$$||Tx|| \leq \sup |\langle u_{n,k(n)}, x \rangle|$$

holds for every $x \in X$. Here Q_n is a class of subsets of X^* with the property that $u_{n,k(n)} \in \{u_{n,k}\}$.

10 Further Questions

We finish this paper by exploring a few open questions. In Approximation theory it is well known that the error of the best approximation is related to the smoothness of functions. The approximation numbers α_n are the error of the best approximation in the space of bounded linear operators. We showed earlier in this paper that compactness is related to the approximation numbers through the measure of non-compactness Γ. Therefore, many open questions center around investigating the relationship between smoothness, compactness, and approximation numbers.

Let A^p_α be an approximation space, that is, the space of sequences of approximation numbers being in a ℓ^p space. We know from Corollary 1 that for compact H-operators, approximation spaces can be defined by sequences of eigenvalues being in ℓ^p spaces. One open question is that can one have inclusion, embedding iteration properties for these spaces? For example, is it true that $A^{p_1}_\alpha \subseteq A^{p_2}_\alpha$ when $p_1 \leq p_2$? In addition to this question, we are also interested in the speed of convergence for these sequences of approximation numbers and eigenvalues.

References

[1] A. G. Aksoy. On a theorem of Terzioğlu. *Turkish Journal of Mathematics*, 43(1):258–267, 2019.

[2] A. Böttcher, A. Chithra, and M. Namboodiri. Approximation of approximation numbers by truncation. *Integral Equations and Operator Theory*, 39(4):387–395, 2001.

[3] R. Chakmak. A formula for eigenvalues on banach and Fréchet spaces. 2018.

[4] K. Deepesh, S. Kulkarni, and M. Nair. Approximation numbers of operators on normed linear spaces. *Integral Equations and operator theory*, 65(4):529, 2009.

[5] G. Godefroy. The Banach space c0. *Extracta Mathematicae*, 16(1):1–25, 2001.

[6] J. Green and F. Valentine. On the Arzela-Ascoli theorem. *Mathematics Magazine*, 34(4):199–202, 1961.

[7] C. Hutton. On the approximation numbers of an operator and its adjoint. *Mathematische Annalen*, 210(4):277–280, 1974.

[8] C. Hutton, J. Morrell, and J. Retherford. Approximation numbers and Kolmogoroff diameters of bounded linear operators. *Bulletin of the American Mathematical Society*, 80(3):462–466, 1974.

[9] R. C. James. A non-reflexive Banach space isometric with its second conjugate space. *Proceedings of the National Academy of Sciences of the United States of America*, 37(3):174, 1951.

[10] M. Krein, M. Krasnoselskii, and D. Milman. On the defect numbers of linear operators in a Banach space and on some geometrical questions (russian), sb. *Tr. Inst. Mat. Akad. Nauk Ukr. SSR*, 11:97–112, 1948.

[11] A. S. Markus. Certain criteria for the completeness of a system of root-vectors of a linear operator in a Banach space. *Matematicheskii Sbornik*, 112(4):526–561, 1966.

[12] A. Martínez-Abejón. An elementary proof of the principle of local reflexivity. *Proceedings of the American Mathematical Society*, 127(5):1397–1398, 1999.

[13] R. Meise and D. Vogt. *Introduction to Functional Analysis.* Clarendon Press, 1997.

[14] R. Meise and D. Vogt. *Introduction to Functional Analysis.* Clarendon Press, 1997.

[15] A. Pietsch. S-numbers of operators in Banach spaces. *Studia Mathematica*, 51:201 – 223, 1974.

[16] E. M. Stein and R. Shakarchi. *Functional analysis: Introduction to Further Topics in Analysis*, volume 4. Princeton University Press, 2011.