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Examining Bias Against Women in Professional Settings through Bifurcation Theory

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Abstract

When it comes to women in professional hierarchies, it is important to recognize the lack of representation at the higher levels. By modeling these situations we hope to draw attention to the issues currently plaguing professional atmospheres. In a paper by Clifton et. al. (2019), they model the fraction of women at any level in a professional hierarchy using the parameters of hiring gender bias and internal homophily on behalf of the applicant. This thesis will focus on a key theory in Clifton et. al.'s analysis and explain its role in the model, specifically bifrucation analysis. In order to analyze the results from Clifton et. al., we give an introduction to bifrucation analysis for the reader. We will then suggest and analyze expansions of Clifton et. al.'s model and discussion of future directions.

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Acknowledgments

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Chapter 1

Introduction

Ever since women entered the work place, we have been climbing the ladders of professional hierarchies. Hierarchies are unavoidable in most professional settings because each institution has a set of professional levels with different requirements. Although each employee has the opportunity to climb through the professional hierarchy, women lack representation in the higher levels (Clifton et al., 2019a, b). The Women's Bureau was established within the United States Department of Labor on June 5, 1920 (Center, 1995). The Bureau's role is to "formulate standards and policies which shall promote the welfare of wage-earning women, improve their working conditions, increase their efficiency, and advance their opportunities for profitable employment" (Women's Buerau, 2021). As of 2021, women make up 8.1% of Fortune 500 Chief Executive Officers (Hinchliffe, 2021), less than 30% of the world's researchers are women (UNESCO, 2021), and 26.7% of Congress is made up of women (Eagleton Institute of Politics, 2021). In 2021, women make up 50.8% of the United States population, yet are still poorly represented in professional and political settings. As positions rise within the hierarchy, the number of women decreases. This phenomenon is called the "leaky pipeline" effect (Shaw and Stanton, 2012). Over the years many have tried to explain the reason for this effect using reasons including inherent differences between the sexes, difference career/life goals, familial obligation, etc (Shaw and Stanton, 2012). It is imperative that we begin to recognize the lack of representation for women in professional settings and initiate change. By initiating this change we will grow our workplaces into more efficient and diverse communities that celebrate talent. By pointing out the gender bias in professional settings we allow for previously unconsidered talent to enter and enhance the company/firm. In order to understand the reasons and

affects of the “leaky pipeline" effect, we must determine how women move and rise within a company and illustrate the bias behind the cause.
To do this we will explore the mathematical model and analysis from "Mathematical Model of Gender Bias and Homophily in Professional Hierarchies" written by Sarah Clifton et. al Clifton et al. (2019b) Clifton et al. (2019a). Clifton et. al. cites many references on women leaders and the role that women pay in different professional hierarchies (Clifton et al., 2019b) (Clifton et al., 2019a). Clifton et. al. refers to the "one-third hypothesis" (Srikantan, 1968) as a basis for the homophilic theory included in her paper. The "one-third hypothesis" suggests that an individual (male or female) feels most comfortable in a group setting when at least 30% of the group is similar to them in demographic. Additionally, Clifton et. al. uses bifurcation theory in order to understand the behavior of the professional hierarchies shown in the graphs. (Data fitting and dervies conclusions)
In Chapter 2, we provided an introduction and explanation of bifurcation theory. Then in Chapter 3 we will analyze Clifton et. al. (Clifton et al., 2019a) (Clifton et al., 2019b) in order to understand how bifurcation theory applies to each of the models discussed in the paper. Lastly in Chapter 4 we will then explore some preliminary expansions to Clifton et. al.'s model, and discuss future directions.

Chapter 2

Bifurcation Analysis for Model Background

A dynamical system of ordinary differential equations display some specific behavior. Consider the following ODE taken from (Strogatz, 2018)

$$\frac{dx}{dt} = f(x;\mu) \tag{2.1}$$

which is dependent on the parameter value $\mu \in \mathbb{R}$. If some parameter value of the system changes, the behavior of the system may change. We care to investigate this because we want to see which parameters affect the system in different ways. The transition point that marks the change of the behavior is called the bifurcation point. The stability of the equilibria points are used to determine the qualitative behavior of one-dimensional continuous dynamical systems, and therefore all bifurcations are associated with the bifurcation of the equilibria. The equilibrium bifurcation point is labeled as (x_0, μ_0) and specified using the critical value of μ at which the transition occurs.

Below we give a brief overview with the aim to understand the analysis in (Clifton et al., 2019a), for a more in depth study of bifurcation analysis we direct the reader to (Strogatz, 2018).

2.1 One-Dimensional Bifurcations

We will focus on the following one-dimensional equilibrium bifurcations in section 2.1 as described by their ODEs (Strogatz, 2018):

$$\text{Saddle-node:} \quad \frac{dx}{dt} = \mu - x^2 \tag{2.2}$$

$$\text{Pitchfork:} \quad \frac{dx}{dt} = \mu x - x^3 \tag{2.3}$$

2.1.1 Saddle-node Bifurcations

Consider the example ODE taken from (Strogatz, 2018)

$$\frac{dx}{dt} = x^2 + \mu. \tag{2.4}$$

First looking at $\mu > 0$ we see that there is no real solution, since there is no value of μ that would allow the above equation (2.4) to equal 0, and therefore no equilibria:

$$\begin{aligned} \frac{dx}{dt} &= 0 = x^2 + \mu \\ x^2 &= -\mu \\ x &= \pm\sqrt{-\mu}. \end{aligned} \tag{2.5}$$

Next, we look at $\mu = 0$:

$$\begin{aligned} \frac{dx}{dt} &= 0 = x^2 + \mu \\ x^2 &= 0 \end{aligned} \tag{2.6}$$

Thus, when $\mu = 0$ there is one equilibrium point, $x = 0$. This fixed point is a half-stable equilibrium point, since this equilibrium point is where the ODE transforms from stable to unstable.
Lastly, we look at when $\mu < 0$ and we see that there are two equilibrium points:

$$\begin{aligned} \frac{dx}{dt} &= 0 = x^2 + \mu \\ x^2 + \mu &= 0 \\ x^2 &= -\mu \\ x &= \pm\sqrt{-\mu} \end{aligned} \tag{2.7}$$

We can identify the stability of these equilibrium points using the stability theorem (Hale, 1963):

$$\text{if } f'(x^*) < 0 \text{ then the equilibrium } x(t) = x^* \text{ is stable} \tag{2.8}$$

$$\text{if } f'(x^*) > 0 \text{ then the equilibrium} x(t) = x^* \text{ is unstable}$$

Thus, the equilibrium point at $\sqrt{-\mu}$ is unstable and the equilibrium point at $-\sqrt{-\mu}$ is stable. This bifurcation is called a saddle-node bifurcation. The two equilibrium points converge at the bifurcation point, combine and disappear.

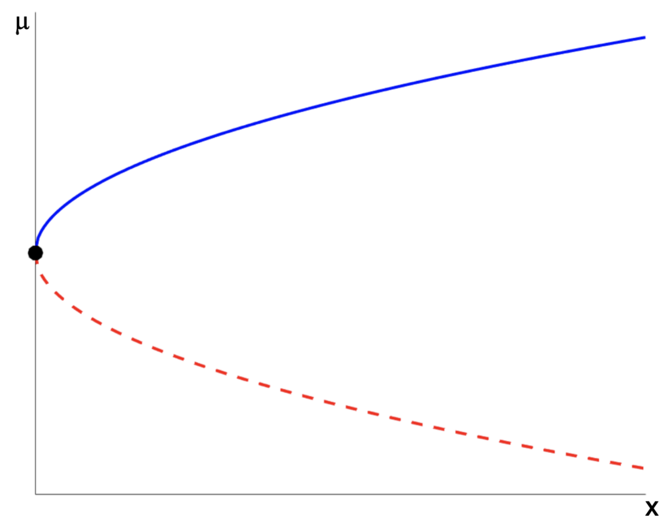

Figure 2.1 An example of a saddle-node bifurcation where the graph changes from stable to unstable at the shown equilibrium point. The dashed line is unstable while the solid line is stable.

Within the category of saddle-node bifurcations, there are subcritical saddle-node bifurcations and supercritical saddle-node bifurcations. We define a subcritical bifurcation as a saddle-node bifurcation point where the equilibria values exist below the bifurcation point. A supercritical saddle-node bifurcation point is defined as a saddle-node bifurcation where the equilibria values exist above the bifurcation point. In the above example [2.4], the bifurcation point is at $(0, 0)$, and because we have our saddle-node bifurcation when $\mu < 0$ we label this bifurcation as a subcritical saddle-node bifurcation. The name saddle-node comes from the two-dimensional dynamical system in which two equilibria, a saddle point and a node, collide and cancel each other out. In a one-dimensional phase plane, the saddle point equilibria will be the stable point, and the node equilibria will be the unstable point.

2.1.2 Pitchfork Bifurcation

Consider the example ODE taken from (Strogatz, 2018):

$$\frac{dx}{dt} = \mu x - x^3 \tag{2.9}$$

First considering the ODE when $\mu > 0$:

$$\begin{gathered}
\frac{dx}{dt} = 0 = \mu x - x^3 \\
\mu x - x^3 = 0 \\
x(\mu - x^2) = 0 \\
x = 0 \quad \& \quad \mu - x^2 = 0 \\
x = 0 \quad \& \quad \mu = x^2 \\
x = 0 \quad \& \quad x = \pm\sqrt{\mu}
\end{gathered} \tag{2.10}$$

When $\mu > 0$, there are three fixed points at $x = 0$ and $x = \pm\sqrt{\mu}$. Using the stability theorem 2.8 we analyze the stability of these fixed points. The fixed points at $x = \pm\sqrt{\mu}$ are stable equilibria, but the fixed point at $x = 0$ is an unstable equilibrium. Thus, when $\mu > 0$ there are two stable equilibria.
Next, let us examine the ODE when $\mu \leq 0$:

$$\begin{gathered}
\frac{dx}{dt} = 0 = \mu x - x^3 \\
\mu x - x^3 = 0 \\
x(\mu - x^2) = 0 \\
x = 0 \quad \& \quad \mu - x^2 = 0 \\
x = 0 \quad \& \quad \mu = x^2 \\
x = 0 \quad \& \quad x = \pm\sqrt{\mu}
\end{gathered} \tag{2.11}$$

Thus, when $\mu \leq 0$ there is only one stable equilibrium point at $x = 0$. This fixed point is a stable equilibrium point that bifurcates into the two stable fixed points when $\mu > 0$.

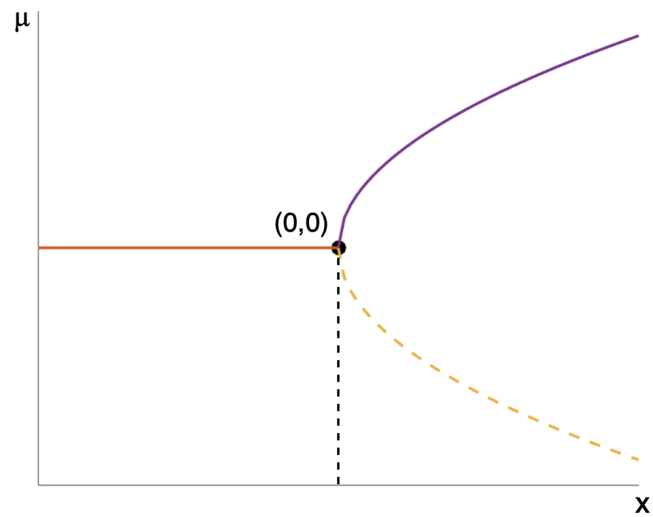

Figure 2.2 An example of a pitchfork bifurcation where a stable graph splits into a saddle-node bifurcation at the shown equilibrium point. The solid orange line represents the stable part of the ODE prior to the pitchfork bifurcation. The point represents the origin, where the bifurcation happens and spits into the stable (solid line) and the unstable (dashed).

This bifurcation from one stable fixed point to two stable fixed points is what creates the "pitchfork" shape of the bifurcation. We now define a pitchfork bifurcation as the division of one stable branch into two stable branches within a solution to an ODE.

2.2 Two-Dimensional Bifurcations

Now that we have described one-dimensional equilibrium bifurcations, we will look at two dimensional bifurcations. Two dimensional bifurcations occur in a planar system (Strogatz, 2018) in contrast to one-dimensional bifurcations seen in section 2.1.

Consider

$$\frac{dx}{dt} = f(x;\mu) \tag{2.12}$$
$$\frac{dy}{dt} = g(y;\mu)$$

where $\mu \in \mathbb{R}$ is a parameter value.
In this section we will analyze two different two-dimensional bifurcations seen in Clifton et. al. (Clifton et al., 2019b, a).

2.2.1 Hopf Bifurcation

A Hopf Bifurcation occurs when a limit cycle that surrounds an equilibrium point appears or disappears as a parameter value, μ varies. Consider the example system of ODEs taken from (Munõz Alicae, 2011):

$$\frac{dx}{dt} = x(\mu - x^2) \tag{2.13}$$
$$\frac{dy}{dt} = -1$$

The origin, $(0, 0)$ is the only fixed point of this system. First we look at the system when $\mu > 0$:

$$\frac{dx}{dt} = x(\mu - x^2)$$

When $\mu > 0$ we see that $\frac{dx}{dt} < 0$ for $x \in (\sqrt{\mu}, \infty)$ and $\frac{dx}{dt} > 0$ for $x \in (0, \sqrt{\mu})$. Thus, using the stability theorem 2.8, we can conclude that the origin is unstable and there is a stable orbit $x = \sqrt{\mu}$. An orbit is formed here because the solution is defined on an interval rather than a single point.
Looking at the equation above, we see that if $\mu = 0$, then:

$$\frac{dx}{dt} = -x^3$$

For any non-zero x, $\frac{dx}{dt} < 0$ which means there are no closed orbits, and our trajectories will approach the origin as $t \longrightarrow \infty$. We know there are no closed orbits because there doesn't exist any point x where $\frac{dx}{dt} = t$ which means there is no closed point of the orbit. From this we can conclude there are no closed orbits.

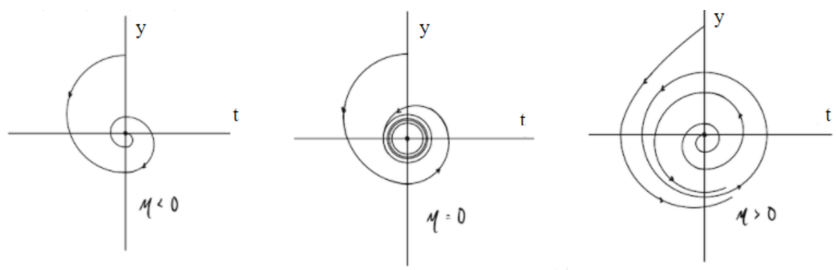

Figure 2.3 An example of a supercritical Hopf bifurcation recreated from figures in (A. Kuznetsov, 2006)

Lastly, we will consider $\mu < 0$:

$$\frac{dx}{dt} = x(\mu - x^2)$$

We see that $\mu - x^2 < 0$ for all x, and therefore, as we saw above when $\mu = 0$, there are no closed orbits. The origin is a stable focus of the system because the system orbits around the origin. The last thing we must consider for our Hopf bifurcation is the meaning of the second part of our system of ODEs (2.13): $\frac{dy}{dt} = -1$. Because $\frac{dy}{dt} < 0$ and is negative, we can say that all trajectories of our bifurcation will move clockwise around the origin.

We can further define Hopf bifurcations by exploring supercritical and subcritical Hopf bifurcations. A supercritical Hopf bifurcation occurs when a stable spiral changes into an unstable spiral that is surrounded by a small limit cycle.

The above example 2.2.1 is classified as a supercritical Hopf Bifurcation because when $\mu < 0$, the system is stable and when μ grows and $\mu > 0$, the system becomes unstable.

In contrast to the supercritical bifurcation, a subcritical Hopf bifurcation the system is destabilizing and the limit cycles oscillate away from the origin. Here we see that as μ increases, the unstable cycle shrinks towards the fixed point.

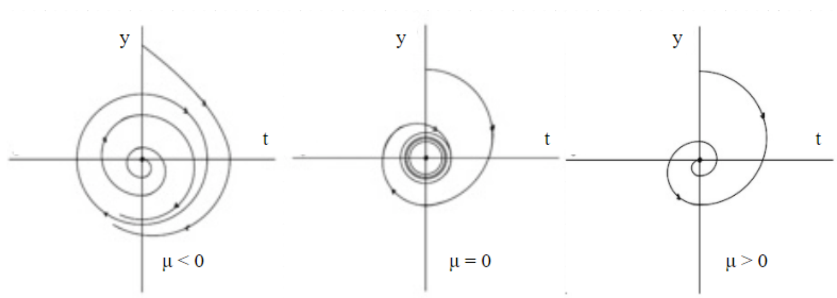

Figure 2.4 An example of a subcritical Hopf bifurcation recreated from figures in (A. Kuznetsov, 2006)

The subcritical Hopf bifurcation occurs at $\mu = 0$ because that is where the unstable cycle has zero amplitude and surrounds the fixed point, making it unstable.

2.2.2 Homoclinic Bifurcation

This is a brief introduction which we will use in Section 3.8 with (Clifton et al., 2019a)(Clifton et al., 2019b). We saw in 2.2.1 with Hopf bifurcations a limit cycle that moves towards a fixed point of the system. Homoclinic bifurcations are similar, except here a limit cycle moves towards a saddle point. Consider the system example from (Strogatz, 2018):

$$\frac{dx}{dt} = y \tag{2.14}$$
$$\frac{dy}{dt} = \mu y + x - x^2 + xy$$

We say that the bifurcation occurs when $\mu = \mu_b$. The value μ_b refers to the bifurcation point. Looking at when $\mu < \mu_b$, we see a stable limit cycle that passes close to a saddle point at the origin. As μ increases towards μ_b the limit cycle grows and pushes towards the saddle, which is what creates the homoclinic orbit. Once $\mu > \mu_b$, the saddle connection breaks and limit cycle disappears. Different than the other bifurcations we discussed, the important characteristic of homoclinic bifurcations is the unstable saddle. This is because as a homoclinic bifurcation occurs, the orbit grows and intersects the saddle and overwhelms it. We focus on the unstable saddle because it is how we can spot that a homoclinic bifurcation occurs.

Chapter 3

Analysis of (Clifton et al., 2019a)

In this section we will examine the paper by Clifton et. al., reproducing model construction and analysis as well as including additional explanations (Clifton et al., 2019a, b). Note that all figures included from (Clifton et al., 2019b) are used with permission from Prof. Sara Clifton.

3.1 Introduction to Paper's Background

Women are poorly represented in many professional disciplines, but there is an apparent lack of representation in professional hierarchies. The paper by Clifton et. al. defines a professional hierarchy as "a field in which an employee enters at a designated low level and gradually moves up the ranks" (Clifton et al., 2019a). The "leaky pipeline" effect has been around for decades and many qualitative theories have been proposed to explain it (Shaw and Stanton, 2012). These qualitative theories, however, are contingent on the assumption that men and women make different decisions due to either biological differences or social persuasion. While there have been attempts at creating quantitative models that explain the leaky pipeline, these models assume logistic growth and eventual gender parity which is unrealistic (Shaw and Stanton, 2012). The Clifton et. al. paper aims to model how both gender bias and homophilic tendencies affect the progression of women within professional hierarchies (Clifton et al., 2019a).
Studies have shown that, taking into account work ethic and attrition, women rise slower in professional hierarchies than men do (Kumra and

Vinnicombe, 2008). From this we can reasonably deduce that gender is a notable factor during the hiring process, and therefore gender bias exists within professional hierarchies. Gender bias is defined as "all conscious or unconscious decisions made by the employer during the hiring process that are affected by the gender of the applicant" (Clifton et al., 2019a). We also define homophily to be the tendency for people to look for and be drawn towards those who are similar to themselves. This means that when an applicant is deciding whether or not to apply for a promotion, they will consider the demographics of the level above them and judge if they "belong" at that higher level. With the understanding that both the applicant and the employer play a role in promoting an applicant within the hierarchy, the following model is proposed.

3.2 Model Set-Up

We understand that the applicant and the employer affect the distribution of promotions through homophily and gender bias, respectively. The Clifton et. al. paper assumes that gendered hiring bias is constant for all levels within the hierarchy (Clifton et al., 2019a). This means that an employer will reduce or improve female applicant's chance of promotion uniformly at all levels of the hierarchy. We also assume that hierarchies remain consistent over time, meaning promotional requirements remain constant and that each individual must rise linearly through the hierarchy.

An important assumption made in the paper is that gender bias and homophily are constant over time and within the hierarchy (Clifton et al., 2019a). Although it is unrealistic to assume that gender bias and homophily do not vary from lower levels to higher levels, this assumption is necessary in order to avoid overfitting the model.

Lastly, the model ignores the fact that women and men might make different decisions (Clifton et al., 2019a). They assume that women and men are equally biased towards a certain gender during the hiring process and that women and men are equally qualified for each position.

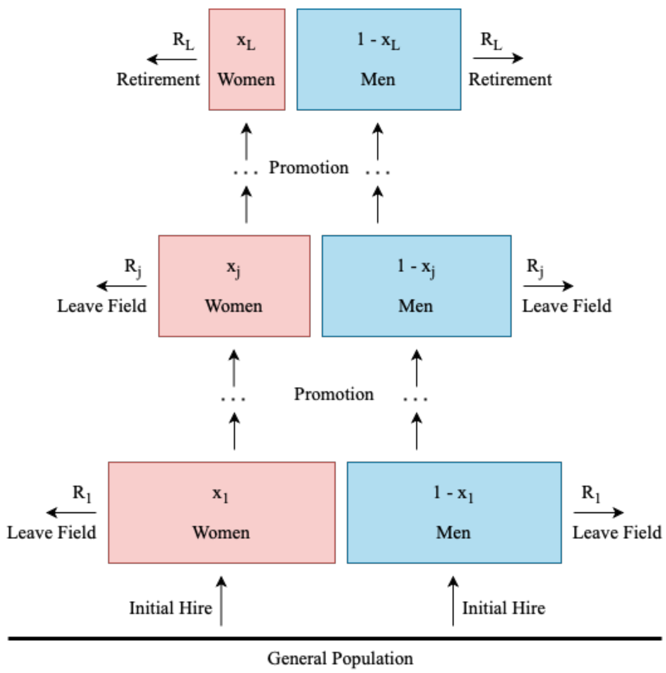

Figure 3.1 This chart visualizes a professional hierarchy and how both gender bias and homophily will affect the promotional opportunities of both genders. Recreated from Figure 2 in (Clifton et al., 2019b)

3.3 Example

Let us consider the promotional process within a financial institution. For the purpose of this example we assume that the institution has only 2 hierarchical levels. If the lower level is 30% women and gender is not a factor in deciding eligibility for promotion, then 30% of women are eligible for promotion. But, if the higher level has less women, then these 30% may be less likely to apply for the promotion due to the lack of similar representation above them (Clifton et al., 2019b).

If these instincts are present, then we can say that the men in the lower level are more likely to apply for a promotion. Say they are twice as likely to apply for a promotion to the higher level. Then, the applicant pool will reduce to only 15% women. If again we consider that gender is not a factor in promotion, then 15% of women will be granted a promotion. If we now include gender bias as a factor in determining promotion, then the percentage of women will shrink further from 15%. As seen in Figure 3.1, we can expand this process to multiple hierarchical levels within an institution (Clifton et al., 2019b).

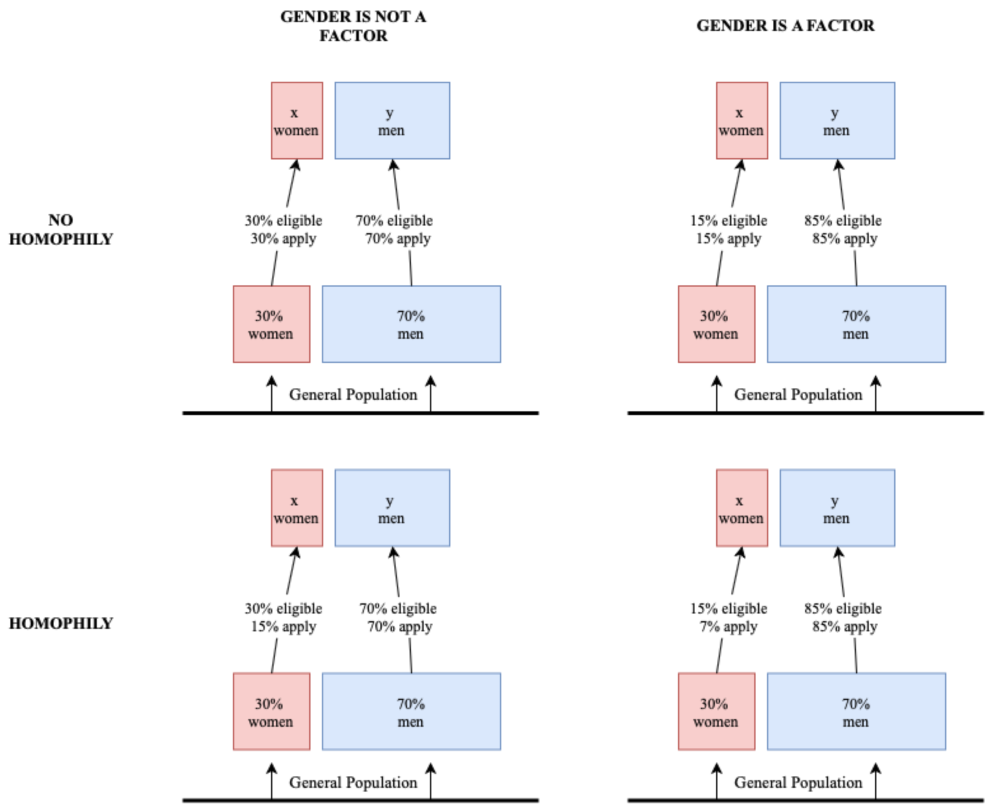

Figure 3.2 Visual representation of the example in Section 3.3. The chart depicts all four comobination models discussed in Clifton et. al. using percentage rates used in the example 3.3 Clifton et al. (2019b, a)

3.4 Derivation

Let us assume that the probability of seeking a promotion, $P(u, v)$, is a function of the fraction of people at the upper level who share the applicant's gender, u, and the fraction of like-gendered people in the applicant's current level, v. The model assumes that there exists a flexible threshold of each individual's comfort at their current level. Including all of these factors into the probability of seeking a promotion, we get the following function

(Clifton et al., 2019b, a):

$$P(u,v) = \frac{1}{1+e^{-\lambda(u-v)}} \tag{3.1}$$

where λ is the strength of an individual's homophilic tendencies. This equation makes sense because the probability of an individual seeking a promotion should depend on both the applicant's similarities with those above them as well as with those in there current level. This would be information that a rational applicant would consider prior to applying for a promotion. This probability function does not include gender specification, therefore we can say that this applies to both women and men. Looking at our example in Section 3.3 the number of people who apply for a promotion depended on homophily which is what is being represented in this function. From here we can calculate the fraction of women seeking a promotion at each level by label u as the fraction of women in the higher level and v is the fraction of women in the current level. Using these variables, we calculate (Clifton et al., 2019b, a):

$$f_0 = (u,v) = \frac{vP(u,v)}{vP(u,v)+(1-v)P(1-u,1-v)} \tag{3.2}$$

This function (3.2) fails to include outside bias from the hiring employers. Therefore we incorporate the constant b into our function as the fraction of women promoted if the applicant pool has an equal number of women and men (Clifton et al., 2019b, a):

$$f_0 = (u,v;b) = \frac{bvP(u,v)}{bvP(u,v)+(1-b)(1-v)P(1-u,1-v)} \tag{3.3}$$

Note that if bias that exceeds $\frac{1}{2}$, $b > \frac{1}{2}$, indicates that women are favored disproportionately to men, and a bias less than $\frac{1}{2}$, $b < \frac{1}{2}$, indicates that men are favored disproportionately to women (Clifton et al., 2019a, b).
From here we can say that the change in the number of women at each level, $x_j N_j$ as seen in Figure 3.1:

$$\underbrace{\frac{d}{dt}(x_L N_L)}_{\text{the change in the number of women at each level}} = \overbrace{R_L N_L f(x_L, X_{L-1}; b)}^{\text{the number of women promoted}} - \underbrace{R_L N_L x_L}_{\text{the number of women retiring}} \tag{3.4}$$

This equation represents the change in the number of women at the j^{th} level and is dependent solely on the women promoted from the lower level and the women who leave the j^{th} level to retire. This makes sense because the number of jobs available at any given level is dependent on the number of people who leave that level, which is described in this equation.

$$\frac{d}{dt}(x_j N_j) = \left(\sum_{k=j}^{L} R_k N_k\right) f(x_j, x_{j-1}; b) - R_j N_j x_j - \left(\sum_{k=j+1}^{L} R_k N_k\right) f(x_{j+1}, x_j; b)$$

$$\text{where } 1 < j < L$$

This equation is a generalized version of the previous equation (3.4). This equation describes the number of jobs available at any given level and therefore takes into account the number of people who are promoted into that level, the number of people who retired out, and then the people who leave the level to be promoted into the next level. The resulting constant is a representation of the number of jobs available at that level.

$$\frac{d}{dt}(x_1 N_1) = \left(\sum_{k=1}^{L} R_k N_k\right) f(x_1, \frac{1}{2}; b) - R_1 N_1 x_1 - \left(\sum_{k=2}^{L} R_k N_k\right) f(x_2, x_1; b)$$

Next, we normalize the system by dividing each equation by the number of people at the j^{th} level who retire or leave the level, $R_j N_j$. We do this so that our previous set of equations (3.4) will all be re-scaled to have a common scale. In order to do this we dividing by a norm, which is the number of people at the j^{th} level who retire or leave the level, $R_j N_j$:

$$\frac{1}{R_L}\frac{dx_L}{dt} = \overbrace{f(x_L, x_{L-1}; b)}^{\text{promoted from lower level j}} - \overbrace{x_L}^{\text{retire out of level L}} \tag{3.5}$$

$$\frac{1}{R_j}\frac{dx_j}{dt} = \underbrace{(1+r_j) f(x_j, x_{j-1}; b)}_{\text{promoted into level } j \text{ from level } j-1} - \underbrace{x_j}_{\text{leave level } j} - \underbrace{r_j f(x_{j+1}, x_j; b)}_{\text{promoted to level } j+1 \text{ from level } j} \tag{3.6}$$

$$\text{where } 1 < j < L$$

$$\frac{1}{R_1}\frac{dx_1}{dt} = \underbrace{(1+r_1) f(x_1, \frac{1}{2}; b)}_{\text{hired from the general pool into level 1}} - \underbrace{x_1}_{\text{leave field}} - \underbrace{r_1 f(x_2, \frac{1}{2}; b)}_{\text{promoted to next level}} \tag{3.7}$$

This system can be condensed if we take $r_L = 0$ and $x_0 = \frac{1}{2}$ (Clifton et al., 2019b, a). We refer to Table 1 for descriptions of the model parameters.

Variable	Meaning
x_j	Fraction of people in the j^{th} level who are women
L	Number of levels in the hierarchy
R_j	Retirement/leave rate at the j^{th} level
N_j	Number of people in the j^{th} level
r_j	Ratio of the total retiring people above the j^{th} level to the retiring people in the j^{th} level
$P(\cdot)$	Likelihood of seeking promotion
$f(\cdot)$	Fraction of people promoted to next level who are women
b	Bias toward or against women ($b = \frac{1}{2}$ is no bias)
λ	Strength of homophilic tendency

Table 3.1 Model variables and parameters. Recreated from Table 1 in (Clifton et al., 2019b)

We refer to the normalized set of equations for the various models we will explore in the rest of this chapter.

3.5 Null Model

First, let us consider the Null model (Clifton et al., 2019b, a) with no hiring bias ($b = \frac{1}{2}$) and no homophily ($\lambda = 0$). Therefore a woman working at a company with this model would have an equal chance of being promoted as a man as well as no internal bias influencing her decision to apply for said promotion. This is called the Null model because it associated with the "zero-value" of each parameter. We start by calculating the probability of an applicant seeking a promotion using (3.1):

$$
\begin{aligned}
P(u,v) &= \frac{1}{1+e^{-\lambda(u-v)}} \\
&= \frac{1}{1+e^{-(0)(u-v)}} \\
&= \frac{1}{1+e^{0}} \\
P(u,v) &= \frac{1}{2}
\end{aligned}
$$

Therefore, we have $P(u,v)$ as a constant, specifically $\frac{1}{2}$ which means that the likelihood of seeking a promotion is equal for both women and men. Next, we can use our calculated value of $P(u,v)$ to determine the fraction of women who are promoted to the next level:

$$
\begin{aligned}
f(u,v;b) &= \frac{bvP(u,v)}{bvP(u,v)+(1-b)(1-v)P(1-u,1-v)} \\
&= \frac{(\frac{1}{2})(v)(\frac{1}{2})}{(\frac{1}{2})(v)(\frac{1}{2})+(\frac{1}{2})(1-v)(\frac{1}{2})} \\
&= \frac{\frac{1}{4}v}{\frac{1}{4}v+\frac{1}{4}(1-v)} \\
f(u,v;b) &= v
\end{aligned}
$$

This means that that the fraction of women promoted to each level is entirely dependent on the fraction of people in the applicant's current level that are like-gendered. This makes sense for the Null model because it is already assumed that the applicant does not consider the fraction of like-gendered people in the level above them.
Lastly, we want to look at the change in the number of women at the j^{th} level. To do this we take our values for $P(u,v)$ and $f(u,v;b)$ and apply them to our model (3.4).

$$
\begin{aligned}
\frac{1}{R_j}\frac{dx_j}{dt} &= (1+r_j)f(x_j,x_{j-1};b) - x_j - r_j f(x_{j+1},x_j;b) \\
\frac{1}{R_j}\frac{dx_j}{dt} &= (1+r_j)x_{j-1} - x_j - r_j x_j
\end{aligned}
\tag{3.8}
$$

We want to understand the different components of this model:

$$\underbrace{\frac{1}{R_j}\frac{dx_j}{dt}}_{\text{Number of women hired from the } (j-1)^{th} \text{ level}} = \underbrace{(1+r_j)x_{j-1}}_{\text{Number of women leaving the field}} - \underbrace{x_j - r_j x_j}_{\text{Number of women promoted to the } j^{th} \text{ level}}$$

The only steady state of this model is $\{x_j^*\} = \{\frac{1}{2}\}$ because that is where the model will converge to over time. Without gender hiring bias and homophily, every level of the hierarchy will converge towards having equal gender representation since applicants won't consider the fraction of like-gendered individuals and gender won't be considered in hiring.

3.6 Homophily-Free Model

Next, we consider the Homophily free model which has no internal bias ($\lambda = 0$). This model includes outside hiring bias which means that $b \neq \frac{1}{2}$. In order to explore this model we start by determining the probability of an applicant applying for a promotion:

$$\begin{aligned} P(u,v) &= \frac{1}{1+e^{-\lambda(u-v)}} \\ &= \frac{1}{1+e^{-(0)(u-v)}} \\ &= \frac{1}{1+e^{0}} \\ P(u,v) &= \frac{1}{2} \end{aligned}$$

As we saw in the Null model (3.5), the probability is a constant. Using this value we can calculate the fraction of women who are promoted to the next level:

$$
\begin{aligned}
f(u,v;b) &= \frac{bvP(u,v)}{bvP(u,v)+(1-b)(1-v)P(1-u,1-v)} \\
&= \frac{(b)(v)(\frac{1}{2})}{(b)(v)(\frac{1}{2})+(1-b)(1-v)(\frac{1}{2})} \\
f(u,v;b) &= \frac{bv}{bv+(1-b)(1-v)}
\end{aligned}
$$

The model reduces to:

$$
\frac{1}{R_j}\frac{dx_j}{dt} = (1+r_j)\frac{bx_{j-1}}{bx_{j-1}+(1-b)(1-x_{j-1}}) - x_j - r_j\frac{bx_j}{bx_j+(1-b)(1-x_j)} \tag{3.9}
$$

with the following breakdown of the components:

$$
\begin{aligned}
\frac{1}{R_j}\frac{dx_j}{dt} = \quad & \overbrace{(1+r_j)\frac{bx_{j-1}}{bx_{j-1}+(1-b)(1-x_{j-1}}}^{\text{number of women hired from the } (j-1)^{th} \text{ level with bias included}} \quad - \quad \overbrace{x_j}^{\text{number of women leaving the field}} \\
- \quad & \underbrace{r_j\frac{bx_j}{bx_j+(1-b)(1-x_j)}}_{\text{number of women promoted to the } j^t h \text{ level with bias included}}
\end{aligned}
$$

As we saw in the Null model (3.5), the Homophily-free model has a single fixed point (Clifton et al., 2019b, a) which tells us that the model converges to a single point. In this model we see that the inclusion of hiring gender bias moves the gender away from gender parity. For a woman working in a company where bias is included, she will have a harder time being eligible for promotion, which will result in less women promoted. This will ultimately lead to a biased work place in favor of men.

3.7 Bias-Free Model

After considering a Homophily-free model Section 3.6 we now want to consider a model where people self-segregate based off of gender, but an employer has no hiring bias ($b = \frac{1}{2}$).
First we want to derive the probability function we will use in this model:

$$
P(u,v) = \frac{1}{1+e^{-\lambda(u-v)}}
$$

In the previous models we saw that the probability was constant; however, in this model, the probability is not constant since $\lambda \neq 0$. This means that the probability of a woman seeking a promotion in a unbiased environment is no longer constant. This means that the probability of a woman applying will depend on the fraction of like-gendered individuals in levels above her. This fraction changes depending on the individual and the level which is what causes the probability to not be constant. This is most likely due to the fact that there is wavering homophily within each individual. We now want to calculate the fraction of women promoted to the next level:

$$
\begin{aligned}
f(u,v;b) &= \frac{bvP(u,v)}{bvP(u,v)+(1-b)(1-v)P(1-u,1-v)} \\
&= \frac{(\frac{1}{2})(v)P(u,v)}{(\frac{1}{2})(v)P(u,v)+(\frac{1}{2})(1-v)P(u,v)} \\
f(u,v;b) &= \frac{vP(u,v)}{vP(u,v)+(1-v)P(u,v)}
\end{aligned}
$$

From this, the model reduces to:

$$
\frac{1}{R_j}\frac{dx_j}{dt} = (1+r_j)\frac{vP(x_j,x_{j-1})}{x_{j-1}P(x_j,x_{j-1})+(1-x_{j-1})P(x_j,x_{j-1})} - x_j - r_j\frac{x_jP(x_{j+1},x_j)}{x_jP(x_{j+1},x_j)+(1-x_j)P(x_{j+1},x_j)}
$$

$$
\frac{1}{R_j}\frac{dx_j}{dt} = (1+r_j)f(x_j,x_{j-1};b) - x_j - r_jf(x_{j+1},x_j;b) \tag{3.10}
$$

$$
\text{for} 1 \leq j \leq L
$$

As we alter the homophily (λ) we see three different models with different qualitative behaviors. The paper describes mild homophily as $\lambda = 2$, moderate homophily as $\lambda = 3$, strong homophily as $\lambda = 4.5$, and finally strongest homophily $\lambda = 5$ (Clifton et al., 2019b, a). We will investigate each of these homophilic tendencies in order to better understand the bifurcation that occurs. First let us evaluate the fraction of women at the j^{th} level for each of the homophily levels listed. Note that Figures 3.3, 3.4, 3.5, and 3.6 are replications from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

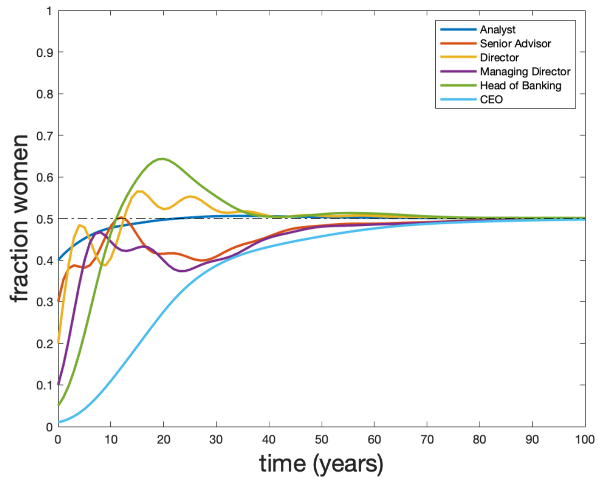

Figure 3.3 We see that when mild homophily ($\lambda = 2$) exists, all levels of the hierarchy converge towards gender equity after oscillating above and below gender equity for some time. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

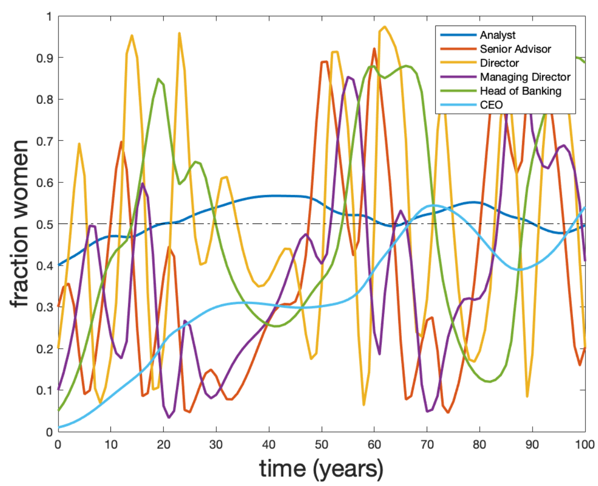

Figure 3.4 We see for moderate homophily ($\lambda = 3$), the fraction of women oscillate about gender equity except for the lowest level which rises towards gender equity. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

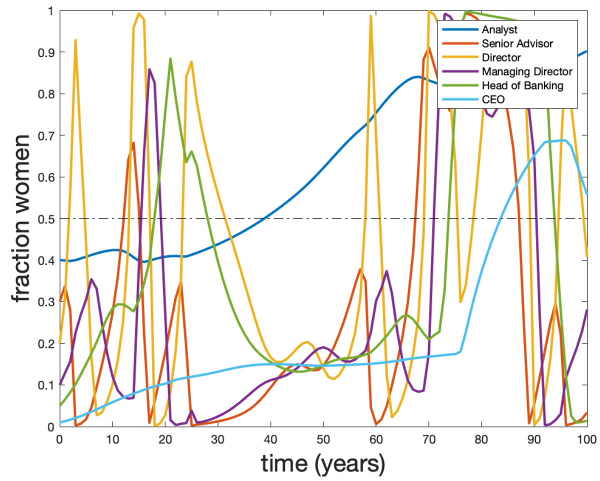

Figure 3.5 For strong homophily ($\lambda = 4.5$), we see a preference towards men until an individual reaches the top level and then the fraction of women continues to rise. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

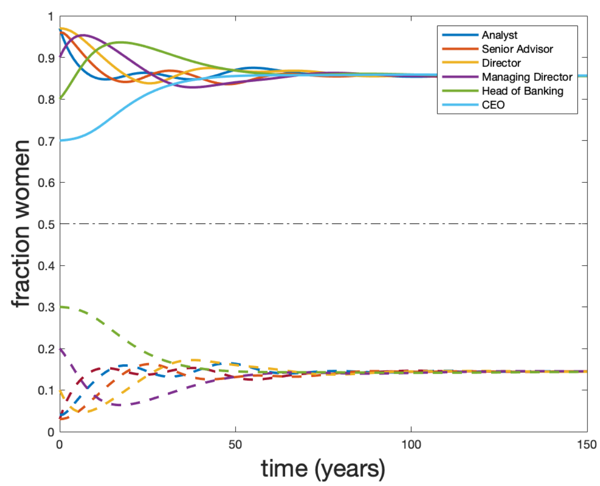

Figure 3.6 For the strongest homophily ($\lambda = 5$), we look at the tendencies of both men and women. The dotted lines represent the behavior of men and the solid lines represent the behavior of women. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

Notice how as the amount of homophily increases, the fraction of women moves from stable equilibrium about gender equity to an unstable oscillation around gender equity and then settles with strong preference toward either gender (Clifton et al., 2019b, a). For a woman at a company with this model the fraction of women and men will eventually trend towards favoring one or the other. This is what creates our bifurcation. We can see this trends put together in Figure 3.7 from (Clifton et al., 2019b). We can see that as homophily increases from $\lambda = 3$ to $\lambda = 4$ that the oscillations become unstable which is why our Hopf bifurcation happens around $\lambda = 3.4$. We interpret this to mean that when homophily is below 4, the fraction of women at a given level is estimated to be $\frac{1}{2}$. This leads us to believe that any homophily under 4 is not strong enough to deter a woman for applying for a promotion which would explain the gender equity seen. We then see that as homophily grows further to $\lambda = 4.5$, the oscillations create a limit cycle and a pitchfork bifurcation happens. We interpret this to mean that homophily becomes strong enough to influence an applicant's decision when homophily is greater than 4. The pitchfork bifurcation can be interpreted as an applicant's non-constant decision to apply for a promotion. The model bifurcates because if a the $(j+1)^{th}$ level has a large fraction of women, then the applicant will apply. But, if the $(j+1)^{th}$ level has a small fraction of women then the applicant will be deterred from applying. Finally when $\lambda = 5$ we see that the two genders become stable again which is shown in Figure 3.7 from (Clifton et al., 2019b).

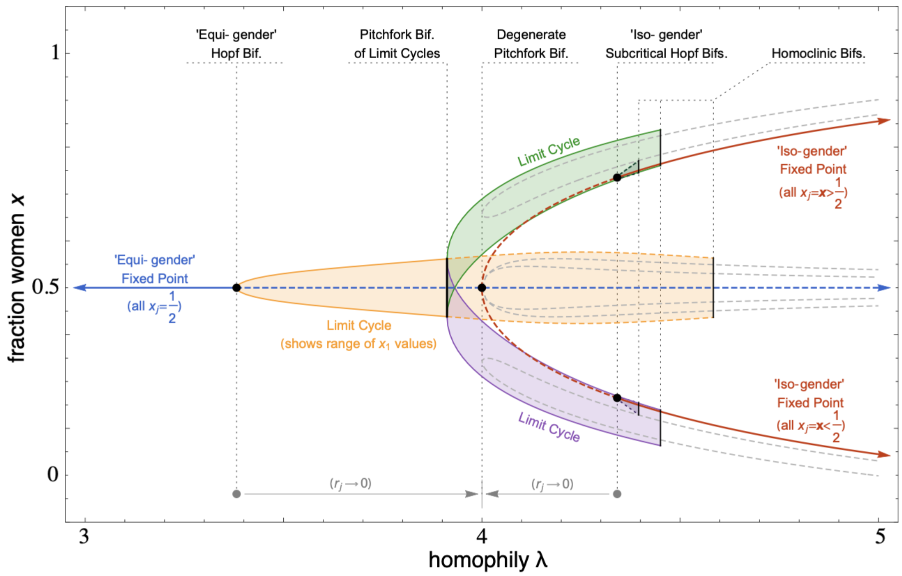

Figure 3.7 Bifurcation graph of the bias-free model from (Clifton et al., 2019b) included with permission from Prof. Sara Clifton.

3.8 Model with Homophily and Bias

To explore the full model including both homophily ($\lambda \neq 0$) and bias ($b \neq \frac{1}{2}$) understand that the the long-term dynamics are similar to the bias-free model. Similar to the bias-free model Section 3.7, the probability that an individual will apply for a promotion is not constant.

$$P(u,v) = \frac{1}{1+e^{-\lambda(u-v)}}$$

We then look at the fraction of people promoted to the next level who are women and see that we cannot substitute in our bias because we do not have a constant value for b.

$$f(u,v;b) = \frac{bvP(u,v)}{bvP(u,v)+(1-b)(1-v)P(1-u,1-v)}$$

Thus, the model reduces to the full model (3.4). Now, we want to alter the homophily and the hiring bias in order to explore how it affects the behavior of the fraction of women in each level. To do this let gender bias be $b = 0.45$

so that we have a slight bias against women and then homophily will vary to mild ($\lambda = 2$), moderate ($\lambda = 3$), strong ($\lambda = 4.5$), and strongest ($\lambda = 5$). Note that Figures 3.8, 3.9, 3.10, and 3.11 are replications from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

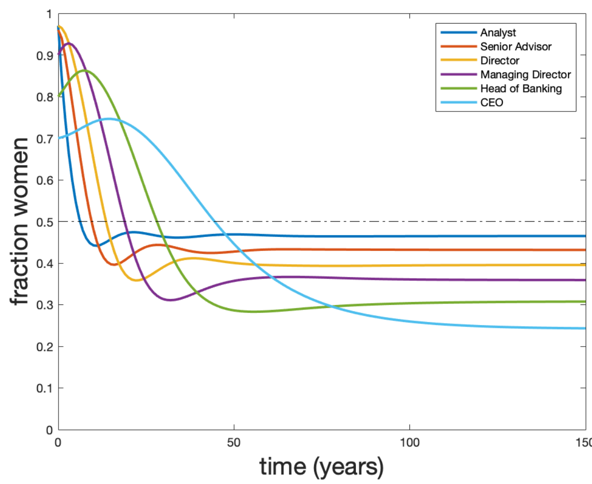

Figure 3.8 Here $\lambda = 2$ and all hierarchies are stable below gender equity. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

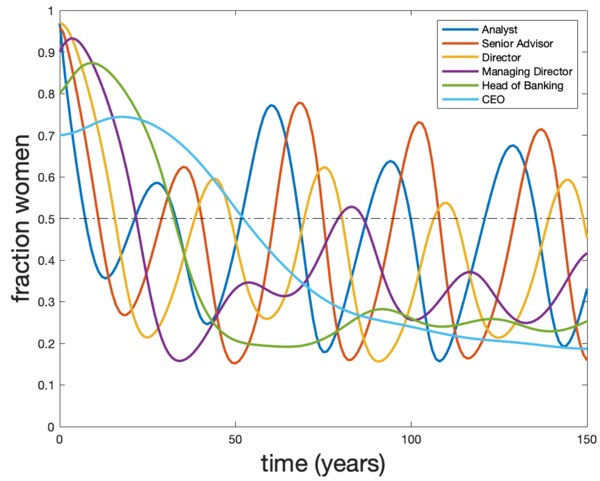

Figure 3.9 Here $\lambda = 3$ and the model becomes unstable and oscillates around gender equity. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

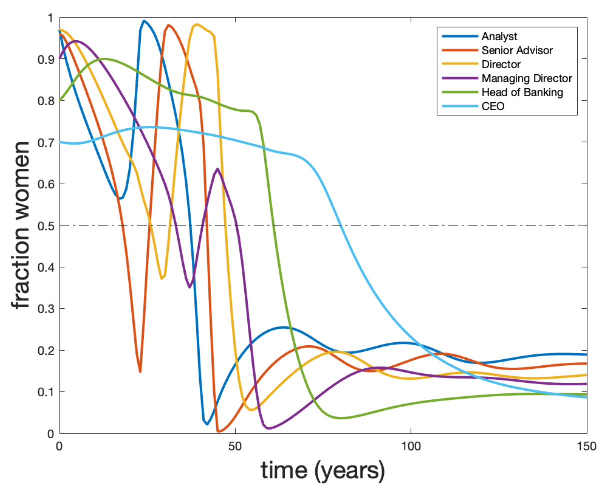

Figure 3.10 Here $\lambda = 4.5$ and the model remains unstable oscillating through a limit cycle. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

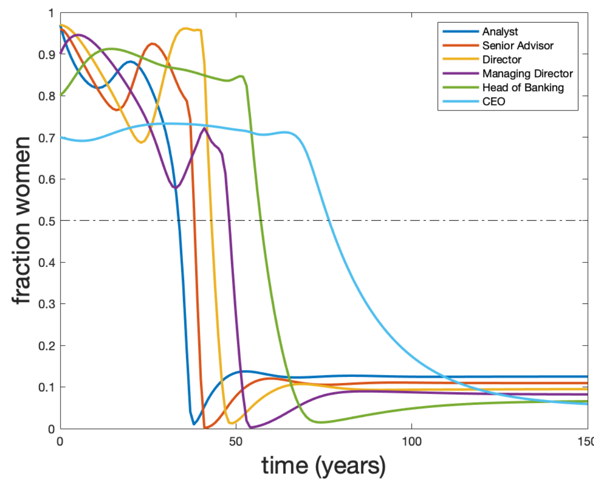

Figure 3.11 Here $\lambda = 5$ and the model starts off unstable but reaches stability at a small fraction of women. This figure is a replication from (Clifton et al., 2019b, a) using Supplementary Material MATLAB codes.

As we did with the bias-free model, we can look at the trends of this model on a larger graph in order to see how they interact. Point (A) on Figure 3.12 is representative of the stable behavior we see when $\lambda = 2$. For a woman in the company this means that there is a very short period of time where homophily is not strong enough to influence her decision. There is a Hopf bifurcation that occurs in the same place that we saw in the bias-free model. After the Hopf bifurcation at point (B), there is an unstable limit cycle that stems from the oscillations we see about gender equilibrium when $\lambda = 3$. We interpret this as a steady decline in the fraction of women at an given level. This is when the model becomes unstable and the homophily begins to affect an applicant's decision. The limit cycle continues on through point (C) where the behavior mirrors the behavior at $\lambda = 4.5$. We see another Hopf bifurcation occur and the model becomes stable again. Point (D) is labeled to be the stable behavior seen when $\lambda = 5$. If we were to change our initial conditions to be larger, then point (D) would move up to the saddle-node bifurcation seen at the top of the graph. If we were to do this, we would see the saddle-node bifurcation that shows the strong preference towards either gender that we saw previously.

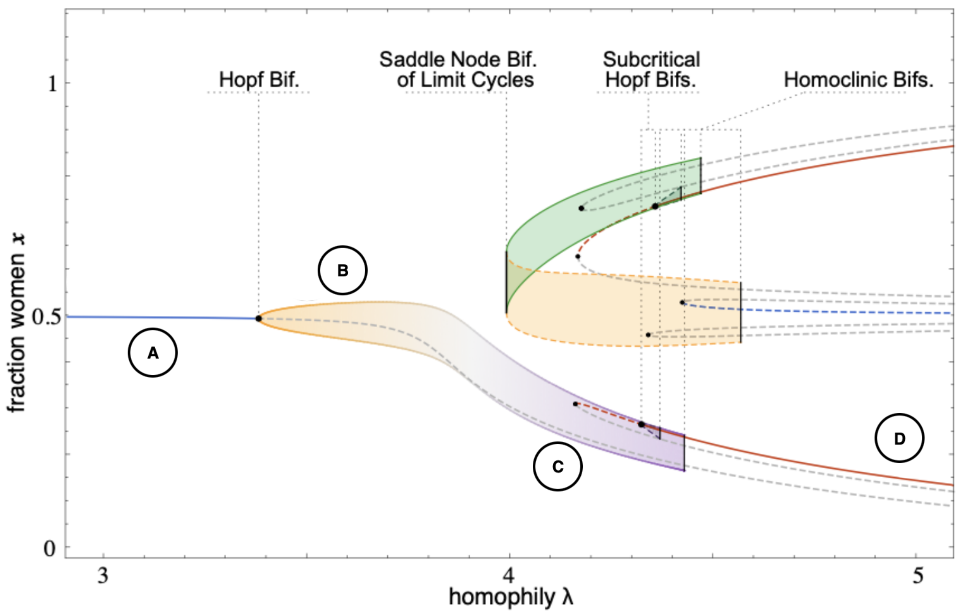

Figure 3.12 The pitchfork bifurcation we saw in the bias-free model degenerates and unfolds itself into a saddle node bifurcation and a fixed point curve. Graph included from (Clifton et al., 2019b) with permission from Prof. Sara Clifton.

3.9 Data and the Model

Now that we have seen all the different variations of gender bias and homophily within a professional hierarchy. We can look at how these two variables compare within different fields.

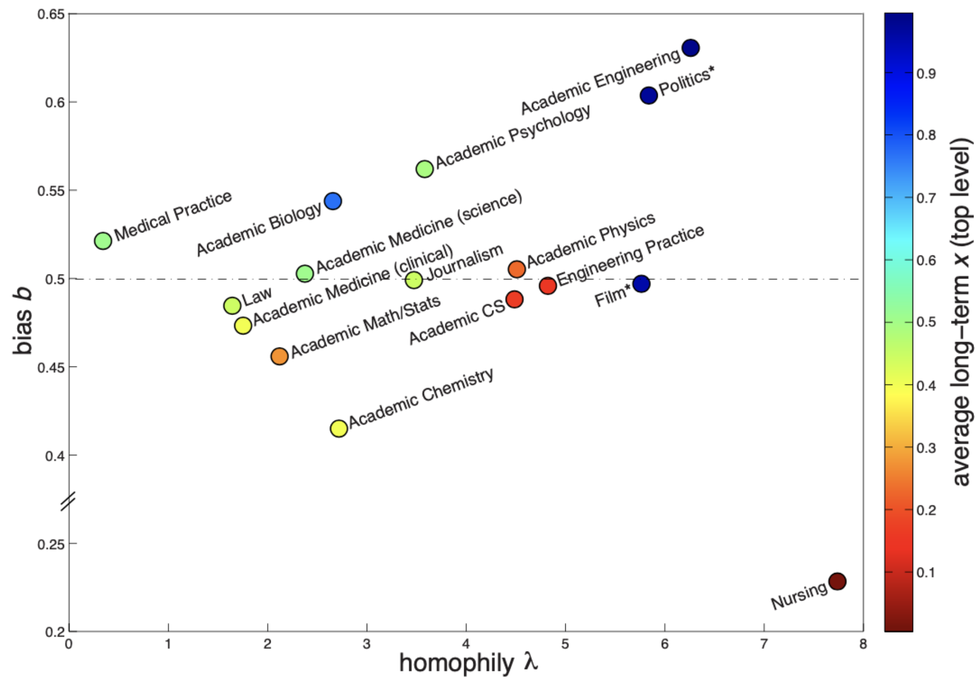

Figure 3.13 Comparison of homophily and gender bias within different professional fields. Figure included from (Clifton et al., 2019b) with permission from Prof. Sara Clifton.

This figure shows us a breakdown of different professional fields by analyzing their bias and general homophily. We see that as homophily increases gender becomes more of a factor when an applicant is deciding to apply for a promotion. We see in the colored graph on the side that when we take these two factors into account we can determine the average long term fraction of women in the given professional field. Professions with warmer color identification have a much smaller long term fraction of women at the top level of the hierarchy.

Many of these professional fields gather around $b = \frac{1}{2}$ and $\lambda = 4$, but some fields fall to extreme values. Nursing has low bias and high homophily meaning that there is extreme hiring bias towards women and gender is a large factor when an applicant is deciding whether or not to apply for an application. In contrast, politics has a high bias meaning it's favorable towards women and a slightly lower homophily than nursing.

3.10 Discussion

This model makes several assumptions that inhibit its application to the real world (Clifton et al., 2019a, b). The first assumption is that men and women can make different decisions when it comes to hiring and promotional decisions. We can not realistically assume that men and women on hiring committees have the same bias towards a specific gender. A possible extension to this model would be to split up the decisions made by men and women on hiring committees.

The extension we will focus on in the next part of the paper addresses the issue of a linear hierarchy, in Section 4.1. The Clifton et. al. model assumes that each individual must ascend linearly through the hierarchy (Clifton et al., 2019a, b). This means that everyone enters at the lowest level and can only be promoted one level at a time. This is unrealistic because there are many instances where people are hired from the outside to fill positions with high authority. In Section 4.1, we explore possible extensions to address this issue.

Chapter 4

Current and Future Directions

4.1 Extensions from (Clifton et al., 2019a)

In this section we will explore the idea of allowing individuals to be hired at all levels of the hierarchy instead of only being able to enter at the lowest level. To do this let us label δ_j to be the rate at which individuals are hired directly into j^{th} level. We are now allowing employees to be hired at any level. This means that the number of people entering at each level is $N_j x_j \delta_j$ where δx_j represents the fraction of people hired at the j^{th} level. Then we can say that the number of people who have the opportunity to be promoted from the $j-1$ level to the j^{th} level is:

the number of people who leave the j^{th} level	-	the number of people who are hired into the j^{th} level

But, this addition to the model creates a waterfall effect. For example, let us explore a 5 tier professional hierarchy. The number of jobs available at the 3^{rd} level can be calculated as follows:

the number of people who leave the 3^{rd} level	-	the number of people who are hired directly into the 3^{rd} level	-	the number of people promoted from the 2^{nd} level

If then we want to know how many women are up for promotion at the j^{th} level then we would multiply this number by the fraction of women in the $(j-1)^{th}$ level. Using these equations we have created using the new variable δ_j, the Figure 4.1 flow chart represents an employee's potential movement throughout a professional hierarchy.

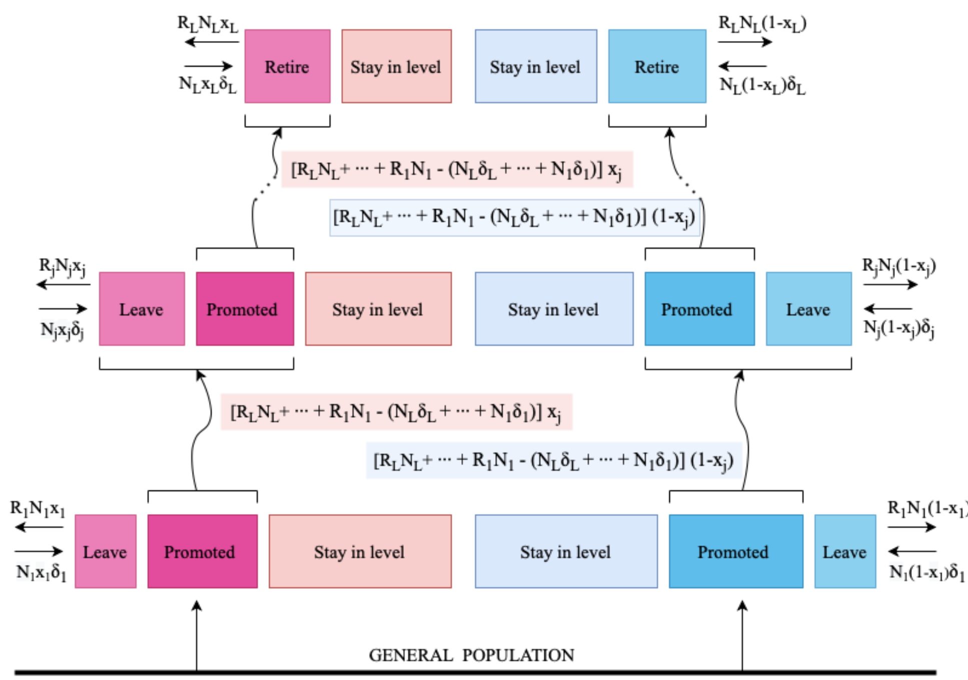

Figure 4.1 A new model flowchart using δ as the rate at which people enter the j^{th} level directly.

We have altered the previous set of equations to now include the rate, δ_j, at which individuals are hired directly into the level. We did this by breaking up the equation 3.4 and replacing every R_j with $R_j - \delta_j$. This changes the fraction of women promoted to include the number of women that are hired into the level directly. This leaves us with a new set of equations:

$$\underbrace{\frac{d}{dt}(x_L N_L)}_{\text{the change in the number of women at each level}} = \overbrace{(R_L - \delta_L)N_L f(x_L, X_{L-1}; b)}^{\text{the number of women promoted}} - \underbrace{(R_L - \delta_L)N_L x_L}_{\text{the number of women retiring}} \tag{4.1}$$

$$\frac{d}{dt}(x_j N_j) = \left(\sum_{k=j}^{L}(R_k - \delta_k)N_k\right) f(x_j, x_{j-1}; b) - (R_j - \delta_j)N_j x_j - \left(\sum_{k=j+1}^{L}(R_k - \delta_k)N_k\right) f(x_{j+1}, x_j; b)$$

$$\text{where } 1 < j < L$$

$$\frac{d}{dt}(x_1 N_1) = \left(\sum_{k=1}^{L}(R_k - \delta_k)N_k\right) f(x_1, \frac{1}{2}; b) - (R_1 - \delta_1)N_1 x_1 - \left(\sum_{k=2}^{L}(R_k - \delta_k)N_k\right) f(x_2, x_1; b)$$

To normalize these equations we want to divide both sides by our similar constant, $(R_j - \delta_j)N_j$:

$$\frac{1}{R_L - \delta_L}\frac{dx_L}{dt} = \overbrace{f(x_L, x_{L-1}; b)}^{\text{promoted from lower level j}} - \overbrace{x_L}^{\text{retire out of level L}}$$

$$\frac{1}{R_j - \delta_j}\frac{dx_j}{dt} = \underbrace{(1 + r_j)f(x_j, x_{j-1}; b)}_{\text{promoted into level } j \text{ from level } j-1} - \underbrace{x_j}_{\text{leave level } j} - \underbrace{r_j f(x_{j+1}, x_j; b)}_{\text{promoted to level } j+1 \text{ from level } j} \tag{4.2}$$

$$\text{where } 1 < j < L$$

$$\frac{1}{R_1 - \delta_1}\frac{dx_1}{dt} = \underbrace{(1 + r_1)f(x_1, \frac{1}{2}; b)}_{\text{hired from the general pool into level 1}} - \underbrace{x_1}_{\text{leave field}} - \underbrace{r_1 f(x_2, \frac{1}{2}; b)}_{\text{promoted to next level}}$$

This new set of equations is similar to the previous model (3.4). The only difference is now our normalized set of equations depends on $R_j - \delta_j$ rather than just R_j. Because the only thing differentiating these two models is our constant δ_j, if we were to run a simulation we would not be able to tell the difference between the two models. δ_j would simply act as another

parameter constant in our model so the behavior at each hierarchical level would be the same.
In order to see a change in the model simulation, we would have to make δ dependent on our bias, b. This would allow us to alter the bias at each hierarchical level rather than having an overall bias for the entire company. $\delta(b)$ would still act as a constant, but the change in bias would allow us to simulate a more realistic version of a professional hierarchy.

4.2 Other Future work

The first unrealistic aspect of this model is the assumption that gender bias and homophily do not vary from lower levels of the hierarchy to higher levels. Clifton et. al. makes this assumption to avoid over-fitting the model, but an interesting extension would be allowing both hiring bias and homophily to vary at each hierarchical level.
Our second idea for an extension would be to parameterize the model and analyze a specific financial firm. From doing this we hope to be able to apply this model and analyze how applicable it is to real life. To do this we would have to gather data about a firm's gender breakdown at each level of the hierarchy, number of levels in the hierarchy, the retirement rates, the promotion rates for women, etc. Due to time constraints we were unable to find sufficient data in order to properly simulate the model. But without this time constraint we would have been able to analyze a firm and discuss the bias that exists within that firm.
Another extension would be to create a new variable to describe the fraction of men. We would also hope to include the fraction of non-binary employees of the company in order to study the bias towards them. This extension opens up a larger discussion of diversity which is important to highlight. The current model has the fraction of men as $1 - x_j$ which is dependent on the fraction of women. By creating this new variable, male, female, and non-binary employees would not have perfect symmetry and both the the male and non-binary employees in the company would have their own behavior meaning they would be making decisions separate from female employees. This would also allow us to apply bias separately to men which would change the behavior of the model significantly.

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