Hercules Meets Laplace in the Classroom

by Mike Huber, Muhlenberg College

Abstract

The mythology surrounding Hercules has been a part of human culture for over two and a half thousand years. Laplace transforms, on the other hand, have been used in solving differential equations for about one-tenth the time. This article treats The Eleventh Labor: The Apples of the Hesperides, in which Hercules must wrestle and defeat the giant Antaeus in order to pass through Libya. Each time Hercules throws Antaeus to the ground, the giant recoups energy from the earth. Hercules then lifts him off the ground and Antaeus' strength decreases. This interdisciplinary problem is a nice application of a model which can use Laplace transforms to determine a solution of how Hercules can defeat the giant. It has been used in the differential equations classroom to enhance learning in a liberal arts environment.

Introduction

The following passage is taken from Apollodorus' *The Library* [1], from the English translation by Sir James George Frazer:

When the [ten] labours had been performed in eight years and a month, Eurystheus ordered Hercules, as an eleventh labour, to fetch golden apples from the Hesperides, for he did not acknowledge the labour of the cattle of Augeas nor that of the hydra. These apples were not, as some have said, in Libya, but on Atlas among the Hyperboreans. They were presented by Earth to Zeus after his marriage with Hera, and guarded by an immortal dragon with a hundred heads, offspring of Typhon and Echidna, which spoke with many and divers sorts of voices. With it the Hesperides also were on guard, to wit, Aegle, Erythia, Hesperia, and Arethusa. So journeying he came to the river Echedorus. And Cycnus, son of Ares and Pyrene, challenged him to single combat. Ares championed the cause of Cycnus and marshalled the combat, but a thunderbolt was hurled between the two and parted the combatants. And going on foot through Illyria and hastening to the river Eridanus he came to the nymphs, the daughters of Zeus and Themis. They revealed Nereus to him, and Hercules seized him while he slept, and though the god turned himself into all kinds of shapes, the hero bound him and did not release him till he had learned from him where were the apples and the Hesperides. Being informed, he traversed Libya. That country was then ruled by Antaeus, son of Poseidon, who used to kill strangers by forcing them to wrestle. Being forced to wrestle with him, Hercules hugged him, lifted him aloft, broke and killed him; for when he touched earth so it was that he waxed stronger, wherefore some said that he was a son of Earth.

Background

Apollodorus is often referenced as "Apollodorus of Athens," and it is believed that he was born around 180 B.C. According to the *The Oxford Classical Dictionary* [3], Apollodorus lived in Alexandria, but he left there in approximately 146 B.C., moving to Athens, where he spent the rest of his life. He had varied interests and was considered to be a great scholar. His *Library* was a study of the ancient Greek heroic mythology. The oldest discovered copy of this book dates to the first or second century A.D. His account of the Herculean myth is commonly accepted by most as the authority.

The Hesperides were known as the "nymphs of the West," and were often compared in mythology to the Three Graces. The apples grew on a magical tree which had golden leaves and a golden bark. These apples supposedly gave eternal life to whoever ate them. Zeus, the king of the gods, had given the tree to Hera as a wedding present. She planted the tree in a garden at the base of Mount Atlas (which gave rise to the Atlantic Ocean, which surrounded the world near Mount Atlas). The Hesperides, daughters of Atlas, liked to pilfer from the tree, so Hera placed a ferocious serpent with one hundred heads named Ladon to guard her precious tree. To successfully accomplish this labor, Hercules must grapple with the giant Antaeus in order to pass through Libya. Each time Hercules throws Antaeus to the ground, the giant recoups energy from the earth. Hercules then lifts him off the ground and Antaeus' strength decreases.

The Problem

Suppose that Hercules and Antaeus wrestle for five minutes before Antaeus is thrown to the ground. In the next minute, his strength is instantly recharged. The ground provides a one-minute pulse of energy to Antaeus (similar to a step or Heaviside function with an amplitude of one), and he again wrestles Hercules for five minutes, and his strength diminishes as before. This periodic forcing function continues throughout the battle. In the absence of the forcing function, the rate of change of Antaeus' strength would decline at a rate proportional to three-fourths of his original strength. Model Antaeus' strength and determine if Hercules can defeat him as long as he is on land.

The Model

We will model Antaeus' strength using an ordinary differential equation with a step forcing function. Often we are faced with solving initial value problems that are differential equations with constant coefficients, but they may have a piece-wise continuous or step forcing function. The forcing function is also known as a "driving" function, since it comes from an external force or source that "drives" the system. One approach to solving such a problem is to transform the equation into another equation that is hopefully easier to work with. We then solve the new equation and reverse the transformation, interpreting the solution back in the context of the original problem. In particular, one method is to use Laplace transforms, which are used to transform the initial value problem into an algebraic equation that can then be easily solved using algebra.

The Laplace transform of a function $a(t)$ uses integration in comparing $a(t)$ to an exponential function of the form e^{st}. We define the Laplace transform function $\mathcal{L}[a](s)$ of the function $a(t)$ to be

$$\mathcal{L}[a](s) = \int_0^\infty a(t)\, e^{-st}\, dt. \tag{1}$$

Mathematically, the Laplace transform is a function that converts $a(t)$ into a new function $A(s)$, but we use the script letter $\mathcal{L}$ to represent that function (we usually write that $A(s) = \mathcal{L}[a](s)$, or simply, $A = \mathcal{L}[a]$. Most calculus and differential equations textbooks have tables of commonly-used Laplace transforms (of polynomials, of exponential and trigonometric functions, of step functions, etc.). In addition, computer algebra systems, such as *Mathematica* and *Maple*, also have built-in algorithms for applying the Laplace transform operator. Here is a procedure for using Laplace transforms with initial value problems containing constant coefficients:

Step 1. Apply the Laplace transform to the initial value problem. This yields an algebraic equation in the transform of the solution. Any initial condition and forcing function is also transformed.

Step 2. Solve the algebraic equation for $\mathcal{L}[a](s)$, the transform. Use algebraic manipulation to ensure each term in the transform's formula is itself in the form of a Laplace transform (see the example).

Step 3. Apply the inverse Laplace transform to obtain the solution to the initial value problem.

The Laplace transform has a very special property involving its derivative that really forms the basis for its use in solving differential equations. Given a function $a(t)$ with associated Laplace transform $\mathcal{L}[a]$, the Laplace transform of $\frac{da}{dt}$ is

$$\mathcal{L}\left[\frac{da}{dt}\right] = s\mathcal{L}[a] - a(0). \tag{2}$$

This is easily verified by using the definition of $\mathcal{L}\left[\frac{da}{dt}\right]$ and evaluating the improper integral using integration by parts.

How does the pulse affect the strength? This is similar to a harvesting problem. Let's define $a(t)$ to be the strength of Antaeus at time t, which is measured in minutes. Since Hercules takes away strength from Antaeus when he lifts him off of the ground, the coefficient for the rate of change of Antaeus' strength (proportional to three-fourths of his original strength) is negative. This gives us the following ordinary differential equation

$$\frac{da}{dt} = -\frac{3}{4}\, a(t) + f(t). \tag{3}$$

This differential equation is first-order, linear, and non-homogeneous. Initially, we assume that Antaeus is at full strength, so $a(0) = 1$. The forcing function of this ODE is a square-wave pulse. A plot of the $f(t)$ is shown in Figure 1, and it can be modeled as

$$f(t) = \begin{cases} 0 & \text{for } 0 \le x < 5 \\ 1 & \text{for } 5 \le x < 6 \\ 0 & \text{for } 6 \le x < 11 \\ 1 & \text{for } 11 \le x < 12 \\ 0 & \text{for } 12 \le x < 17 \\ 1 & \text{for } 17 \le x < 18 \\ 0 & \text{for } 18 \le x < 23 \\ & \text{etc.} \end{cases}.$$

Now that we have the initial value problem, we need to solve it.

The Solution

The Laplace Transform of the homogeneous portion of Equation 3 is

$$\mathcal{L}\left[\frac{da}{dt} = -\frac{3}{4}\, a(t)\right] \Longrightarrow s\mathcal{L}\,[a] - a(0) = \frac{3}{4}\,\mathcal{L}\,[a]\,.$$

The forcing function $f(t)$ looks like a series of unit step functions. A unit step function has a discontinuity where it jumps from 0 to 1 or back from 1 to 0. It is commonly used in differential equations to model discontinuous processes. The step function's main use is to turn other functions on and off, like a light switch. Since the function we are modeling is Antaeus' strength, which is equal to either 1 or 0, our model needs to turn the value of the strength on and off at the appropriate time. For example, $f(t)\cdot\text{Step}(t,\alpha)$ turns the function $f(t)$ on at $t = \alpha$, setting the function to a value of 1. Further, $f(t)\cdot\{\text{Step}(t,\alpha) - \text{Step}(t,\beta)\}$ turns $f(t)$ on at $t = \alpha$ and off at $t = \beta$. Our function $f(t)$ can therefore be modeled as

$$\begin{aligned} f(t) &= 1 \times \{\text{Step}(t,5) - \text{Step}(t,6) + \text{Step}(t,11) \\ &\quad -\text{Step}(t,12) + \text{Step}(t,17) - \text{Step}(t,18) + \cdots\}. \end{aligned}$$

The Laplace Transform of $\text{Step}(t,5)$ as a function of s is

$$\mathcal{L}\,[\text{Step}(t,5)] = \frac{1}{s} \times e^{-5s},$$

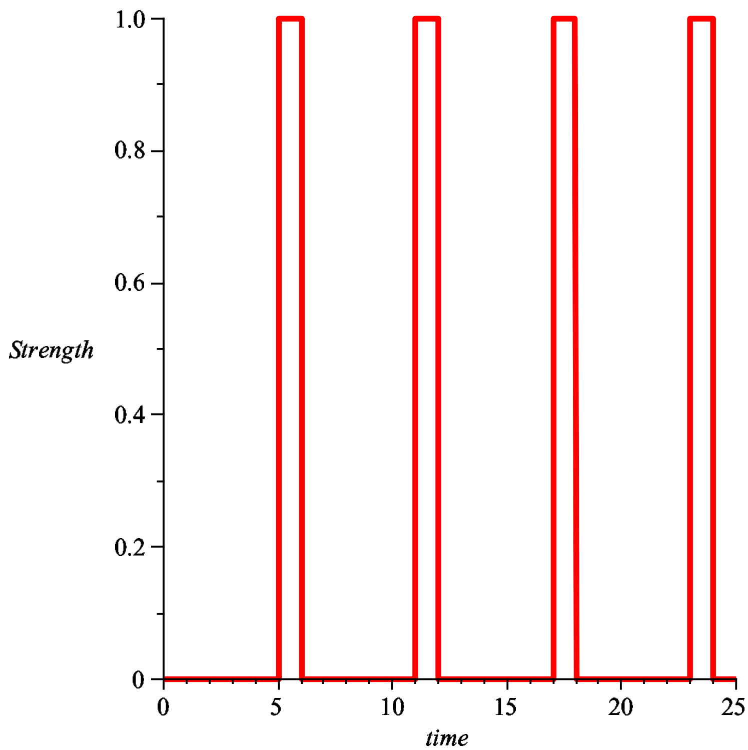

Figure 1: The forcing function for Antaeus' strength

and when we add up all the terms, the Laplace Transform of $f(t)$ becomes

$$\mathcal{L}\left[f(t)\right] = \frac{1}{s}\ \left[e^{-5s} - e^{-6s} + e^{-11s} - e^{-12s} + e^{-17s} - e^{-18s} + \cdots\right].$$

We could truncate the forcing function at $t = 18$ and see if we can determine any trend, but let's keep a generalized form for the moment. Thus, the transformed differential equation becomes

$$\begin{aligned} s\mathcal{L}\left[a\right] - a(0) &= -\frac{3}{4}\,\mathcal{L}\left[a\right] + \frac{1}{s}\ \left[e^{-5s} - e^{-6s}\right. \\ &\quad \left. + e^{-11s} - e^{-12s} + e^{-17s} - e^{-18s} + \cdots\right]. \end{aligned} \tag{4}$$

Assign the value of $a(0) = 1$ and solve Equation 4 for $\mathcal{L}\left[a\right]$.

$$\left(s + \frac{3}{4}\right)\mathcal{L}\left[a\right] = 1 + \frac{1}{s}\ \left[e^{-5s} - e^{-6s} + e^{-11s} - e^{-12s} + e^{-17s} - e^{-18s} + \cdots\right].$$

So,

$$\mathcal{L}[a] = \frac{1}{s+\frac{3}{4}} + \frac{1}{s\,(s+\frac{3}{4})}\;\left[e^{-5s} - e^{-6s} + e^{-11s} - e^{-12s} + e^{-17s} - e^{-18s} + \cdots\right]. \quad (5)$$

All that's left to do now is take the inverse Laplace transform of the right-hand side of Equation 5 and we have the solution. The inverse transforms take the following forms:

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s-k}\right] &= e^{kt}, \\ \mathcal{L}^{-1}\left[\frac{1}{s\,(s-k)}\,e^{-rs}\right] &= \frac{1}{k}\,\mathrm{Step}(t,r)\,(1-e^{k(t-r)}). \end{aligned}$$

So, the solution to the initial value problem transforms to

$$\begin{aligned} a(t) = {} & e^{-\frac{3}{4}t} + \frac{4}{3}\left[\mathrm{Step}(t,5)\,\left(1-e^{-0.75t+3.75}\right)\right] \\ & -\frac{4}{3}\left[\mathrm{Step}(t,6)\,\left(1-e^{-0.75t+4.50}\right)\right] \\ & +\frac{4}{3}\left[\mathrm{Step}(t,11)\,\left(1-e^{-0.75t+8.25}\right)\right] \\ & -\frac{4}{3}\left[\mathrm{Step}(t,12)\,\left(1-e^{-0.75t+9.00}\right)\right] \\ & +\frac{4}{3}\left[\mathrm{Step}(t,17)\,\left(1-e^{-0.75t+12.75}\right)\right] \\ & -\frac{4}{3}\left[\mathrm{Step}(t,18)\,\left(1-e^{-0.75t+13.50}\right)\right] \\ & +\cdots. \end{aligned} \quad (6)$$

Equation 6 represents the changing strength on the giant Antaeus as the wrestling match progresses, and this solution is plotted in Figure 2. We see that there is exponential decay while Hercules holds Antaeus above the ground for the first five minutes. Then, as his strength is almost at nothing, Antaeus is thrown to the ground, and he regains his strength (notice the positive steep rate of change). Hercules wrestles again with Antaeus, and the cycle repeats and repeats. The only way in which Hercules will win is if he can either strangle Antaeus while he is in the air or throw Antaeus into the water, where he will not gain strength from the earth. Apollodorus tells us the Hercules kills Antaeus while holding him in the air.

In the Classroom

This is an appropriate interdisciplinary project to use in a differential equations course. Many textbooks are concerned with offering "practical applications"

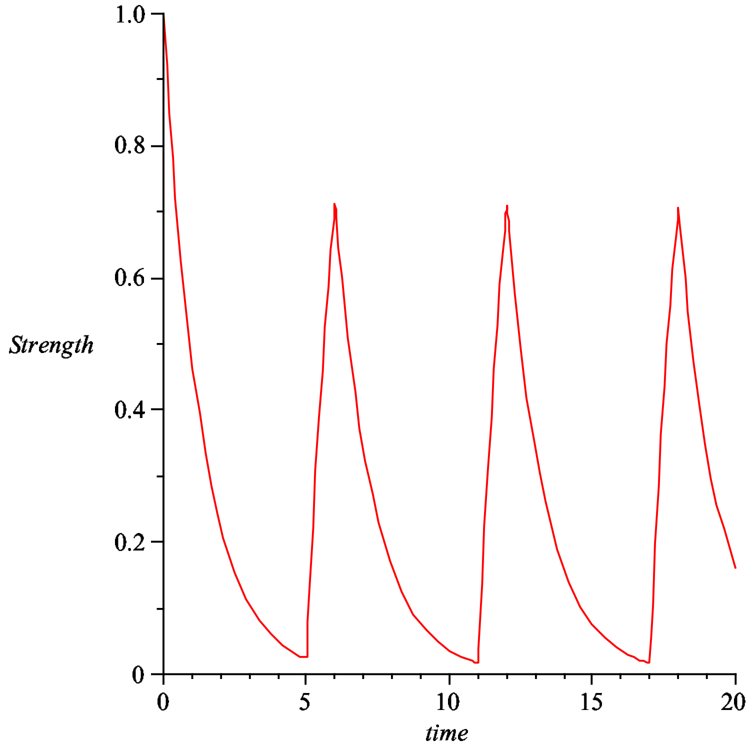

Figure 2: Antaeus' strength as he wrestles Hercules

or "real world" problems, and when Laplace transform applications problems are described, they usually involve an energy pulse or engineering flavor. This scenario, although contrived by the author, combines a nice approach using Laplace transforms with Greek mythology. In a liberal arts setting, students can relate the problem to a course outside the science or mathematics department. The labor has been in existence for 2000 years. By inserting some ideas to model the rise/fall of Antaeus' energy, this becomes an exercise which my students have enjoyed solving. They know who Hercules is and can relate their solutions to the myth itself, as Apollodorus does not go into details.

References

[1] Apollodorus. 1939. *Apollodorus, The Library*, with an English Translation by Sir J. G. Frazer, in 2 Volumes. Cambridge, Massachusetts: Harvard University Press. Includes Frazer's notes.

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[3] M. Huber. 2009. *Mythematics: Solving the 12 Labors of Hercules*. Princeton, New Jersey: Princeton University Press.

[4] S. Hornblower and A. Spawforth, editors. 1996. *The Oxford Classical Dictionary*. New York: Oxford University Press.