

**RESEARCH PROPOSAL:  
FINDING THE DISTANCE BETWEEN A TRIANGLE'S CONFORMAL  
CENTER AND THE SECOND MORLEY POINT**

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For my senior thesis, I would like to continue work I began last summer with Professor Walden at Santa Clara University. In every triangle there is a unique point from which a Brownian path is equally likely to exit the triangle from any of its three sides. We've named this point the conformal center because it is preserved by conformal maps. Over the summer we found some formulations of the point's barycentric coordinates, and discovered it lies very close to the better-known second Morley point. I would like to continue the work I did with Professor Walden, investigating the conformal center and its relation to the second Morley point.

## **1 PRIOR WORK**

Define a function on the interior of a triangle to be, at each point, the probability that a Brownian path begun at that basepoint will exit the triangle by a particular side. This is called the harmonic measure of that side from the given basepoint, as the probability is a harmonic function of the basepoint. If we define such a probability for each of the triangle's three sides, then the conformal center is the point where all three probabilities equal  $1/3$ . These probabilities are invariant under conformal mappings, so the conformal center is also invariant (hence the name). Specifically, it is invariant under the Schwartz-Christoffel map from the half-plane to a triangle. By locating the point in the half-plane which maps to a triangle's conformal center, we were able to find a formula for the conformal center's barycentric coordinates. Over the summer, I used this formulation to write a Java-based program, which estimated and plotted the conformal center to arbitrary precision, for any given triangle. We used this program to compare the conformal center to other known triangle centers and lines. We found that the conformal center lies strikingly close to the second Morley point, a well-known triangle center. We suspect there may be a deeper reason for this similarity, as well as a closed-form expression that bounds the distance between the two points.

## **2 PRIOR READING**

Since I haven't yet taken complex analysis, I did some reading this summer on conformal maps, especially the Schwartz-Christoffel map, in *Complex Analysis*, by Lars V. Ahlfors.

## **3 PLANS FOR ORIGINAL RESEARCH**

Although it is known that a unique conformal center exists in every triangle, there are no elementary formulae for its coordinates. However, there are relatively straightforward formulae involving sums of beta functions or equivalently hypergeometric series, which yield excellent approximations that hold promise for establishing tight bounds for the distance between the conformal center and the second Morley point. We plan to explore the interplay between what these formulae can say about the conformal center and what knowledge of the conformal center and harmonic measure can say about the hypergeometric series.

## **4 INTENDED READING**

My intended reading includes *Special Functions*, by Andrews, Askey and Roy. I also intend to take *Complex Analysis* from Professor Ward this semester, using the text, *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering*, 2nd edition, E.B. Saff and A.D. Snider, Prentice Hall, 1993.