Math 197: Senior Thesis

Sensitivity of a Laplacian Family of Ranking Methods

Claire Chang

Introduction

We focus on a family of ranking methods for pairwise comparisons introduced by Devlin and Treloar (2018) which encompasses the well-known Massey, Colley, and Markov methods.

Through the lens of sports, we study systems of n teams and examine the

- *maximal upset*: the last ranked team beats the first, and the
- *perfect season*: the ith team plays one game against each team and beats all lower ranked teams $i+1, i+2, \dots, n$

Objectives

We will investigate this family in order to

1. connect it to the graph Laplacian to motivate a network diffusion interpretation, and
2. analyze its sensitivity by studying the maximal upset.

Methods

$W_i,\ L_i$	The total number of wins and losses for i
$W_{ij},\ L_{ij}$	The number of wins and losses for i against j

Equation for this family (parametrized by p):

$$\mathscr{L}_p \mathbf{v}_p = \mathbf{s}_p, \tag{1}$$

with rating vector $\mathbf{v}_p$.

Entries in right hand side vector $(\mathbf{s}_p)_{n\times 1}$:

- General: $W_i - L_i$[1]
- Perfect season: $n - 2i + 1$

The matrix is a variant of the graph Laplacian:

$$\mathscr{L}_p = W + pL^1, \tag{2}$$

and W and L are defined entrywise as

$$W_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ L_i & \text{if } i = j \end{cases}, \quad \text{and} \quad L_{ij} = \begin{cases} -l_{ij} & \text{if } i \neq j \\ W_i & \text{if } i = j \end{cases}.$$

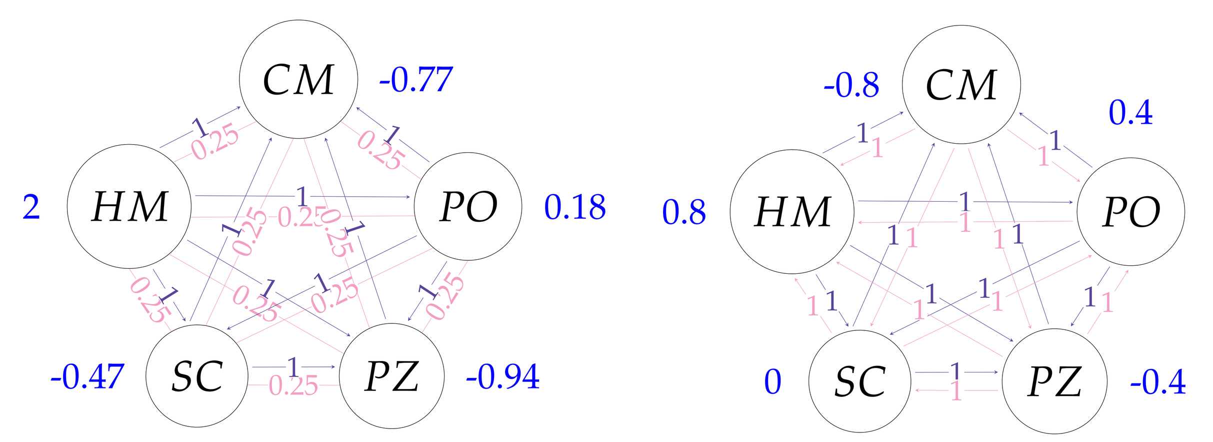
Network Diffusion

In Figure 1, notice (true for any n):

❶ p increases $\to$ rating magnitudes decrease.

❷ low $p \to$ ratings for high ranked teams highly separated ratings for low ranked teams very similar

❸ high $p \to$ ratings evenly spaced overall



(a) Diffusion with $p = 0.25$. (b) Diffusion with $p = 1$.

Figure 1: The network diffusion with ratings in blue outside each node. The flows in blue are from wins, and in pink are from losses.

Imagine rank is a liquid/gas diffusing among the nodes. Why are ❶, ❷, and ❸ true, and what conclusions can we then draw?

❶: p increases $\to$ overall flow of system increases (especially for strong teams) $\to$ rank "leaks" out from losses $\to$ rating magnitudes are lowered

❷ and ❸:

- **Partial ranking** (top k) $\to$ lower p desirable, as top k teams will have more distinct ratings
- **Data with high variability** $\to$ higher p desirable, as n teams are more evenly distributed so the rankings are more stable

Sensitivity

Partially inspired by Chartier et al. (2011),

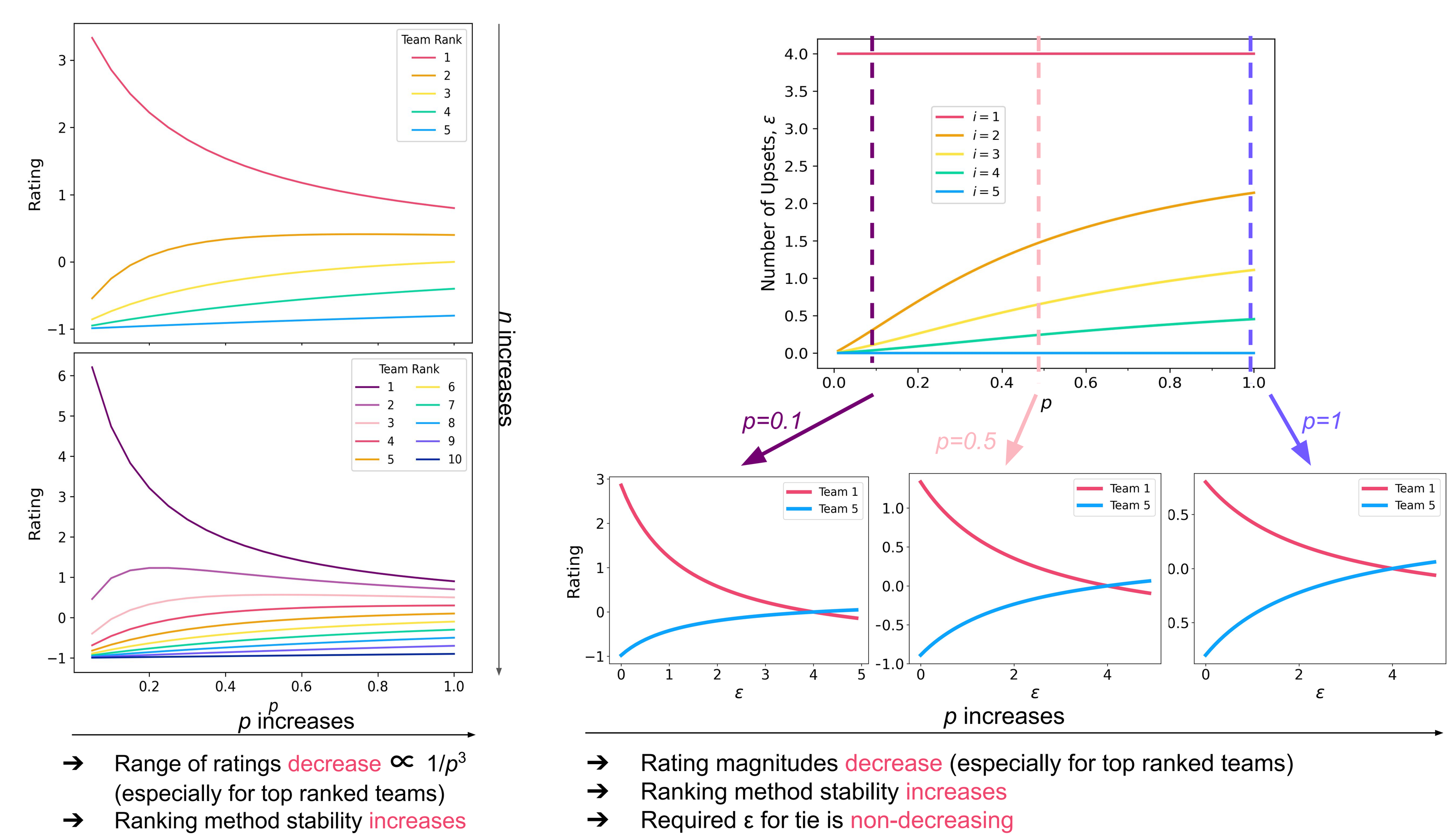
1. Apply Sherman-Morrison inversion formula to find $(\mathscr{L}_p)^{-1}$
2. Multiply by $\mathbf{s}_p$ to determine ratings in the perfect season (Figure 2a)
3. Prove rating formula with Gaussian elimination and induction
4. Find ϵ of wins needed for Team j against Team $i < j$ for Team j to start to overtake Team i in rank (Figure 2b)

For n teams and $p \neq 0$[2], the ith rating in the perfect season is

$$v_i = \frac{(n-i+1)(n-i)p - i(i-1)}{(((n-i+1)p + i - 1))((n-i)p + i)}. \tag{3}$$

The perturbed rating vector is

$$\tilde{\mathbf{v}}_p = \mathbf{v}_p + \Big[-\epsilon + \Big(\frac{1}{4p+1} + \frac{p}{p+4} \Big) \frac{\epsilon(4p+1)(p+4)}{(4p+1)(p+4) + 2\epsilon(2p^2+p+2)} (\epsilon - 4) \Big] \mathbf{a}_i. \tag{4}$$



(a) Ratings from (3).[3] (b) After ϵ wins of Team j against Team i, they are tied.

Figure 2: Sensitivity (a) in the perfect season and (b) with upsets.

Conclusions

- Proved a general formula for the ratings of all n alternatives for any point in our Laplacian ranking family
- Lower p values are desirable for partial ranking applications and higher p values for ranking with high variability data.

References

Chartier, Timothy P., Erich Kreutzer, Amy N. Langville, and Kathryn E. Pedings. 2011. Sensitivity and stability of ranking vectors. *SIAM J Sci Comput* 33(3):1077–1102.

Devlin, Stephen, and Thomas Treloar. 2018. A network diffusion ranking family that includes the methods of Markov, Massey, and Colley. *J Quant Sport* 14(3):91–101.

Acknowledgments

Thank you so much to Professor Jamie Haddock for all the mentorship and support, and to the rest of the math department (especially Professors Michael Orrison and Jon Jacobsen)!

[1] As originally defined in Devlin and Treloar (2018), their matrix $\mathscr{L}_p$ (which we denote $\overline{\mathscr{L}_p}$ for clarity) is not full rank, so an additional constraint that all the ratings must sum to zero is added in accordance with the convention of the Massey method, which creates the matrix $\mathscr{L}_p^+$. We also remove the last row of $\overline{\mathscr{L}_p}$, thereby redefining $\mathscr{L}_p$ by *replacing* the last row of $\overline{\mathscr{L}_p}$ with n ones and a zero rather than *adding* this row of ones and a zero.

[2] For $p = 0$, the right hand side vector is $\mathbf{0}$, which means the method described with calculating $(\mathscr{L}_p)^{-1}$ does not work. Furthermore, the Markov method at $p = 0$ has a different scale where all the ratings are positive and sum to 1, whereas for $0 < p \leq 1$ ratings can also be negative and sum to 0.

[3] In the range $p = (0, 1]$, since as previously noted, there is a discontinuity in the way the Markov method ratings are scaled at $p = 0$, so we do not include them in this plot.

Advisor: Jamie Haddock

Reader: Michael Orrison

Email: cschang@g.hmc.edu