Math 197: Senior Thesis
Sensitivity of a Laplacian Family of Ranking Methods
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Introduction
We focus on a family of ranking methods for pairwise comparisons introduced by Devlin and Treloar (2018) which encompasses the well-known Massey, Colley, and Markov methods.

Through the lens of sports, we study systems of n teams and examine the
• maximal upset: the last ranked team beats the first, and the
• perfect season: the i-th team plays one game against each team

and beats all lower ranked teams $i + 1, i + 2, \ldots, n$

Objective
We will investigate this family in order to
1. connect it to the graph Laplacian to motivate a network diffusion interpretation, and
2. analyze its sensitivity by studying the maximal upset.

Methods
For any family (parametrized by $p$):

$$L_p \mathbf{v}_p = \mathbf{s}_p,$$

with rating vector $\mathbf{v}_p$, Entries in right hand side vector ($\mathbf{s}_p$) $n \times 1$:

• General: $W_i - L_i$
• Perfect season: $n - 2i + 1$

The matrix is a variant of the graph Laplacian:

$$L_p = W + pL_1,$$

and $W$ and $L$ are defined entrywise as

$$W_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ L_i & \text{if } i = j \end{cases} \quad \text{and} \quad L_{ij} = \begin{cases} -l_{ij} & \text{if } i \neq j \\ W_i & \text{if } i = j \end{cases}$$

1. Apply Sherman-Morrison inversion formula to find $L_p^{-1}$
2. Multiply by $s_p$ to determine ratings in the perfect season
3. Prove rating formula with Gaussian elimination and induction
4. Find $e$ wins needed for Team $j$ against Team $i$ for $i < j$ to start to overtake Team $i$ in rank

Network Diffusion
In Figure 1 notice (true for any $n$):

• $p$ increases $\rightarrow$ rating magnitudes decrease.
• low $p$ $\rightarrow$ ratings for high ranked teams highly separated ratings for low ranked teams very similar
• high $p$ $\rightarrow$ ratings evenly spaced overall

Figure 1: The network diffusion with ratings in blue outside each node. The flows in blue are from wins, and in pink are from losses.

Imagining rank as a liquid/gas diffusing among the nodes. Why are $\bullet$, $\bullet$, and $\bullet$ true, and what conclusions can we then draw?

• $p$ increases $\rightarrow$ overall flow of system increases (especially for strong teams) $\rightarrow$ rank “leaks” out from losses $\rightarrow$ rating magnitudes are lowered
• $\bullet$ and $\bullet$

• Partial ranking (top $k$) $\rightarrow$ lower $p$ desirable, as top $k$ teams will have more distinct ratings
• Data with high variability $\rightarrow$ higher $p$ desirable, as $n$ teams are more evenly distributed so the rankings are more stable

Sensitivity
Partially inspired by Chartier et al. (2011),

1. Apply Sherman-Morrison inversion formula to find $L_p^{-1}$
2. Multiply by $s_p$ to determine ratings in the perfect season (Figure 2)
3. Prove rating formula with Gaussian elimination and induction
4. Find $e$ wins needed for Team $j$ against Team $i < j$ for Team $j$ to start to overtake Team $i$ in rank (Figure 2)

For $n$ teams and $p \neq 0$, the $i$-th rating in the perfect season is

$$v_i = \frac{(n - i + 1)(n - i) + i - 1}{((n - i + 1)p + i - 1)((n - i)p + i - 1)}.$$

The perturbed rating vector is

$$v_p = v_0 + [-e + \frac{1}{4p + 1} + \frac{1}{p + 4} - \frac{e}{4(p + 1)}(p + 4) + 2(b)^2 + p + 2(e - 4)]a_e.$$ (4)

Figure 2: Sensitivity (a) in the perfect season and (b) with upsets.

Conclusions
• Proved a general formula for the ratings of all $n$ alternatives for any point in our Laplacian ranking family
• Lower $p$ values are desirable for partial ranking applications
• For $0 < p < 1$ ratings can also be negative and sum to 0.

References

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