Feed a Fever...,
or
"How long should I leave a thermometer in my mouth to take my body temperature accurately?"

Abstract:

The purpose of this project is to apply Newton's Law of Cooling to study the rate of change of the body's temperature. To be used in a classroom setting, this project requires three ordinary mercury-filled glass thermometers, three modern electronic thermometers (the kind which give an accurate temperature within a fraction of a minute in a digital readout), a pack of chewing gum, and a few glasses of cold water (preferably with ice). In addition, some material for cleaning the thermometers (alcohol wipes, for example) will be needed if the thermometers are to be reused in a classroom environment. This Interdisciplinary Lively Applications Project is based upon the article "Fever," published by Elmo Moore and Charles M. Biles in *The UMAP Journal*, and it has been revised by Michael Huber, Department of Mathematics and Computer Science at Muhlenberg College.

Introduction

In this project we will study the rate of change of a body's temperature. The following excerpt from Moore and Biles in *The UMAP Journal* [1] presents an interesting situation to ponder:

> *Normal body temperature is commonly understood to be* $98.6°F$ *(*$37°C$*). However, normal temperature fluctuates* $2°F$ *to* $3°F$ *daily between early morning and late afternoon;* $98.6°F$ *is simply a statistical average. Normal body temperature can be elevated by muscular activity. For example, strenuous athletic activity can raise body temperature as much as* $5°F$*, and vigorous gum chewing can raise an oral temperature by as much as* $1°F$*. Similarly, drinking a glass of ice water can lower an oral temperature by* $1.6°F$ *.*

How can we accurately measure a person's body temperature? The hypothalmus gland regulates a person's body temperature, responding to both heat loss and heat production of the body. According to [1], each 1°C increase in your body's temperature increases your body's heat production by 13%. This in turn causes an increase in your need for water by 15%. Think about how many times you have heard the following remedy for the flu: "Take aspirin, drink lots of fluids, and rest." The aspirin is designed to lower the body's temperature, as will the fluids. The rest will not cause an increase due to strenuous activity.

We have all learned that a person's temperature can be measured with a glass thermometer in three different ways: oral, rectal, and axilla (under the armpit). The rectal temperature is considered the most accurate, usually measuring $\approx 0.7°$F higher than an oral measurement and about 0.3°F lower than an axilla measurement. We will probably all agree that the oral method for measuring a body's temperature is the most frequently used, both by medical professionals and at home, and it will be the choice in this experiment. Modern electronic technology allows for measurement inside the ear as well, but we will confine our work to using two types of oral thermometers.

The length of time the thermometer is left in place is critical to the accuracy of the reading. What is your experience? Do you remember someone shaking the thermometer to get the mercury to fall somewhere around 92 or 94°F, then placing the thermometer under your tongue and being told to keep it there (without talking) for two to three minutes? Nursing practice textbooks recommended keeping the mercury-filled thermometers under the tongue for eight to nine minutes. So, in the first experiment, we will determine how long we should keep the glass thermometer in the mouth to *accurately* take the temperature.

Newton's Law of Cooling

Let T be the temperature of a person's body and t be time. The rate of change of an object's temperature is directly proportional to the difference of the object's temperature and the temperature of the surrounding environment. Assuming that the latter temperature (T_E) is constant, this relationship is known as Newton's Law of Cooling. In the experiments we will consider, Moore and Biles refer to it as "Newton's Law of Warming." Symbolically, we write Newton's Law as

$$\frac{dT}{dt} = k(T - T_E), \tag{1}$$

where k is a constant of proportionality.

The First Experiment

We will use Equation 1 to experimentally determine temperatures and times. Break the class into three groups. Group A gets to choose a volunteer who is given a piece of gum and told to "chew vigorously" for a few minutes. Group B's volunteer drinks a glass (or bottle) of cold water. The subject from Group C does jumping jacks for a few minutes. Then the temperature readings begin.

Designate someone as timekeeper. While the activity (chewing, drinking, etc.) is taking place, each group shakes its thermometer until the mercury reads below 94°F. When the timekeeper calls, "Time!" each group reads the temperature on the thermometer. This is T_0.

The thermometer is then inserted into the mouth of the group's volunteer (under the tongue — no talking!). After one minute, the timekeeper calls, "One minute!" and the thermometer is read again (while still in the mouth). How long until the temperature stabilizes? Continue keeping the thermometer in the mouth for at least two more minutes.

Back to Equation 1. Separate variables and integrate

$$\int \frac{dT}{T(t) - T_E} = \int k dt$$

to obtain

$$\ln(T(t) - T_E) = kt + c,$$

where c is a constant of integration. Using $T(0) = T_0$ (the "shake-down temperature") and solving for $T(t)$ yields

$$T(t) = T_E + (T_0 - T_E)\, e^{kt}. \tag{2}$$

We could rewrite Equation 2 as

$$T(t) - T_E = (T_0 - T_E)\, e^{kt}. \tag{3}$$

The goal is to solve for the parameter k. Using natural logarithms,

$$k = \frac{1}{t} \ln\left(\frac{T(t) - T_E}{T_0 - T_E}\right). \tag{4}$$

So, how long do we leave the thermometer in place? In most instances, since the graduations on the glass thermometer are every 0.2°F, the accuracy will be 0.1°F. Therefore, we will need to continue reading the temperature until $T(t) - T_E < 0.1$. Using Equation 3,

$$\begin{aligned} T(t) - T_E &< 0.1 \\ (T_0 - T_E)\, e^{kt} &< 0.1 \\ kt &< \ln\left(\frac{0.1}{T_0 - T_E}\right) \\ t &> \frac{1}{k} \ln\left(\frac{0.1}{T_0 - T_E}\right). \end{aligned}$$

Let's try an example. Suppose we use Group C with $T_0 = 93.0$°F, and that after one minute, $T(1) = T_1 = 96.0$°F. Further, suppose Group C's volunteer's temperature stabilized at $T_E = 100.0$°F (slightly above normal due to the vigorous exercise). Using these conditions with Equation 4 yields

$$k = \left(\frac{1}{1}\right) \ln\left(\frac{T_1 - T_E}{T_0 - T_E}\right) = \ln\left(\frac{96.0 - 100.0}{93.0 - 100.0}\right) \approx -0.5596,$$

so

$$T(t) = 100.0 - 7.0\, e^{-0.5596t}, \tag{5}$$

for any time t. This solution is plotted in Figure 1. We calculate the time needed to stabilize the temperature reading as

$$t > -\frac{1}{0.5596} \ln\left(\frac{0.1}{93.0 - 100.0}\right) \approx 3.4773,$$

or, the thermometer must remain under the tongue for at least three and a half minutes. This seems to be much less than the required "eight to nine" minutes. As T_E increases into a real fever, the time to stabilization will increase.

The Second Experiment

In this case, we will repeat the groups and experiments, given three electronic thermometers (which advertise that they reach stabilization in under a minute — several on the market today claim accuracy within less than ten seconds).

The goal of the second experiment is to calculate the value of k, since the time to stabilization is so short. Many of these thermometers have a built-in "Beep!" They tend to beep at regular intervals. When the stabilization temperature is reached, the thermometers tend to beep rapidly. The trick is to have students read the thermometer initially and then immediately after the set of rapid beeping. Measuring T initially and then after staibilation will give the two conditions necessary to solve for the unknowns (c and k).

Repeat Experiment One, except now the timekeeper must be in each group. One member will call "Time!" and then record the temperature on the thermometer just before it is inserted into the subject's mouth (again — under the tongue and no talking!). However, now the timekeeper must be prepared to capture the temperature every ten seconds, until the rapid beeping occurs. Each group therefore has values for $T(0)$ and $T(i)$, where i is the time in seconds. Record also the stabilization temperature, T_E.

Now, have different members of each group solve the ODE (Equation 1 using T_0 and a different intermediate temperature and time. For instance, one student uses T_0 and $T(10)$, the time after ten seconds. Another uses T_0 and $T(20)$, the time after twenty seconds. You get the idea. Calculate k and the stabilization time as in Experiment One. Compare answers. Are the proportionality constants (k) close? Is the stabilization time the same? This leads to a nice discussion about test conditions and Newton's Law.

By using the electronic thermometers, the experiment takes less time and offers more time for calculations and discussion. Also, you could do it again for different members of the group. One more point: students can compare ratios of k values from the two experiments. Group A compares its k_1 / k_2 with those of Groups B and C. Notice anything? As the time to stabilize shortens, what happens to k?

To incorporate writing into this assignment, all data can be taken in class and students can be asked to develop a written report, outlining their findings, showing a graph or two of the temperature versus time scenarios, and explaining their theories about Newton's Law. In the event of a real fever, with T_E several degrees above "normal," what might we expect to see in terms of k and t? If k is kept constant, what happens to t? Even ask the students to come up with a question or two on their own about this experiment. Finally, getting back to the very modern temperature-taking apparatus: how accurate is that device that's put in our ear (you know, the one which gives a temperature in about a second)?

References

[1] E. Moore and C. M. Biles. 1983. "Fever." *The Journal of Undergraduate Mathematics and Its Applications*. Volume IV, Number 3, pages 253 - 258.

[2] R. Borelli and C. Coleman. 2004. *Differential Equations: A Modeling Perspective*. 2d Edition. New York: John Wiley & Sons, Inc.