An Overview of Thin Fluid Films

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Overview

The Problem

Inquiring Minds Want To Know

- How do these behave?
- Height of the fluid is $h(x, t)$.

Now Presenting...

Our Hero, the Thin Film Equation

$$h_t = -(h^n h_{xxx})_x$$

Which Is the Same As...

$$h_t = -nh^{n-1} h_x h_{xxx} - h^n h_{xxxx}$$

Why We Care

- Thin film equation is basis for lubrication theory
  - Would like to know if lubricant will dry up

Conservation

Some energies correspond to physical properties.

Mass

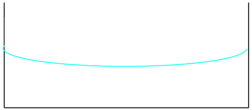
- Mass $= \int h dx$
- Physically, mass is **conserved**:
  - Mass does not change over time
  - $\frac{\partial}{\partial t} \int h dx = 0$

Shaded region indicates mass

Dissipation

Surface Area

- Usually $\int \sqrt{1 + h_x^2} dx$
- Equivalent to consider $\int h_x^2 dx$.
- Physically and theoretically, surface area is **dissipated**:
  - Surface area will decrease if it can
  - $\frac{\partial}{\partial t} \int h_x^2 dx \leq 0$

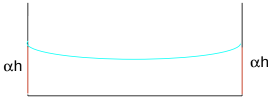


Blue segment indicates surface area

Balance

Coating Energy

- Physically we observe fluid sticking to walls
- If α sets contact angle, coating energy per side is αh
- Coating energy wants to increase, surface area wants to decrease; they must balance each other



Red segments indicate coating energy; blue, surface area

Statement of Theorem

Theorem

There is no rupture in any film for $n > 3.5$.

Note

We say the film has "ruptured" if its height becomes 0 at any point.

Useful Facts

- Fact #1

$$\int a^2 \cdot \int b^2 \geq \int |ab|$$

- Fact #2

  If $a < 0$ and $x \leq y$, then $x^a \geq y^a$.

Proof

Proof

Define

$$E = \int h_x^2 dx, \qquad P_m = \int h^m dx.$$

Then according to Useful Fact #1,

$$\sqrt{EP_m} \geq \int \sqrt{h_x^2 h^m} dx$$

But $h^{m/2} \cdot h_x = C \cdot \partial / \partial x (h^{m/2+1})$.

Bounding

Note

For a periodic function f, $\int |f_x| dx \geq f_{max} - f_{min}$.

Substitution...

So $\sqrt{EP_m} \geq C[max(h^{m/2+1}) - min(h^{m/2+1})]$, or

$$C \cdot min(h^{m/2+1}) + \sqrt{EP_m} \geq max(h^{m/2+1})$$

More Bounding

The Crucial Assumption!

Suppose $m/2+1<0$ and E and P_m are bounded.

And That Means...

- We know $max(h) \geq \bar{h}$
- By Useful Fact #2: $min(h^{m/2+1}) \leq \bar{h}^{m/2+1}$

And More Bounding

- $min(h^{m/2+1})$ is bounded, $\sqrt{EP_m}$ is bounded
- This means $max(h^{m/2+1})$ is bounded
- If $h \to 0$, $h^{m/2+1} \to \infty$. Contradiction!
- So $h \not\to 0$.

Calculation

A Little Calculation

- $m = 3/2 - n$
  - (That's the m from P_m and the n from the equation!)
- So if $m/2 + 1 < 0$, then $3/4 - n/2 + 1 < 0$...
- Which means $n > 7/2$.

References

On the Web

Senior thesis website:
⟨http://www.math.hmc.edu/∼rmbaur/thesis/⟩

In Print

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